On the commutation properties of finite convolution and differential operators II: sesquicommutation.

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Abstract

We introduce and fully analyze a new commutation relation $\overline{K}L_1 = L_2K$ between finite convolution integral operator K and differential operators L_1 and L_2, that has implications for spectral properties of K. This work complements our explicit characterization of commuting pairs $KL = LK$ and provides an exhaustive list of kernels admitting commuting or sesquicommuting differential operators.

1 Introduction

In many applications it is important to understanding spectral properties of finite convolution integral operators

$$(Ku)(x) = \int_{-1}^{1} k(x-y)u(y)\mathrm{d}y, \tag{1.1}$$

especially, when such operators are compact and self-adjoint, i.e. when $k(z)$ is smooth and $k(-z) = \langle k(z)|$, $z \in [-2, 2]$. No general algorithm exists for answering this question. One approach that can work in certain cases calls for comparison of a given operator to a special one that commutes with a differential operator as was done in [4], for example. In the first part of this work [1] we have examined all such operators with possibly complex $k(z)$, extending an earlier result of Morrison [3] (see also [2, 5]) for real-valued $k(z)$. Unfortunately, no essentially new cases of commutation were discovered: all self-adjoint compact operators (1.1) with complex-valued $k(z)$ that commute with differential ones were conjugate to Morrison's. In an attempt to significantly enlarge the set of special operators we introduce a new type of commutation that we call *sesquicommutation*:

$$\begin{cases} \overline{K}L_1 = L_2K, \\ L_j^T = L_j, \end{cases} \qquad j = 1, 2, \tag{C}$$

where L_1, L_2 are differential operators with complex coefficients.

We note that Morrison's result lies in the intersection of commutation and sesquicommutation (with $L_1 = L_2$), when K is real and self-adjoint, since in this case sesquicommutation reduces to commutation.

The main case of interest is for self-adjoint operator K. However, even if K is not self-adjoint (but compact) the sesquicommutation (C) permits us to relate singular values and functions of K to solutions of differential equations. It can be easily checked that (C) implies

$$L_1 K^* K = \overline{K^* K} L_1. \tag{1.2}$$

Let now λ be a singular value of K corresponding to singular function u, i.e. $K^*Ku = \lambda u$, clearly $\lambda \in \mathbb{R}$ and therefore we find $\lambda \overline{L_1 u} = K^*K\overline{L_1 u}$. It follows that $\overline{L_1 u}$ is either zero, or an eigenfunction of K^*K with the same eigenvalue λ. If the corresponding eigenspace of K^*K is one-dimensional, then there exists a complex number σ such that

$$L_1 u = \sigma \overline{u}.$$

Otherwise, applying (1.2) to $\overline{L_1 u}$ we find that

$$K^*K(L_1^* L_1 u) = \lambda L_1^* L_1 u,$$

hence eigenspaces of K^*K are invariant under the fourth order self-adjoint operator $L_1^* L_1$. In particular, there exists an eigenbasis of K^*K consisting of eigenfunctions of $L_1^* L_1$. Moreover, transposing the sesquicommutation relation and then taking adjoint we find $KL_1^* = L_2^* \overline{K}$, which along with (C) implies

$$KL_1^* L_1 = L_2^* L_2 K.$$

In particular if $L_1 = L_2 =: L$ we see that L^*L commutes with K (and also with K^*), hence eigenspaces of L^*L are invariant under K and K^*.

Under the assumption that K is self-adjoint we prove in Theorem 1 that k is trivial (see Definition 1), unless $L_1 = L_2$ or $L_1 = -L_2$. We then show in Theorem 4 that the latter case yields only trivial kernels. The results in the former case are listed in Theorem 2, which presents a new class of finite convolution operators whose spectral properties will be amenable to analysis by means of differential equations.

As a particularly interesting example derived from sesquicommutation, we mention that the eigenfunctions of the compact self-adjoint integral operator K with kernel $k(z) = \dfrac{e^{-i\frac{\pi}{4}z}}{\cos\frac{\pi}{4}z} + \dfrac{ze^{i\frac{\pi}{4}z}}{\sin\frac{\pi}{2}z}$ are eigenfunctions of the fourth order self-adjoint differential operator L^*L, where

$$L = -\frac{\mathrm{d}}{\mathrm{d}y}\left[\cos\left(\frac{\pi y}{2}\right)\frac{\mathrm{d}}{\mathrm{d}y}\right] + \frac{\pi^2}{32}e^{i\frac{\pi y}{2}}.$$

The corresponding integral operator K is self-adjoint and compact, since singularities at $z = \pm 2$ of $k(z)$ are removable.

The strategy for obtaining the complete list of sesquicommuting pairs in Theorem 2 is the same as in analyzing commutation in Part I of this work [1]. Sesquicommutation is written in terms of the kernel $k(z)$ and coefficients of the differential operator L. From this relation we obtain differential equations satisfied by the coefficients of L and $k(z)$.

2 Preliminaries

We assume that $k(z) \in L^2((-2, 2), \mathbb{C})$ is analytic in a neighborhood of 0. Further, assume that L_j are second order differential operators:

$$\begin{cases} Lu = \mathscr{a}u'' + \mathscr{b}u' + \mathscr{c}u, \\ \mathscr{a}(\pm 1) = 0,\ \mathscr{b}(\pm 1) = \mathscr{a}'(\pm 1), \end{cases} \tag{2.1}$$

where the indicated boundary conditions are necessary for the sesquicommutation relation to hold. They are also necessary for symmetry of differential operators, in which case we will only be specifying additional constraints on the coefficients of L, always assuming that the boundary conditions in (2.1) hold. In particular operators L_j have to be of Sturm-Liouville type, since $L = L^T$ implies that $\mathscr{b} = \mathscr{a}'$. Thus

$$\begin{cases} L_j u = (\mathscr{b}_j u')' + \mathscr{c}_j u, \\ \mathscr{b}_j(\pm 1) = 0, \end{cases} \qquad j = 1, 2. \tag{2.2}$$

Due to the imposed boundary conditions it is a matter of integration by parts to rewrite (C) as

$$\begin{aligned} \mathscr{b}_1(y)\overline{k''(z)} - \mathscr{b}_2(y+z)k''(z) - \mathscr{b}_1'(y)\overline{k'(z)} - \mathscr{b}_2'(y+z)k'(z) + \\ + \mathscr{c}_1(y)\overline{k(z)} - \mathscr{c}_2(y+z)k(z) = 0. \end{aligned} \tag{R}$$

The main idea of the proof is to analyze (R) by differentiating it w.r.t. z sufficient number of times and evaluating the result at $z = 0$. This allows one to find relations between the coefficient functions of the differential operators, and an ODE for the highest order coefficient. Once the form of the highest order coefficient is determined, we consequently find the forms of all the other coefficient functions. It turns out that the coefficient functions satisfy linear ODEs with constant coefficients, and therefore are equal to linear combinations of polynomials multiplied by exponentials. We then substitute these expressions into (R) and using the linear independence of functions $y^j e^{y\lambda_l}$, obtain equations for k. Then the task becomes to analyze how many of these equations can be satisfied by k and how its form changes from one equation to another.

Remark 1. The reason that reduction of (C) to $L_1 = \pm L_2$ (see Section 5) works, is the self-adjointness assumption on K. This induces symmetry in (R). More precisely, (R) becomes a relation involving the even and odd parts (and their derivatives) of the function $k(z)e^{\frac{\lambda}{2}z}$. And as a result the relations for even and odd parts separate. We then prove that if $L_1 \neq \pm L_2$, then both even and odd parts of k are determined in a way that k becomes trivial.

3 Main Results

Definition 1. We will say that k (or operator K) is *trivial*, if it is a finite linear combination of exponentials $e^{\alpha z}$ or has the form $e^{\alpha z}p(z)$, where $p(z)$ is a polynomial. Note that in this case K is a finite-rank operator.

Let us assume that

(A) K is self-adjoint, so $k(-z) = \overline{k(z)}$, $\qquad z \in [-2, 2]$.

Theorem 1 (Reduction of sesquicommutation)
Let K, L_1, L_2 be given by (1.1) *and* (2.2) *with $\mathscr{b}_j, \mathscr{c}_j, k$ smooth in $[-2,2]$. Assume k is nontrivial, (A) holds, and k is analytic at* 0*, but not identically zero near* 0*. Then* (C) *implies either $L_1 = L_2$ or $L_1 = -L_2$.*

Remark 2. Let M be the multiplication operator by $z \mapsto e^{\tau z}$ with $\tau \in i\mathbb{R}$, then MKM^{-1} is a finite convolution operator with kernel $k(z)e^{\tau z}$ (where k is the kernel of K), which is also self-adjoint since so is K. If K sesquicommutes with L, i.e. $\overline{K}L = LK$, then MKM^{-1} sesquicommutes with $M^{-1}LM^{-1}$. With this observation the results of Theorem 2 are stated up to multiplication of k by $e^{\tau z}$, i.e. we chose a convenient constant τ in order to more concisely state the results.

Theorem 2 ($L_1 = L_2$)
Let K, L_1, L_2 be given by (1.1) *and* (2.2)*, with $L_1 = L_2$ and let their coefficient functions be $\mathscr{b}$ and $\mathscr{c}$. Let $\mathscr{b}, \mathscr{c}, k$ be smooth in $[-2,2]$. Further, assume k is nontrivial, (A) holds, k is analytic at* 0*, but not identically zero near* 0*. Then* (C) *implies (all the used parameters are real, unless stated otherwise)*

1. $k(z) = \dfrac{\gamma \sinh \mu z}{\mu \sinh \gamma z}$.

$$\begin{cases} \mathscr{b}(y) = \frac{1}{2\gamma^2}\left[\cosh(2\gamma y) - \cosh(2\gamma)\right], \\ \mathscr{c}(y) = (\gamma^2 - \mu^2)\mathscr{b}(y) + c_0, \end{cases}$$

   where $\mu \in \mathbb{R} \cup i\mathbb{R}$ and $c_0 \in \mathbb{C}$.

