Benjamin Franklin and His Magic Squares

If you think Benjamin Franklin was a man of science only because of his experiments with electricity, you are wrong. A little known fact is that he has also contributed to mathematics – via magic squares. A couple of these squares have been published, as well as a magic circle, whereas others remained buried in the 35+ volumes of the *Papers of Benjamin Franklin* stored at the American Philosophical Society in Philadelphia.

The man who founded the University of Pennsylvania was not a born mathematician. In fact, he discussed his troubles in his autobiography, admitting, "I acquired fair writing pretty soon, but I failed in the arithmetic, and made no progress in it." Nevertheless, Franklin continued to teach himself mathematics, and one may find several references to calculus among the letters of Franklin. His attitude toward mathematicians, however, was not as positive. About his friend Thomas Godfrey, Franklin assessed, "He was not a pleasing companion; as, like most great mathematicians I have met with, he expected universal precision in everything said, or was for ever denying or distinguishing upon trifles, to the disturbance of all conversation." Despite Franklin's first unsuccessful scholastic attempts and his opinion of mathematicians in general, he still managed to make fascinating contributions to mathematics. On one hand, his *Poor Richard's Almanac*, aside from witty adages, included eclipse predictions and other astronomical details, even word problems, mainly in the form of multiplication riddles.

Perhaps the most remarkable of Franklin's contributions is the collection of magic squares. A set of six squares and one circle was quite unique. His magic squares did not match the characteristics of the traditional magic square. A *semi-magic square* is a square matrix whose row and column sums are all equal. If a square was fully magic, the diagonal sums would also be equal to the same constant. A *Franklin magic square* is a semi-magic square with each of the four main bent row sums equal to the magic constant. A "semi" is associated if k and its complement $n^2 + 1 – k$ are symmetrically placed with respect to the center, for $1 < k < n$; this would become automatically fully magic. Lastly, if all 2n diagonals of a fully magic square sum to the magic constant, it is called *pandiagonal*.

Only two pandiagonal squares were published during Franklin's lifetime. The first is a square of eight blocks. In the *Letters of Franklin,* he points out its various properties. It is not associated, but semi-magic. Row and column sums are 260, each half-row and half-column sums to 130, and all of the parallel bent rows sum to 260. He notes, "The four corner numbers, with the four middle numbers, make 260." Also, any 2 x 2 block adds up to 130 and the twelve disconnected bent rows add up to 260.

One of Franklin's friends pointed out that there were far greater squares, such as the 16-square presented by M. Stifel in *Arithmetica Integra* (1544). In his autobiography, Franklin noted, "Not willing to be outdone by Mr. Stifelius … I went home, and made, that evening, the following magical square of sixteen." After seeing this square, his friend, James Logan, refers to it as "the most magically magical of any magic square ever made by any magician." In this square, the rows, columns, and four main bent rows sum to the same number. All 36 connected bent rows and 28 disconnected bent rows work as well. Every 2 x 2 square sums to one-quarter the magic sum. Also, the half-rows and half-columns add to half the magic sum. This square was first published in Ferguson's *Tables and Tracts Relative to Several Arts and Sciences* (1767). It was also published in the *Gentleman's Magazine*, and the fourth edition of *Franklin's Experiments and Observations in Electricity.*

The next square was curiously found on a piece of scrap paper among the Franklin Papers at the American Philosophical Society, undated and unsigned. Its properties are similar to those of the first two. Here, there are also many sets of eight numbers that add to a magic sum when combined in a "knight's move" pattern. But since there was no text or correspondence with this square, it appears it was only for personal use.

Many questions remained about the formulation of Franklin magic squares. Why use bent rows? How are these formed? In a letter from 1768, Franklin noted, "I am told they have occasioned a good deal of puzzling among the mathematicians … no one has desired me to show him my method … they wish rather to investigate themselves." Nevertheless, several mathematicians attempted to decode the construction of these squares, including Van Doren, Siegmund Günther, J. Emory McClintock, and J. Moran.

Like magic squares, magic circles date back as far as Yang Hui in the 1200s. Unlike squares, magic circles are not standardized. Some have consisted of cells lining nonconcentric circles, some include numbers that are placed at the intersections of interlocking coplanar circles, and others are placed in concentric annuli. Franklin's circle combines the traits of interlocking circles and concentric circles, with a boggling array of properties. Numbers 12 to 75 are arranged in concentric annuli, with 12 in the center. The sum of any of the numbers in any annulus with the central number is 360. This is also the sum along a radius and the center. The sum of the upper half of an annulus with the center is the number of degrees in a semi-circle. Twenty "excentric" annuli, where the numbers inside and outside the central number, also make 360. Furthermore, "any four adjoining numbers, standing nearly in a square … and added with half the central number, they make 180." Franklin also stated, "There is no one of the numbers but what belongs to at least two different circles."

Paul C. Pasles, an authority on Franklin's magic [and a Ph.D. from Temple—editor] has published a method for the construction of the circle that also answers the previous questions about the squares. Using the aforementioned, unpublished Franklin square, Pasles used permutations and matrix operations and physically morphed the square so that the right and left edges meet. He then changed a few of the numbers so that the sums would be those of above. Pasles does, in fact, keep track of the construction in a way that is agreeable with methods Franklin might have used. The result is the circle. And so it seems the unsigned square was a draft to the circle. In fact, this method also explains why Franklin used his bent rows to the traditional ones. According to Pasles, "A fully magic square produces a magic circle with spiral magic sums, not excentric circles. The 20 excentric circles come from the 20 internal bent rows."

Besides these figures, Franklin produced three more magic squares, which he hints at in a letter dated 1765 to friend John Canton. Unfortunately these papers were "lost" for quite a bit of time; in fact, the originals are still missing. Luckily, a copy was found in the library of the Royal Society. The properties of these squares are entirely different from those of the previous ones.

The 6-square is a Franklin magic one; four more parallel bent rows also have the magic sum. In the upper half, each triplet exhibits the "knight's move" pattern within the respective 3 x 6 rectangle. Also, if the square is separated into quadrants, two quadrants are fully magic and the others are very close.

The 4-square, not being Franklin magic but fully magic, is the same as Dürer's *Melancholia* square. It is likely Ben did not know it already existed. It is associated and each quadrant adds up to the magic sum. Every other parallel diagonal works as well.

The last square is a square of 16. The most interesting thing about this square is that it is both fully and Franklin magic. It is also pandiagonal. The parallel bent rows and some nonparallel bent rows also work. The half-rows and half-columns sum to 1028. The 2 x 2 block property is also satisfied. Also noteworthy is that there is an unusual variation of the knight's move. Here, the square was built two rows at a time, instead of columns, and the integers are partitioned into subsets, as well as the complements. They form the square in row pairs from the top using knight's moves.

From this example it seems obvious that Franklin knew a number of methods to generate magic squares. After all, he was an expert in a multitude of fields. In an excerpt from his writings, however, he reflects, "The magical squares are what I cannot value myself upon, but am rather ashamed to have it known I spent any part of my time in an employment that cannot possibly be of any use." Franklin, like many mathematics majors today, just did not fully realize the extent of his own mathematical skill.

Kate Ploskina
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Sources

1. Benjamin Franklin, *The Autobiography and Other Writings* (Ed. Kenneth Silverman), New York: Penguin, 1986.
2. Pasles, Paul C., The lost squares of Dr. Franklin, *American Mathematical Monthly* **108** (June-July 2001) 489-511.

Readers are encouraged to visit the web site of Paul Pasles at

http://www.pasles.org/Franklin.html

in order to view Franklin's magic squares and circle.