2. $k(z) = \alpha e^{-i\mu z} + \dfrac{\sin \mu z}{z}$, $\alpha \neq 0$ *and*

$$\begin{cases} \mathscr{b}(y) = y^2 - 1, \\ \mathscr{c}(y) = i\mu\mathscr{b}'(y) + \mu^2\mathscr{b}(y) + \frac{\mu}{\alpha}. \end{cases}$$

3. $k(z) = \dfrac{\sinh(2\mu_2)\sinh(\mu_1 z)e^{-\frac{i\pi}{4}z} + \sinh(2\mu_1)\sinh(\mu_2 z)e^{\frac{i\pi}{4}z}}{\mu_1\mu_2 \sin\frac{\pi z}{2}}$ *and*

$$\begin{cases} \mathscr{b}(y) = -\cos\frac{\pi y}{2}, \\ \mathscr{c}(y) = i\frac{\mu_2^2-\mu_1^2}{\pi}\mathscr{b}'(y) - \left(\frac{\pi^2}{16} + \frac{\mu_1^2+\mu_2^2}{2}\right)\mathscr{b}(y), \end{cases} \tag{3.1}$$

   where $\mu_1, \mu_2 \in \mathbb{R} \cup i\mathbb{R}$. In the special case $\mu_1 = i\mu$; $\mu_2 = i(\mu \pm \frac{\pi}{2})$ with $\mu \in \mathbb{R}$, to $\mathscr{c}(y)$ a complex multiple of $e^{-2i(\frac{\pi}{4} \pm \mu)y}$ can be added.

Remark 3.

(i) In items 1 and 3, if μ, μ_j or $\gamma = 0$, one takes appropriate limits. Note that k can be multiplied by arbitrary real constant and $L_1 = L_2$ by a complex one.

(ii) Using the same proof techniques one can easily check that under the given assumptions of the theorem, no kernel would satisfy the sesquicommutation relation, when $L_1 = L_2$ is a first order operator.

(iii) In item 1, K is real valued and self-adjoint, in particular sesquicommutation reduces to commutation and we recover Morrison's result.

(iv) Widom's theory of asymptotics of eigenvalues applies only if $k(z)$ has an even extension to $\mathbb{R}$ such that $\hat{k}(\xi)$ is nonnegative and monotone decreasing, at least when $\xi \to \infty$. Item 2 corresponds to $\hat{k}(\xi)$ being a characteristic function of an interval plus a delta-function, centered anywhere one likes. Item 3 is the most puzzling, it is unknown if there is an extension whose Fourier transform is nonnegative and monotone decreasing. Item 1 are all even kernels.

From the discussion in the introduction we immediately obtain:

Corollary 3. Let K be one of the operators of Theorem 2 and let L be corresponding operator that sesquicommutes with it (i.e. $\overline{K}L = LK$), then L^*L commutes with K. In particular, the eigenfunctions of K are eigenfunctions of the fourth order self-adjoint differential operator L^*L. Moreover, if eigenspaces of K are one-dimensional, then eigenfunction u of K satisfies second order differential equation $Lu = \sigma\overline{u}$ for some $\sigma \in \mathbb{C}$.

Remark 4. The example mentioned in the introduction is obtained from item 3 of Theorem 2 by choosing $\mu_2 = 0,\ \mu_1 = \frac{i\pi}{4}$.

Theorem 4 ($L_1 = -L_2$)
Let K, L_1, L_2 be given by (1.1) *and* (2.2)*, with $L_1 = -L_2$ and let the coefficients of L_1 be $\not{b}$ and $\not{c}$. Let $\not{b}, \not{c}, k$ be smooth in $[-2, 2]$. Further, assume (A) holds, k is analytic at* 0*, but not identically zero near* 0*. If* (C) *holds true, then k is trivial.*

4 Relations for coefficients

In this section we consider (C) with L_1, L_2 given by (2.2). We assume (A) holds, k is analytic at 0, but not identically zero near 0 and finally k is not of the form $e^{\alpha z}$. We aim to find the relations that the coefficient functions $\not{b}_j, \not{c}_j$ must satisfy. Write $k(z) = \sum_{n=0}^{\infty} \frac{k_n}{n!} z^n$ near $z = 0$. The n-th derivative of (R) w.r.t. z at $z = 0$ gives

$$(-1)^n[\not{b}_1 k_{n+2} + \not{b}_1' k_{n+1} + \not{c}_1 k_n] - \sum_{j=0}^{n} C_j^n \not{b}_2^{(n-j)} k_{j+2} - \sum_{j=0}^{n} C_j^n \not{b}_2^{(n-j+1)} k_{j+1} - \sum_{j=0}^{n} C_j^n \not{c}_2^{(n-j)} k_j = 0, \tag{4.1}$$

where $C_j^n = \binom{n}{j}$, when $n = 0$ we get

$$k_1(\not{b}_1' - \not{b}_2') + k_2(\not{b}_1 - \not{b}_2) + k_0(\not{c}_1 - \not{c}_2) = 0.$$

• If $k_0 = k_1 = 0$, then let us show that k is trivial. Assume first $\mathscr{b}_1 \neq \pm\mathscr{b}_2$, then clearly $k_2 = 0$. Let us prove by induction that all $k_j = 0$, which contradicts to the assumption that k doesn't vanish near 0. Assume $k_j = 0$ for $j = 0, ..., m$, then (4.1) for $n = m - 1$ reads $[(-1)^{m-1}\mathscr{b}_1 - \mathscr{b}_2]\, k_{m+1} = 0$, therefore $k_{m+1} = 0$. Let now $\mathscr{b}_1 = \mathscr{b}_2$, assume for the induction step that $k_j = 0$ for $j = 0, ..., n$, then (4.1) reads

$$[(-1)^n - 1]\, k_{n+2}\mathscr{b}_1 + [(-1)^n - n - 1]\, k_{n+1}\mathscr{b}_1' = 0.$$

When n is odd we immediately obtain $k_{n+1} = 0$. When n is even we get $(n+2)k_{n+1}\mathscr{b}_1' + 2k_{n+2}\mathscr{b}_1 = 0$ and because of boundary conditions $\mathscr{b}_1(\pm 1) = 0$ we deduce $k_{n+1} = k_{n+2} = 0$. Finally, the case $\mathscr{b}_1 = -\mathscr{b}_2$ can be done analogously.

• If $k_0 = 0, k_1 \neq 0$, by rescaling let $k_1 = 1$ and by considering $e^{-\frac{k_2}{2}z}k(z)$ instead of $k(z)$ (see Remark 2) we may assume $k_2 = 0$. Now, $\mathscr{b}_2(y) = \mathscr{b}_1(y) + \alpha$ for some $\alpha \in \mathbb{C}$. From (4.1) with $n = 1$ we find $\mathscr{c}_2 = -\mathscr{b}_1'' - 2k_3\mathscr{b}_1 - \mathscr{c}_1 - k_3\alpha$. Using the obtained expressions, from the relation corresponding to $n = 2$ we get

$$\mathscr{c}_1' = -\tfrac{1}{2}\mathscr{b}_1''' - k_3\mathscr{b}_1' + \tfrac{k_4\alpha}{2}. \tag{4.2}$$

Now, (4.1) with $n = 3$ reads

$$2\mathscr{b}_1^{(4)} + k_3\mathscr{b}_1'' - 5k_4\mathscr{b}_1' + 2(k_3^2 - k_5)\mathscr{b}_1 + 3\mathscr{c}_1'' + \alpha(k_3^2 - k_5) = 0.$$

Let us now replace $\mathscr{c}_1''$ using (4.2). The result becomes an ODE for $\mathscr{b}_1$: for some constants α_j,

$$\mathscr{b}_1^{(4)} + \sum_{j=0}^{3}\alpha_j\mathscr{b}_1^{(j)} = \alpha_4.$$

• If $k_0 \neq 0$, by rescaling let $k_0 = 1$ and by considering $e^{-k_1 z}k(z)$ instead of $k(z)$ (see Remark 2) we may assume $k_1 = 0$. Note that $\mathscr{c}_2 = \mathscr{c}_1 + k_2(\mathscr{b}_1 - \mathscr{b}_2)$, using this in (4.1) with $n = 1$, we get

$$\mathscr{c}_1' = -k_3(\mathscr{b}_1 + \mathscr{b}_2) - k_2(2\mathscr{b}_1' + \mathscr{b}_2'). \tag{4.3}$$

The relation for $n = 2$ reads

$$-k_2(\mathscr{b}_1'' + 2\mathscr{b}_2'') + k_3(\mathscr{b}_1' - 3\mathscr{b}_2') + (k_4 - k_2^2)(\mathscr{b}_1 - \mathscr{b}_2) - \mathscr{c}_1'' = 0,$$

and replacing $\mathscr{c}_1''$ using (4.3) we obtain

$$k_2(\mathscr{b}_1'' - \mathscr{b}_2'') + 2k_3(\mathscr{b}_1' - \mathscr{b}_2') + (k_4 - k_2^2)(\mathscr{b}_1 - \mathscr{b}_2) = 0.$$

Consider the following cases:

1. If $k_2 = k_3 = 0$, then we are going to show that k is trivial. Assume first that $\mathscr{b}_1 \neq \pm\mathscr{b}_2$, so from the above equation $k_4 = 0$. Further, we see that in this case $\mathscr{c}_1 = \mathscr{c}_2 = \text{const}$. Let now $k_j = 0$ for $j = 1, ..., n+1$, then (4.1) reads

$$k_{n+2}\left[(-1)^n \mathscr{b}_1 - \mathscr{b}_2\right] = 0,$$

so $k_{n+2} = 0$ and by induction $k_j = 0$ for any $j \neq 0$, i.e. k is trivial. When $\mathscr{b}_1 = \mathscr{b}_2$, or $\mathscr{b}_1 = -\mathscr{b}_2$ the result follows analogously.

2. If $k_2 = 0$ and $k_3 \neq 0$, then $\mathscr{b}_2(y) = \mathscr{b}_1(y) + \alpha e^{\tau y}$ with $\tau = -\frac{k_4}{2k_3}$ and some $\alpha \in \mathbb{C}$. From (4.1) with $n = 3$ (by replacing $\mathscr{c}_1'''$ using (4.3)) we find

$$\mathscr{c}_1 = -\tfrac{\alpha}{2k_3}(5\tau^2 k_3 + 4\tau k_4 + k_5)e^{\tau y} - 2\mathscr{b}_1'' - \tfrac{5k_4}{2k_3}\mathscr{b}_1' - \tfrac{k_5}{k_3}\mathscr{b}_1.$$

Finally we replace this and $\mathscr{b}_2$ in (4.3) to obtain, for some other constants α_j

$$\mathscr{b}_1^{(3)} + \sum_{j=0}^{2} \alpha_j \mathscr{b}_1^{(j)} = \alpha_3 e^{\tau y}.$$

3. If $k_2 \neq 0$, then $\mathscr{b}_2(y) = \mathscr{b}_1(y) + f(y)$ and f solves $k_2 f'' + 2k_3 f' + (k_4 - k_2^2)f = 0$, so either $f(y) = \lambda_1 e^{\tau_1 y} + \lambda_2 e^{\tau_2 y}$ or $f(y) = (\lambda_1 y + \lambda_2)e^{\tau y}$. Using the ODE for f, (4.1) for $n = 3$ can be written as

$$4k_2\mathscr{b}_1''' + 6k_3\mathscr{b}_1'' + 5k_4\mathscr{b}_1' + 2k_5\mathscr{b}_1 + \mathscr{c}_1''' + 3k_2\mathscr{c}_1' + 2k_3\mathscr{c}_1 = -k_4 f' + (k_2k_3 - k_5)f.$$

Let us now replace $\mathscr{c}_1'''$ and $\mathscr{c}_1'$ in the above relation using (4.3). The result becomes

$$\begin{aligned} 2k_3\mathscr{c}_1 = &- k_2\mathscr{b}_1''' - 4k_3\mathscr{b}_1'' + (9k_2^2 - 5k_4)\mathscr{b}_1' + (6k_2k_3 - 2k_5)\mathscr{b}_1 + \\ &+ (4k_2^2 - 2k_4 + 2\tfrac{k_3^2}{k_2})f' + (3k_2k_3 - k_5 + \tfrac{k_3k_4}{k_2})f, \end{aligned}$$

but because f' has the same form as f we can rewrite the above relation as

$$2k_3\mathscr{c}_1(y) = -k_2\mathscr{b}_1''' + \sum_{j=0}^{2} \gamma_j \mathscr{b}_1^{(j)}(y) + f(y),$$

with different constants λ_j in f and γ_j are some constants. Now if $k_3 = 0$ we got an ODE for $\mathscr{b}_1$, otherwise divide by it and substitute the obtained expression and the expression of $\mathscr{b}_2$ into (4.3), the result is (with different constants)

$$\mathscr{b}_1^{(4)} + \sum_{j=0}^{3} \gamma_j \mathscr{b}_1^{(j)} = f(y).$$

5 Reduction of the general case

In this section we prove Theorem 1, i.e. if k is nontrivial, then $L_1 = L_2$ or $L_1 = -L_2$. Analysis of the previous section shows that $\mathscr{b}_j, \mathscr{c}_j$ are linear combinations of polynomials multiplied with an exponential, moreover the polynomials have degree at most five. So let us consider a typical such term:

$$\mathscr{b}_1(y) \leftrightarrow \left(\sum_{j=0}^{5} b_j y^j\right) e^{\lambda y}, \qquad \mathscr{c}_1(y) \leftrightarrow \left(\sum_{j=0}^{5} c_j y^j\right) e^{\lambda y},$$

and analogous terms in $\mathscr{b}_2, \mathscr{c}_2$ only with possibly different coefficients $\tilde{b}_j, \tilde{c}_j$ respectively. Set $k(z) = \kappa(z) e^{-\frac{\lambda}{2}z}$ and let

$$\kappa_+(z) = \tfrac{1}{2}[\kappa(z) + \kappa(-z)], \qquad \kappa_-(z) = \tfrac{1}{2}[\kappa(z) - \kappa(-z)]. \tag{5.1}$$

Substituting the expressions for $\mathscr{b}_j, \mathscr{c}_j$ and $k(z) = e^{-\frac{\lambda}{2}z}[\kappa_+(z) + \kappa_-(z)]$ into (R), we obtain that a linear combination of terms $y^j e^{\lambda y}$ is zero. From linear independence we conclude that each coefficient must vanish. In particular, the relation corresponding to $y^5 e^{\lambda y}$ reads

$$(b_5 - \tilde{b}_5)\kappa_+'' - \left((b_5 - \tilde{b}_5)\tfrac{\lambda^2}{4} + \tilde{c}_5 - c_5\right)\kappa_+ - (b_5 + \tilde{b}_5)\kappa_-'' + \left((b_5 + \tilde{b}_5)\tfrac{\lambda^2}{4} - \tilde{c}_5 - c_5\right)\kappa_- = 0.$$

Because κ_+ is even, and κ_- is odd we can add the above relation, with z replaced by $-z$, to itself. Like this we separate the above relation into two ODEs one for κ_+ and the other for κ_-:

$$\begin{cases} (b_5 - \tilde{b}_5)\kappa_+'' - \left((b_5 - \tilde{b}_5)\frac{\lambda^2}{4} + \tilde{c}_5 - c_5\right)\kappa_+ = 0, \\ (b_5 + \tilde{b}_5)\kappa_-'' - \left((b_5 + \tilde{b}_5)\frac{\lambda^2}{4} - \tilde{c}_5 - c_5\right)\kappa_- = 0. \end{cases}$$

If $b_5 \neq \pm\tilde{b}_5$, then $\kappa_+ = \cosh(\mu z)$ and κ_- is either z or $\sinh(\mu z)$ for some $\mu \in \mathbb{C}$, therefore k is trivial. Therefore, we consider the following cases:

• $b_5 = \tilde{b}_5$, then obviously $c_5 = \tilde{c}_5$ and we get $b_5\kappa_-'' - \left(\frac{b_5\lambda^2}{4} - c_5\right)\kappa_- = 0$. Assume $b_5 \neq 0$, then by normalization we can make $b_5 = 1$, now with $\mu^2 = \frac{\lambda^2}{4} - c_5$

$$\kappa_-(z) = \begin{cases} \alpha z, & \mu = 0, \\ \alpha \sinh(\mu z), & \mu \neq 0. \end{cases}$$

Using the ODE that κ_- solves, the even part of the relation corresponding to $y^4 e^{\lambda y}$ reads

$$(b_4 - \tilde{b}_4)\kappa_+'' - \left((b_4 - \tilde{b}_4)\tfrac{\lambda^2}{4} + \tilde{c}_4 - c_4\right)\kappa_+ = 0,$$

which immediately implies $b_4 = \tilde{b}_4$, and hence $c_4 = \tilde{c}_4$. Odd part of that relation is

$$z\kappa_+'' + 2\kappa_+' - \mu^2 z\kappa_+ = -\tfrac{2b_4}{5}\kappa_-'' + \left(\tfrac{b_4\lambda^2}{10} - \tfrac{2c_4}{5} + \lambda\right)\kappa_-.$$

Making the change of variables $\kappa_+(z) = \frac{u(z)}{z}$, the left-hand side of the above relation becomes $u'' - \mu^2 u$, therefore using the expression for κ_- and the evenness of κ_+ we find

$$\kappa_+(z) = \begin{cases} \alpha_1 z^2 + \alpha_0, & \mu = 0, \\ \alpha_1 \cosh(\mu z) + \alpha_0 \frac{\sinh \mu z}{z}, & \mu \neq 0. \end{cases}$$

If κ_+ is given by the first formulas, then k is trivial. Therefore, we assume $\mu \neq 0$ and the second formula holds. The even part of the relation for $y^3 e^{\lambda y}$ is

$$(-10z^2 + b_3 - \tilde{b}_3)\kappa_+'' - 20z\kappa_+' + \left[\left(\frac{5\lambda^2}{2} - 10c_5\right) z^2 - (b_3 - \tilde{b}_3)\frac{\lambda^2}{4} + c_3 - \tilde{c}_3\right] \kappa_+ = \\ = 4b_4 z\kappa_-'' - (b_4\lambda^2 - 4c_4 + 10\lambda) z\kappa_-.$$

When we substitute the formulas for $\kappa_\pm$ and multiply the relation by z^3, the result has the form

$$p(z)e^{\mu z} - p(-z)e^{-\mu z} = 0,$$

where $p(z) = \sum_{j=0}^4 p_j z^j$, therefore by linear independence we conclude that all the coefficients of p vanish, in particular one can compute that $p_0 = -2\alpha_0(b_3 - \tilde{b}_3)$ and $p_2 = \alpha_0\left(-(b_3 - \tilde{b}_3)\mu^2 + (b_3 - \tilde{b}_3)\frac{\lambda^2}{4} + \tilde{c}_3 - c_3\right)$, if $\alpha_0 = 0$, then obviously k is trivial, so $p_0 = 0$ implies $b_3 = \tilde{b}_3$, but then $p_2 = 0$ implies $c_3 = \tilde{c}_3$. Looking at the even part of the relation coming from $y^2 e^{\lambda y}$ we obtain an analogous equation, where the polynomial p may be of 5th order, but expressions of p_0, p_2 stay the same, only the subscripts of $b_3, \tilde{b}_3, c_3, \tilde{c}_3$ change to two. And we conclude $b_2 = \tilde{b}_2$ and $c_2 = \tilde{c}_2$. Likewise looking at the even parts of the relations coming from $ye^{\lambda y}, e^{\lambda y}$ we find $b_j = \tilde{b}_j$ and $c_j = \tilde{c}_j$ for $j = 1, 0$.

When we look at another term with $\left(\sum_{j=0}^5 b_j' y^j\right) e^{\lambda' y}$ in the coefficient $\mathscr{b}_1$ (and similar terms for other coefficient functions) we must have $b_5' = \tilde{b}_5'$, otherwise k is trivial.

If $b_5 = 0$, the same procedure applies, we only need to relabel the coefficients in the above equations. Thus our conclusion is that $L_1 = L_2$.

- $b_5 = -\tilde{b}_5$, this case is analogous to the previous one and the conclusion is $L_1 = -L_2$.

6 $L_1 = L_2$

In this section we aim to prove Theorem 2. Item 1 (in the limiting case $\gamma = 0$) and item 2 of Theorem 2 are derived in Corollary 7. Item 1 (in the case $\gamma \neq 0$) and item 3 are derived in Sections 6.3, 6.4. So let us assume the setting of Theorem 2.

The analysis in the beginning of Section 4 shows that $\mathscr{b}$ solves a linear homogeneous ODE with constant coefficients of order at most 4. Hence $\mathscr{b}(y)$ is a linear combination of terms like $y^l e^{\lambda_j y}$, where λ_j (called also a *mode*) is a root of fourth order polynomial. We will see that there are two major cases: $\operatorname{Re} \lambda_j = 0$ (*type 1*) or $\operatorname{Re} \lambda_j \neq 0$ (*type 2*). In the former case $k(z)$ is given in three possible forms featuring a free real-valued and even function (cf. (6.8)). In the latter case $k(z)$ is determined and has two possible forms (cf. (6.9)).

In Section 6.1 we analyze the multiplicity of the mode λ_j, in particular type 2 mode cannot have multiplicity larger than one, as is shown in Lemma 9, while type 1 mode can have multiplicity at most 3 as established in Lemma 8.

Finally, in Section 6.2 we turn to the question of analyzing possibilities of having multiple modes, i.e. distinct roots λ_j. In Corollary 11 we show that having three distinct type 1 modes is impossible. In Corollary 15 we show that having three distinct type 2 modes is impossible. In Lemma 12 we show that two distinct type 1 modes with one of them having multiplicity at least 2 leads to trivial kernels. And in Lemma 16 we show that having type 1 mode with multiplicity at least 2 and a type 2 mode again leads to trivial kernels. So the only cases leading to nontrivial kernels are: two type 2 and one type 1 mode all with multiplicity one analyzed in Section 6.3; and two type 1 modes with multiplicity 1 analyzed in Section 6.4.

Throughout this section, until Section 7 we will be working with $k(-z)$ and with an abuse of notation it will be denoted by $k(z)$. We will remember about this notational abuse when collecting the results in Theorem 2. In particular (R) becomes

$$\mathscr{b}(y)k''(z)-\mathscr{b}(y+z)k''(-z)-\mathscr{b}'(y)k'(z)+\mathscr{b}'(y+z)k'(-z)+\mathscr{c}(y)k(z)-\mathscr{c}(y+z)k(-z)=0. \quad (6.1)$$

The analysis in the beginning of the Section 4 shows that $\mathscr{b}$ solves a linear homogeneous ODE with constant coefficients of order at most 4, and that

$$-k_0\mathscr{c}'(y)+2k_1\mathscr{c}(y)+k_1\mathscr{b}''(y)-3k_2\mathscr{b}'(y)+2k_3\mathscr{b}(y)=0. \quad (6.2)$$

So $\mathscr{b}$ has the following form

$$\mathscr{b}(y)=\sum_{j=1}^{\nu}p_{d_j}(y)e^{\lambda_j y}, \quad (6.3)$$

where $\lambda_1,...,\lambda_\nu$ are distinct complex numbers and p_{d_j} are polynomials of degree d_j, so that

$$\nu+\sum_{j=1}^{\nu}d_j\le 4.$$

Then $\mathscr{c}(y)$ satisfying (6.2) must also have the same form, except the polynomials are different and there could be an extra exponential term $e^{\frac{2k_1}{k_0}y}$, if $\frac{2k_1}{k_0}\notin\{\lambda_1,...,\lambda_\nu\}$. Because we also require $\mathscr{b}(\pm1)=0$, then either

I. $\nu=1,\ d_1\ge 1$;

II. $\nu=2,\ d_1\ge 1$;

III. $\nu=2,\ d_1=d_2=0,\ \mathscr{b}(y)=e^{i\beta y}\sin(\pi n(y-1)/2)$ for some $\beta\in\mathbb{R}$ and $n\ge 1$;

IV. $\nu\ge 3$.

6.1 Single mode and multiplicities

In this section we concentrate on the single mode λ and analyze its multiplicity. So suppose $p(y)e^{\lambda y}$ is one of the terms in (6.3), while $q(y)e^{\lambda y}$ is one of the terms in $\mathscr{c}(y)$. Where $p(y) = \sum_{j=0}^{4} p_j y^j$ and $q(y) = \sum_{j=0}^{4} q_j y^j$. We are going to show that type 2 mode cannot have multiplicity larger than one (see Lemma 9), while type 1 mode cannot have multiplicity larger than 3 (see Lemma 8). Finally, here we also derive item 1 (in the limiting case $\gamma = 0$) and item 2 of Theorem 2 (see Corollary 7).

After substitution of the corresponding expressions for $\mathscr{b}, \mathscr{c}$ into (6.1), we collect the coefficients of $y^j e^{\lambda y}$ and from linear independence conclude that they must be zero. Like this we obtain 5 relations involving k. Let us first change the variables $k(z) = \kappa(z)e^{\lambda z/2}$, then the relation corresponding to $y^j e^{\lambda y}$ can be conveniently written as

$$\begin{aligned} p_j\kappa''(z) - \tfrac{p^{(j)}(z)}{j!}\kappa''(-z) + \tfrac{p^{(j+1)}(z)}{j!}\kappa'(-z) - (j+1)p_{j+1}\kappa'(z) + \\ + \tfrac{\varepsilon^{(j)}(z)}{j!}\kappa(-z) - \varepsilon_j\kappa(z) = 0, \quad j = 0, \dots, 4, \end{aligned} \tag{6.4}$$

with the convention that $p_5 = 0$, and the notation

$$\varepsilon(z) = \sum_{j=0}^{4} \varepsilon_j z^j, \qquad \varepsilon_j = \tfrac{\lambda^2 p_j}{4} - q_j + \tfrac{(j+1)}{2}\lambda p_{j+1}.$$

Let $\deg(p) = m$ and $\deg(q) = n$, and κ_+, κ_- be the even and odd parts of κ, respectively. If $n > m$ the relation in (6.4) for $j = n$ reads $q_n\kappa_-(z) = 0$, so $k(z) = \kappa_+(z)e^{\lambda z/2}$, the symmetry (A) implies $\lambda = 2i\beta$ for some $\beta \in \mathbb{R}$ and that κ_+ is real valued.

Let now $n \le m$, then (6.4) for $j = m$ reads

$$\kappa_-''(z) - \mu^2\kappa_-(z) = 0, \qquad \mu = \sqrt{\tfrac{\lambda^2}{4} - \tfrac{q_m}{p_m}}, \tag{6.5}$$

hence there are two possibilities: if $\mu = 0$, then $\kappa_-(z) = \alpha z + \beta$ and if $\mu \neq 0$, then $\kappa_-(z) = \alpha e^{\mu z} + \beta e^{-\mu z}$, using that κ_- is an odd function we conclude

$$\kappa_-(z) = \begin{cases} \alpha z, & \mu = 0, \\ \alpha \sinh(\mu z), & \mu \neq 0. \end{cases} \tag{6.6}$$

Thus, $k(z) = e^{\lambda z/2}(\kappa_+(z) + \kappa_-(z))$, where κ_+ is a free even function. Now the symmetry condition (A) says

$$e^{\bar{\lambda} z/2}\left(\overline{\kappa_+(z)} + \overline{\kappa_-(z)}\right) = e^{-\lambda z/2}(\kappa_+(z) - \kappa_-(z)). \tag{6.7}$$

This equation can be solved uniquely for κ_+ if and only if $\operatorname{Re}\lambda \neq 0$.

If $\lambda = 2i\beta$, then κ_+ can be arbitrary real and even function, while solvability implies that

$$k(z) = e^{i\beta z}\left(\kappa_+(z) + \begin{cases} i\alpha z, & \mu = 0 \\ i\alpha\sinh(\mu z), & \mu \neq 0 \\ i\alpha\sin(\mu z), & \mu \neq 0 \end{cases}\right), \tag{6.8}$$

where $\alpha, \mu \in \mathbb{R}$. Observe that the case $n > m$ is included here when we take $\alpha = 0$, therefore we may assume $m \geq n$.

Remark 5. When κ_- is given by the second formula of (6.6), then (6.7) implies that there are two cases, either $\alpha \in i\mathbb{R}$ and $\mu \in \mathbb{R}$ which gives the second formula of (6.8), or $\alpha \in \mathbb{R}$ and $\mu \in i\mathbb{R}$, which gives the third one, where with the abuse of notation we denoted the imaginary part of μ again by μ.

If $\lambda = 2\gamma + 2i\beta$ with $\gamma \neq 0$, then

$$k(z) = \begin{cases} ze^{i\beta z}\dfrac{\alpha e^{-\gamma z} + \overline{\alpha} e^{\gamma z}}{\sinh(2\gamma z)}, & \mu = 0, \\ e^{i\beta z}\dfrac{\alpha e^{-\gamma z}\sinh(\mu z) + \overline{\alpha} e^{\gamma z}\sinh(\overline{\mu} z)}{\sinh(2\gamma z)}, & \mu \neq 0, \end{cases} \tag{6.9}$$

where $\alpha, \mu \in \mathbb{C}$.

So far we have analyzed only one of the relations from (6.4) and deduced the possible forms of k. When the mode λ has multiplicity at least two we have $m \geq 1$, and therefore there are more relations in (6.4) that k has to satisfy (in particular the one corresponding to $j = m - 1$). In the two subsections below we analyze these possibilities.

6.1.1 Type 1 mode and multiplicities

Proposition 5. Let $\operatorname{Re}\lambda = 0$ and $m \geq 1$, then with $\lambda = 2i\beta$ and $\alpha, \mu, \varkappa, \kappa_0 \in \mathbb{R}$ we have (in fact $\varkappa = i\alpha\omega$ with ω defined in (6.11) below)

$$k(z) = e^{i\beta z} \cdot \begin{cases} i\alpha z + \kappa_0 + \frac{\varkappa}{6} z^2, & \mu = 0, \\ i\alpha \sinh(\mu z) + \kappa_0 \frac{\sinh \mu z}{z} + \frac{\varkappa}{2\mu}\cosh \mu z, & \mu \neq 0, \\ i\alpha \sin(\mu z) + \kappa_0 \frac{\sin \mu z}{z} - \frac{\varkappa}{2\mu}\cos \mu z, & \mu \neq 0. \end{cases} \tag{6.10}$$

Proof. So we see that the function κ_+ in (6.8) is not arbitrary and we are going to find it from the relation (6.4) with $j = m - 1$ (because $m \neq 0$ we can consider the index $m - 1$). Recall that w.l.o.g. we assumed $m \geq n$, note that $p^{(m-1)}(z) = m!p_m z + (m-1)!p_{m-1}$, $\varepsilon_m = \frac{\lambda^2 p_m}{4} - q_m$ and $\varepsilon_{m-1} = \frac{\lambda^2 p_{m-1}}{4} - q_{m-1} + \frac{m}{2}\lambda p_m$ so we obtain

$$p_{m-1}\kappa''(z) - (mp_m z + p_{m-1})\kappa''(-z) + mp_m[\kappa'(-z) - \kappa'(z)] + \\ + [m\varepsilon_m z + \varepsilon_{m-1}]\kappa(-z) - \varepsilon_{m-1}\kappa(z) = 0.$$

Now using (6.5) we can rewrite the above relation as

$$z\kappa_+'' + 2\kappa_+' - \mu^2 z\kappa_+ = \omega\kappa_-, \qquad \omega = -\lambda + \frac{2}{mp_m}\left(q_{m-1} - \frac{q_m p_{m-1}}{p_m}\right), \tag{6.11}$$

where κ_- appears in the three formulas from (6.8).

According to Remark 5, when $\kappa_-(z) = i\alpha \sin \mu z$, in the above relation μ should be replaced by $i\mu$, which changes the sign of the last term on LHS from negative to positive.

This explains the difference of the sign in the second and third formulas of (6.10). Solving the obtained ODE, recalling that κ_+ is even and real valued, we find (6.10) with $\varkappa = i\alpha\omega$. □

When $m \geq 2$, we can consider (6.4) with $j = m-2$, moreover we know that (6.5) and (6.11) also hold, and using these and $p^{(m-2)}(z) = \frac{m!}{2}p_m z^2 + (m-1)!p_{m-1}z + (m-2)!p_{m-2}$, the relation with $j = m-2$ can be simplified to

$$z\kappa'_- + \eta_1\kappa_- = \eta_2 z\kappa_+, \qquad \eta_2 = \frac{\omega}{2}, \tag{6.12}$$

where ω is defined in (6.11) and η_1 is a constant whose precise expression is not important.

Proposition 6. Let $\operatorname{Re}\lambda = 0$ and $m \geq 2$, then with $\lambda = 2i\beta$ and $\alpha, \kappa_0, \mu \in \mathbb{R}$

$$k(z) = e^{i\beta z}\cdot\begin{cases}\kappa_0\dfrac{\sinh\mu z}{z}, \\ \alpha e^{i\mu z} + \kappa_0\dfrac{\sin\mu z}{z}.\end{cases} \tag{6.13}$$

Moreover, in the second case the following relations between the involved parameters must be satisfied

$$\kappa_0\eta_2 = i\alpha\eta_1, \qquad \eta_2 = \pm i\mu. \tag{6.14}$$

Proof. By Proposition 5 we know what are the functions κ_- and κ_+ that satisfy the two relations (6.4) with $j = m, m-1$ (they are given in the three formulas in (6.10), with $\varkappa = i\alpha\omega$). Here we want to see which of these satisfy the third relation (6.12). First note that $\varkappa \in \mathbb{R}$ implies ω and hence also $\eta_2 = \frac{\omega}{2}$ are purely imaginary. The case (6.10)a implies that k has rank at most three and so, is trivial.

If (6.10)b holds, then (6.12) after multiplying by 2μ reads

$$z(2i\alpha\mu^2 - \eta_2\varkappa)\cosh(\mu z) + 2\mu(i\alpha\eta_1 - \eta_2\kappa_0)\sinh(\mu z) = 0.$$

By linear independence we conclude that the two coefficients must vanish: $2i\alpha\mu^2 - \eta_2\varkappa = 0$ and $i\alpha\eta_1 - \eta_2\kappa_0 = 0$. Let us ignore the second equation (it just gives some restrictions on q_j's), using the expression for $\varkappa$ the first one becomes $\alpha(\mu^2 - \eta_2^2) = 0$. If $\alpha \neq 0$, because $\eta_2 \in i\mathbb{R}$, we conclude $\mu = \eta_2 = 0$ which is a contradiction. Thus $\alpha = 0$, which gives the first formula of (6.13).

If (6.10)c holds, then (6.12) reads

$$z(2i\alpha\mu^2 + \eta_2\varkappa)\cos(\mu z) + 2\mu(i\alpha\eta_1 - \eta_2\kappa_0)\sin(\mu z) = 0.$$

Again the two coefficients must be zero, the second one implies the first relation of (6.14) and the first one gives $\alpha(\mu^2 + \eta_2^2) = 0$. One possibility is $\alpha = 0$, another one: when $\alpha \neq 0$, then $\operatorname{Im}\eta_2 = \pm\mu$, hence we may write $\kappa(z) = \pm\alpha(\cos\mu z \pm i\sin\mu z) + \kappa_0\frac{\sin\mu z}{z} = \pm\alpha e^{\pm i\mu z} + \kappa_0\frac{\sin\mu z}{z}$. These cases can be unified in the second formula of (6.13). □

Corollary 7. When there is one type 1 root with multiplicity three (i.e. $\nu = 1$, $m = 2$ and $\lambda = 2i\beta$), we obtain item 1 (in the limiting case $\gamma = 0$) and item 2 of Theorem 2.

Proof. Using the boundary conditions $\mathscr{b}(y) = (y^2-1)e^{\lambda y}$, we know k from the above proposition so it only remains to find $\mathscr{c}$. Before that let us invoke Remark 2 and w.l.o.g. assume that $\beta = 0$, or equivalently $\lambda = 0$.

From (6.2) we know that $\mathscr{c}(y) = \sum_{j=0}^{3} c_j y^j + c_4 e^{\tau y}$ with $\tau \neq 0$. Clearly $\mu \neq 0$, otherwise k is trivial (see (6.13)). We substitute these expressions into (6.1) and obtain that a linear combination of $e^{\tau y}$ and monomials y^j is zero, hence by linear independence each of the coefficients must vanish. The equation coming from the term $e^{\tau y}$ reads

$$c_4\left[k(z) - e^{\tau z}k(-z)\right] = 0. \tag{6.15}$$

Equations coming from the terms $y^3, ..., 1$, respectively are

$$\begin{aligned} c_3\left[k(z) - k(-z)\right] &= 0, \\ k''(z) - k''(-z) + c_2 k(z) - (3c_3 z + c_2)k(-z) &= 0, \\ 2zk''(-z) + 2k'(z) - 2k'(-z) - c_1 k(z) + (3c_3 z^2 + 2c_2 z + c_1)k(-z) &= 0, \\ k''(z) + (z^2-1)k''(-z) - 2zk'(-z) - c_0 k(z) + (c_3 z^3 + c_2 z^2 + c_1 z + c_0)k(-z) &= 0. \end{aligned} \tag{6.16}$$

Assume k is given by the first formula of (6.13), in particular it is even and (6.15) implies $c_4 = 0$. The first equation of (6.16) is identity, the second one implies $c_3 \sinh(\mu z) = 0$ and hence $c_3 = 0$. Third one reads $(c_2 + \mu^2)\sinh(\mu z) = 0$, hence $c_2 = -\mu^2$. Finally, the fourth relation simplifies to $c_1 \sinh(\mu z) = 0$, so that $c_1 = 0$. We note that c_0 remains free. Thus, we conclude that $\mathscr{c}(y) = -\mu^2 y^2 + c_0$ and since we are free to choose c_0, we can rewrite $\mathscr{c}$ as $\mathscr{c}(y) = -\mu^2 \mathscr{b}(y) + c_0$, which proves item 1 of Theorem 2 in the case $\gamma = 0$ and $\mu \in \mathbb{R}$.

Assume k is given by the second formula of (6.13). Because $\kappa_0 \neq 0$, we may normalize it to be one. (6.15) reads

$$c_4\left[e^{-i\mu z} - e^{i\mu z} + e^{(i\mu+\tau)z} - e^{(-i\mu+\tau)z} + i\alpha z(e^{(-i\mu+\tau)z} - e^{i\mu z})\right] = 0,$$

and from the linear independence of the involved exponentials we get $c_4 = 0$. The first equation of (6.16) reads $c_3 \alpha \sin(\mu z) = 0$, and there are two cases to consider.

If $\alpha = 0$, the second equation reads $c_3 \sin(\mu z) = 0$, so $c_3 = 0$. The third equation becomes $(c_2 - \mu^2)\sin(\mu z) = 0$, hence $c_2 = \mu^2$. Finally, the fourth equation implies $c_1 = 0$ and again c_0 is free. So we find $\mathscr{c}(y) = \mu^2 \mathscr{b}(y) + c_0$, which proves item 1 of Theorem 2 in the case $\gamma = 0$ and $\mu \in i\mathbb{R}$.

If $\alpha \neq 0$, then $c_3 = 0$. The second equation of (6.16) implies $c_2 = \mu^2$, the third one: $c_1 = 2i\mu$ and finally the fourth one implies $c_0 = -\mu^2 + \frac{2\mu}{\alpha}$. Thus, $\mathscr{c}(y) = \mu^2(y^2-1) + 2i\mu y + \frac{2\mu}{\alpha}$, which proves item 2 of Theorem 2.

□

Lemma 8. Let $\operatorname{Re}\lambda = 0$ and $m \geq 3$, then k is trivial.

Proof. By the previous proposition we know that $\kappa(z)$ has two possible forms coming from (6.13). The goal is to show that it cannot solve (6.4) with $j = m-3$. Using the equations (6.5), (6.11) and (6.12) we can rewrite the relation for $j = m-3$ as

$$(\eta_2 z^2 + \eta_3)\kappa_- = z^2\kappa_+' + 3\eta_1 z\kappa_+, \tag{6.17}$$

where η_1, η_2 are the same as in (6.12) and the expression for η_3 is not important.

When k is given by the first formula of (6.13), $\kappa_-(z) = 0$ and $\kappa_+(z) = \kappa_0 \frac{\sinh(\mu z)}{z}$ so (6.17) implies $\mu = 0$ and hence $k = 0$.

When k is given by the second formula of (6.13), let us w.l.o.g. take $\kappa_0 = 1$. As we saw in the previous proposition $\kappa_-(z) = i\alpha \sin(\mu z)$ and $\kappa_+(z) = \frac{\sin(\mu z)}{z} - \frac{i\alpha\eta_2}{\mu}\cos(\mu z)$ with $\eta_2 = \pm i\mu$ and $i\alpha\eta_1 = \eta_2$. Let first $\eta_2 = i\mu$, then substituting $\kappa_\pm$ into (6.17) we get

$$[i\alpha\eta_3 + \kappa_0(1 - 3\eta_1)]\sin(\mu z) - z(\mu + 3\alpha\eta_1)\cos(\mu z) = 0.$$

But then $\mu + 3\alpha\eta_1 = 4\mu$ which must be zero, hence k is trivial. The case $\eta_2 = -i\mu$ is done analogously.

□

6.1.2 Type 2 mode and multiplicities

Lemma 9. Let $\operatorname{Re}\lambda \neq 0$ and $m \geq 1$, then $k = 0$.

Proof. Let $\lambda = \gamma + i\beta$, with $\gamma \neq 0$, (6.7) implies

$$\begin{cases} \kappa_+ - \overline{\kappa}_+ e^{\gamma z} = \overline{\kappa}_- e^{\gamma z} + \kappa_-, \\ \overline{\kappa}_+ - \kappa_+ e^{\gamma z} = \kappa_- e^{\gamma z} + \overline{\kappa}_-, \end{cases}$$

where the second equation was obtained by conjugating the first one, then

$$\kappa_+ = -\coth(\gamma z)\kappa_- - \operatorname{csch}(\gamma z)\overline{\kappa}_-. \tag{6.18}$$

We know that both of the relations (6.5) and (6.11) hold. When $\mu = 0$, we have $\kappa_-(z) = \varkappa z$, hence $\kappa_+(z) = \frac{\omega\alpha}{6} z^2 + \kappa_0$ and comparing this with (6.18) we conclude $k = 0$. So let us assume $\mu \neq 0$, then from (6.6), $\kappa_-(z) = \alpha\sinh(\mu z)$, hence solving the ODE (6.11) we get

$$\kappa_+(z) = c_2 \frac{\sinh(\mu z)}{z} + \frac{\varkappa\alpha}{2\mu}\cosh(\mu z),$$

substitute this into (6.18) divide the result by $\sinh(\mu z)$ to get

$$\frac{c_2}{z} + \frac{\varkappa\alpha}{2\mu}\coth(\mu z) = -\alpha\coth(\gamma z) - \overline{\alpha}\frac{\sinh(\overline{\mu}z)}{\sinh(\mu z)}\operatorname{csch}(\gamma z).$$

Assume $\gamma > 0$ (otherwise negate $(\gamma, \alpha, \varkappa)$), write $\mu = \mu_1 + i\mu_2$, assume $\mu_1 \neq 0$, then we may assume $\mu_1 > 0$, otherwise multiply the equation by -1. Now consider the asymptotics as $z \to +\infty$,

$$\frac{c_2}{z} + \frac{\varkappa\alpha}{2\mu} = -\alpha - 2\overline{\alpha}e^{-\gamma z}e^{-2i\mu_2 z},$$

clearly this implies $\alpha = c_2 = 0$, so $k = 0$. Let now $\mu_1 = 0$, then the relation reads

$$\frac{c_2}{z} - \frac{\varkappa\alpha}{2\mu_2}\cot(\mu_2 z) = -\alpha\coth(\gamma z) + \overline{\alpha}\operatorname{csch}(\gamma z),$$

and asymptotics at $+\infty$ gives $\frac{c_2}{z} - \frac{\varkappa\alpha}{2\mu_2}\cot(\mu_2 z) = -\alpha + 2\overline{\alpha}e^{-\gamma z}$ which again implies $\alpha = c_2 = 0$.

□

6.2 Multiple modes

Before we start to analyze the possibilities of having multiple distinct modes λ_j in (6.3), we state that in view of Lemmas 8 and 9 the cases I and II can be rewritten

I. $\nu = 1,\ d_1 = 2,\ \operatorname{Re}\lambda_1 = 0;$

IIa. $\nu = 2,\ d_1 \geq 1,\ \operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = 0;$

IIb. $\nu = 2,\ d_1 \geq 1,\ \operatorname{Re}\lambda_1 = 0,\ \operatorname{Re}\lambda_2 \neq 0.$

The case I was analyzed in Corollary 7, so it remains to consider cases IIa,b and III, IV. We will see in Lemmas 12 and 16 that the cases IIa,b lead to trivial kernels k. Case III will be analyzed in Section 6.4. We will show that case IV is only possible when there are exactly three modes: two type 1 and one type 2, all with multiplicity one. This case will then be analyzed in Section 6.3.

When $\lambda_j = 2i\beta_j$ (of course $\beta_1 \neq \beta_2$) then (6.8) holds true for both of the modes λ_j and we determine the free functions and conclude

$$k(z) = \frac{\alpha_1 k_s(\mu_1 z)e^{i\beta_1 z} + \alpha_2 k_r(\mu_2 z)e^{i\beta_2 z}}{\sin(\beta_1 - \beta_2)z}, \qquad r, s \in \{1, 2, 3\}, \tag{6.19}$$

where all the constants are real, $\mu_j \neq 0$ and k_r is given by

$$k_1(t) = t, \quad k_2(t) = \sin t, \quad k_3(t) = \sinh t. \tag{6.20}$$

Proposition 10. Let k be given by (6.19), then β_1 and β_2 are determined by k.

Proof. W.l.o.g. let $\beta_1 - \beta_2 > 0$, otherwise swap β_1 with β_2; r with s; μ_1 with μ_2 and replace (α_1, α_2) by $(-\alpha_2, -\alpha_1)$. There are six cases to consider.

- If $(s, r) = (3, 3)$; we have

$$k(it) = e^{-\beta_1 t} \cdot \frac{\alpha_1 \sin(\mu_1 t) + \alpha_2 \sin(\mu_2 t)e^{(\beta_1 - \beta_2)t}}{\sinh(\beta_1 - \beta_2)t},$$

therefore

$$k(it) \sim \begin{cases} 2\alpha_1 \sin(\mu_1 t)e^{(\beta_2 - 2\beta_1)t} + 2\alpha_2 e^{-\beta_1 t}\sin(\mu_2 t), & t \to +\infty, \\ 2\alpha_1 \sin(\mu_1 t)e^{-\beta_2 t} + 2\alpha_2 e^{(\beta_1 - 2\beta_2)t}\sin(\mu_2 t), & t \to -\infty. \end{cases}$$

When $(s, r) = (1, 1)$ the same formulas hold with $\sin(\mu_j t)$ replaced by t for $j = 1, 2$. And when $(s, r) = (1, 3)$ the same formulas hold with $\sin(\mu_1 t)$ replaced by t. The above asymptotics immediately conclude the proof in this case.

- If $(s, r) = (2, 3)$, we may assume $\mu_1 > 0$, otherwise negate α_1, so

$$k(it) = e^{-\beta_1 t} \cdot \frac{\alpha_1 \sinh(\mu_1 t) + \alpha_2 \sin(\mu_2 t)e^{(\beta_1 - \beta_2)t}}{\sinh(\beta_1 - \beta_2)t},$$

and therefore

$$k(it) \sim \begin{cases} \alpha_1 e^{(\mu_1+\beta_2-2\beta_1)t} + 2\alpha_2 \sin(\mu_2 t) e^{-\beta_1 t}, & t \to +\infty, \\ \alpha_1 e^{-(\mu_1+\beta_2)t} + 2\alpha_2 \sin(\mu_2 t) e^{(\beta_1-2\beta_2)t}, & t \to -\infty. \end{cases}$$

If $\alpha_2 \neq 0$ clearly β_1 and β_2 are determined. So assume $\alpha_2 = 0$, then from the above asymptotics we conclude that α_1, $\mu_1 + \beta_2$ and β_1 are determined. But note that $k_0 := k(0) = \frac{\mu_1 \alpha_1}{\beta_1 - \beta_2}$, so we have a system ($k_1$ denotes a parameter determined by k)

$$\begin{cases} \alpha_1 \mu_1 + k_0 \beta_2 = k_0 \beta_1 \\ \mu_1 + \beta_2 = k_1 \end{cases}$$

Which is not solvable w.r.t. μ_1 and β_2 if and only if $k_0 = \alpha_1$, but in this case the first equation implies $\beta_1 - \beta_2 = \mu_1$, therefore $k(z) = \alpha_1 e^{i\beta_1 z}$ which is trivial. When $(s, r) = (2, 1)$ the asymptotic formulas hold with $\sin(\mu_2 t)$ replaced by t and the same argument applies.

• If $(s, r) = (2, 2)$, we may assume $\mu_1, \mu_2 > 0$, otherwise negate α_1, α_2, so

$$k(it) = e^{-\beta_1 t} \cdot \frac{\alpha_1 \sinh(\mu_1 t) + \alpha_2 \sinh(\mu_2 t) e^{(\beta_1-\beta_2)t}}{\sinh(\beta_1 - \beta_2)t},$$

therefore

$$k(it) \sim \begin{cases} \alpha_1 e^{(\mu_1+\beta_2-2\beta_1)t} + \alpha_2 e^{(\mu_2-\beta_1)t}, & t \to +\infty, \\ \alpha_1 e^{-(\mu_1+\beta_2)t} + \alpha_2 e^{-(\mu_2-\beta_1+2\beta_2)t}, & t \to -\infty. \end{cases}$$

If $\alpha_1, \alpha_2 \neq 0$, clearly β_1 and β_2 are determined. Assume $\alpha_1 = 0$, then from the above asymptotics we conclude that $\alpha_2, \mu_2 - \beta_1$ and β_2 are determined. Next, as above we look at $k(0) = \frac{\mu_2 \alpha_2}{\beta_1 - \beta_2}$, and conclude that β_1, μ_2 are not determined if and only if $\mu_2 = \beta_1 - \beta_2$ in which case k is trivial. Analogous conclusion holds in the case $\alpha_2 = 0$.

□

Corollary 11. Having three distinct modes $\lambda_1, \lambda_2, \lambda_3 \in i\mathbb{R}$ is impossible.

Lemma 12. Having two distinct type 1 modes, one of them with multiplicity at least two leads to a trivial kernel. In other words, if $k(z)$ can be written in the form (6.10) and (6.19), then k is trivial.

Proof. The denominator in (6.19) is zero when $z = \pi n/(\beta_1 - \beta_2)$. If the numerator does not vanish at all of these values then the function in (6.19) is not entire, while all functions (6.10) are entire. Thus it must hold

$$\alpha_1 k_s \left(\tfrac{\pi \mu_1 n}{\beta_1 - \beta_2}\right) + (-1)^n \alpha_2 k_r \left(\tfrac{\pi \mu_2 n}{\beta_1 - \beta_2}\right) = 0 \qquad \forall n \in \mathbb{Z}.$$

This equation can hold in three cases $(r, s) = (2, 2), (2, 3)$ or $(1, 2)$. Let us consider the first one, the other two can be analyzed similarly, and in fact are simpler. The solutions of the above equation for $r = s = 2$ are

(a) $\mu_j = m_j(\beta_1 - \beta_2)$ with $m_j \in \mathbb{Z}$ for $j = 1, 2$,

(b) $\alpha_1 = \pm\alpha_2$, $\mu_2 = (2m_1 + 1)(\beta_1 - \beta_2) \mp \mu_1$.

In both of these cases k is a trigonometric polynomial. But if k is given by (6.10) and is a trigonometric polynomial, then $k(z) = e^{i\beta z}(i\alpha \sin \mu z + \alpha' \cos \mu z)$ for some constants α, α', β and μ. Showing that k is trivial.

□

Lemma 13. Let k be given by (6.9), then the pair $(|\gamma|, \beta)$ is determined by k.

Proof. Let k be given by the first formula, assume $\gamma > 0$, otherwise replace (γ, α) with $(-\gamma, -\overline{\alpha})$, then

$$k(z) \sim 2\overline{\alpha} z e^{-\gamma z} e^{i\beta z}, \qquad \text{as } z \to +\infty, \tag{6.21}$$

so α, γ, β are determined by k. But note that the sign of γ is not determined.

Let now k be given by the second formula, write $\mu = \mu_1 + i\mu_2$ and $\alpha = \alpha_1 + i\alpha_2$,

1. let $\mu_1 \neq 0$, we may assume $\mu_1 > 0$, otherwise we replace (α, μ) with $(-\alpha, -\mu)$. Also assume $\gamma > 0$, otherwise we replace (γ, α, μ) with $(-\gamma, -\overline{\alpha}, \overline{\mu})$, then

$$k(z) \sim \overline{\alpha} e^{(-\gamma+\mu_1)z} e^{i(\beta-\mu_2)z}, \qquad \text{as } z \to +\infty, \tag{6.22}$$

so $\alpha, -\gamma + \mu_1$ and $\beta - \mu_2$ are determined by k. We then note that $k(0) = \frac{\mathrm{Re}(\alpha\mu)}{\gamma}$ and $k'(0) = i\beta k(0) - i\,\mathrm{Im}(\alpha\mu)$. Because of the symmetry of k, we know that $k(0) \in \mathbb{R}$ and $k'(0) \in i\mathbb{R}$, so let us set $k_0 = k(0)$ and $k_1 = \frac{k'(0)}{i}$, then we obtain the system

$$\begin{cases} \alpha_1\mu_1 - \alpha_2\mu_2 - k_0\gamma = 0, \\ -\alpha_2\mu_1 - \alpha_1\mu_2 + k_0\beta = k_1, \\ \mu_1 - \gamma = k_2, \\ -\mu_2 + \beta = k_3, \end{cases} \qquad A = \begin{pmatrix} \alpha_1 & -\alpha_2 & -k_0 & 0 \\ -\alpha_2 & -\alpha_1 & 0 & k_0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix},$$

where the unknowns are $\mu_1, \mu_2, \gamma, \beta$ and k_2, k_3 are parameters determined by k. The system is linear and one can compute $\det(A) = (\alpha_1 - k_0)^2 + \alpha_2^2$. If $\det(A) \neq 0$, then the system has a unique solution and all the constants $\mu_1, \mu_2, \gamma, \beta$ are determined by the function k. Of course we see that the signs of γ and μ_1 are not determined.

When $\det(A) = 0$, we get $\alpha_1 = k_0$ and $\alpha_2 = 0$, then (note that $k_0 \neq 0$, because otherwise $k = 0$). Now we must have $k_2 = 0$ and $k_3 = \frac{k_1}{k_0}$ and the above system reduces to

$$\begin{cases} \mu_1 - \gamma = 0, \\ -\mu_2 + \beta = k_3. \end{cases}$$

So α is real and $\mu_1 = \gamma$, and in this case one can check that the formula reduces to $k(z) = \alpha e^{i(\beta+\mu_2)z}$ which is a trivial kernel.

2. $\mu_1 = 0$, we may assume $\gamma > 0$, otherwise replace (γ, α) by $(-\gamma, \overline{\alpha})$, then

$$k(z) \sim \overline{\alpha} e^{-\gamma z} \left[e^{i(\beta-\mu_2)z} - e^{i(\beta+\mu_2)z} \right] \qquad \text{as } z \to +\infty, \tag{6.23}$$

so $\alpha, \gamma, \beta, \mu_2$ are determined by k. And again we see that the sign of γ is not determined.

□

Corollary 14. Let $\lambda_j = 2\gamma_j + i2\beta_j$, with $\gamma_j \neq 0$ for $j = 1, 2$. Assume $\lambda_1 \neq \lambda_2$, then $\lambda_2 = -\overline{\lambda_1}$.

Proof. For each λ_j, k can be given by two formulas from (6.9), let us refer to them as "a" and "b". There are three cases to consider: (a,a); (b,b) and (a,b). By comparing the asymptotics (6.22) and (6.23) with (6.21) we see that they cannot be matched, hence the third case is impossible. Consider the first one, then

$$k(z) = ze^{i\beta_j z} \cdot \frac{\alpha_j e^{-\gamma_j z} + \overline{\alpha_j} e^{\gamma_j z}}{\sinh(2\gamma_j z)}, \qquad j = 1, 2.$$

As we saw $|\gamma_j|$ and β_j are determined by k, hence we conclude $|\gamma_1| = |\gamma_2|$ and $\beta_1 = \beta_2$. Because $\lambda_1 \neq \lambda_2$ we must have $\gamma_1 = -\gamma_2$. The second case is done analogously.

□

Corollary 15. Having three distinct modes $\lambda_1, \lambda_2, \lambda_3 \notin i\mathbb{R}$ leads to trivial k..

Lemma 16. Having a type 2 mode and a type 1 mode of multiplicity at least two leads to a trivial kernel. In other words, if $k(z)$ can be written in the form (6.10) and (6.9), then k is trivial.

Proof. So $\lambda_1 = i2\beta_1$ and $\lambda_2 = 2\gamma + i2\beta_2$ with $\gamma \neq 0$. All the functions in (6.10) are entire, and one can easily check that the first function of (6.9) is entire if and only if $\alpha = 0$, which leads to $k = 0$. So let us consider the case when k is given by the second formula:

$$k(z) = e^{i\beta_2 z} \cdot \frac{\alpha_2 e^{-\gamma z} \sinh(\mu z) + \overline{\alpha_2} e^{\gamma z} \sinh(\overline{\mu} z)}{\sinh(2\gamma z)} = e^{i\beta_1 z} \begin{cases} i\alpha_1 z + \kappa_0 + \frac{\varkappa}{6} z^2, \\ i\alpha_1 \sinh \mu_0 z + \kappa_0 \frac{\sinh \mu_0 z}{z} + \frac{\varkappa}{2\mu_0} \cosh \mu_0 z, \\ i\alpha_1 \sin \mu_0 z + \kappa_0 \frac{\sin \mu_0 z}{z} - \frac{\varkappa}{2\mu_0} \cos \mu_0 z, \end{cases} \tag{6.24}$$

where $\mu_0 (\neq 0), \alpha_1, \kappa_0, \varkappa \in \mathbb{R}$, and write $\mu = \mu_1 + i\mu_2$.

Case 1: if $\mu_1 \neq 0$, may assume $\mu_1 > 0$ and $\gamma > 0$. If k is given by the

1. 1st formula, then comparing the asymptotics we see that $\alpha_1 = \varkappa = 0$, then for the LHS $k(z) \sim \kappa_0 e^{i\beta_1 z}$. Again comparing we find $\overline{\alpha_2} = \kappa_0$, $-\gamma + \mu_1 = 0$ and $\beta_2 - \mu_2 = \beta_1$. The last two conditions can be rewritten as $\lambda_2 - \lambda_1 = 2\mu$, and so $k(z) = \kappa_0 e^{i\beta_1 z}$, which is trivial.

2. 2nd formula, we may assume $\mu_0 > 0$, otherwise negate $(\alpha_1, \kappa_0, \varkappa)$, then $k(z) \sim \frac{1}{2}(i\alpha_1 + \frac{\varkappa}{2\mu_0})e^{\mu_0 z}e^{i\beta_1 z}$, comparing with (6.22) we conclude

$$-\gamma + \mu_1 = \mu_0, \qquad \beta_2 - \mu_2 = \beta_1, \qquad i\alpha_1 + \tfrac{\varkappa}{2\mu_0} = 2\overline{\alpha_2},$$

with these, in (6.24) we express sinh and cosh in terms of exponentials, by linear independence we conclude that $\kappa_0 = 0$, and obtain

$$-\overline{\alpha_2}e^{(\gamma-\mu_1)z} + \alpha_2 e^{(\gamma-\mu_1)z} = e^{i2\mu_2 z}\left[\alpha_2 e^{(-3\gamma+\mu_1)z} - \overline{\alpha_2}e^{(-3\gamma-\mu_1)z}\right].$$

Hence $\mu_2 = 0$, then using that $\gamma, \mu_1 \neq 0$ we deduce that the above relation is possible (with $\alpha_2 \neq 0$) if and only if $\mu_1 = 2\gamma$. Thus $k(z) = e^{i\beta_1 z}\left[i\alpha_1 \sinh \mu_0 z + \frac{\varkappa}{2\mu_0}\cosh \mu_0 z\right]$ is trivial.

3. 3rd formula, we may assume $\mu_0 > 0$, otherwise negate $(\alpha_1, \kappa_0, \varkappa)$, then $k(z) \sim e^{i\beta_1 z}\left[(\frac{\alpha_1}{2} - \frac{\varkappa}{4\mu_0})e^{i\mu_0 z} - (\frac{\alpha_1}{2} + \frac{\varkappa}{4\mu_0})e^{-i\mu_0 z}\right]$, comparing this with (6.22) we conclude $-\gamma + \mu_1 = 0$ and

   (a) $\beta_1 + \mu_0 = \beta_2 - \mu_2$, $\frac{\alpha_1}{2} - \frac{\varkappa}{4\mu_0} = \overline{\alpha_2}$ and $\frac{\alpha_1}{2} + \frac{\varkappa}{4\mu_0} = 0$, or

   (b) $\beta_1 - \mu_0 = \beta_2 - \mu_2$, $\frac{\alpha_1}{2} - \frac{\varkappa}{4\mu_0} = 0$ and $\frac{\alpha_1}{2} + \frac{\varkappa}{4\mu_0} = -\overline{\alpha_2}$

   Let us consider the first option, in that case (6.24) simplifies to $\kappa_0 e^{i\beta_1 z}\frac{\sin \mu_0 z}{z} = 0$ which implies $\kappa_0 = 0$, and so $k(z) = \alpha_1 e^{i(\beta_1+\mu_0)}$. The other case is done analogously.

Case 2: if $\mu_1 = 0$, we may assume $\gamma > 0$. If k is given by the 1st or 3rd formulas, comparing the asymptotics of LHS with (6.23) we conclude $\gamma = 0$, which is a contradiction, so these cases lead to $k = 0$. Now let k be given by the second formula, again w.l.o.g let $\mu_0 > 0$, then we see that the asymptotics cannot be matched because in (6.23) $e^{i(\beta_2 \pm \mu_2)z}$ are linearly independent, hence $k = 0$.

□

Lemma 17. Let $\lambda_1 = i2\beta_1$ and $\lambda_2 = 2\gamma + i2\beta_2$, with $\gamma \neq 0$, then $\beta_1 = \beta_2 =: \beta$ and

$$k(z) = \alpha e^{i\beta z}\frac{k_r(\mu z)}{\sinh \gamma z}, \quad r \in \{1, 2, 3\}, \tag{6.25}$$

where $\alpha, \mu \in \mathbb{R}$ and k_r is defined in (6.20).

Proof. So k is given by both of the forms (6.9) and (6.8). Assume k is given by the first formula of (6.9), then we can find

$$\kappa_+(z) = ze^{i\Delta\beta z}\frac{\alpha e^{-\gamma z} + \overline{\alpha}e^{\gamma z}}{\sinh(2\gamma z)} - i\alpha' k_r(\mu' z), \quad r \in \{1, 2, 3\},$$

where $\Delta\beta = \beta_2 - \beta_1$, $0 \neq \mu', \alpha' \in \mathbb{R}$. It is easy to check that κ_+ as above satisfies $\overline{\kappa_+(-z)} = \overline{\kappa_+(z)}$, hence κ_+ is real valued if and only if it is even, and with $\alpha = \alpha_1 + i\alpha_2$ the imaginary part of κ_+ being zero reads

$$z\alpha_1 \frac{\sin(\Delta\beta z)}{\sinh(\gamma z)} - z\alpha_2 \frac{\cos(\Delta\beta z)}{\cosh(\gamma z)} = \alpha' k_r(\mu' z). \tag{6.26}$$

We may assume $\gamma > 0$, otherwise replace (γ, α_1) with $(-\gamma, -\alpha_1)$. Assume $k \neq 0$, note that

$$\text{LHS} \sim 2ze^{-\gamma z}[\alpha_1 \sin(\Delta\beta z) - \alpha_2 \cos(\Delta\beta z)], \qquad \text{as } z \to +\infty.$$

Comparing this with the asymptotic of RHS for $r = 1, 2, 3$ we conclude that (6.26) is possible if and only if $\Delta\beta = 0$ and $\alpha_2 = \alpha' = 0$. And we see that k is given by (6.25) with $r = 1$.

Assume now k is given by the second formula of (6.9), then

$$\kappa_+(z) = e^{i\Delta\beta z} \cdot \frac{\alpha e^{-\gamma z} \sinh(\mu z) + \overline{\alpha} e^{\gamma z} \sinh(\overline{\mu} z)}{\sinh(2\gamma z)} - i\alpha' k_r(\mu' z), \qquad r \in \{1, 2, 3\}.$$

Write $\mu = \mu_1 + i\mu_2$ and $\alpha = \alpha_1 + i\alpha_2$, w.l.o.g. let $\gamma > 0$, assume $\mu_1 \neq 0$ then we can assume $\mu_1 > 0$; again κ_+ being even and real valued are equivalent and $\operatorname{Im} \kappa_+ = 0$ reads

$$\begin{aligned} &\frac{\sin(\Delta\beta z)}{\sinh(\gamma z)} [\alpha_1 \sinh(\mu_1 z) \cos(\mu_2 z) - \alpha_2 \cosh(\mu_1 z) \sin(\mu_2 z)] - \\ -&\frac{\cos(\Delta\beta z)}{\cosh(\gamma z)} [\alpha_1 \cosh(\mu_1 z) \sin(\mu_2 z) + \alpha_2 \sinh(\mu_1 z) \cos(\mu_2 z)] = \alpha' k_r(\mu' z). \end{aligned} \tag{6.27}$$

We note that as $z \to \infty$

$$\text{LHS} \sim e^{(-\gamma + \mu_1) z} \left[\alpha_1 \sin(\Delta\beta - \mu_2) z - \alpha_2 \cos(\Delta\beta - \mu_2) z \right],$$

comparing this with the asymptotic of RHS for r=1,2,3 we conclude that (6.27) is possible for non-trivial k if and only if $\Delta\beta = \mu_2$ and $\alpha_2 = \alpha' = 0$. (For example when $r = 2$, (6.27) is also possible when $\mu_1 = \gamma$, $\alpha_2 = 0$, $\alpha' = \alpha_1$ and $\Delta\beta - \mu_2 = \mu'$ but in this case one easily checks that k is trivial). Now (6.27) reduces to

$$\sin(2\mu_2 z) \left[\frac{\sinh \mu_1 z}{\sinh \gamma z} - \frac{\cosh \mu_1 z}{\cosh \gamma z} \right] = 0.$$

If the second factor is zero, we must have $\gamma = \mu_1$ and in this case k reduces to a trivial kernel. So $\mu_2 = 0$, and k is given by (6.25) with $r = 3$. Let now $\mu_1 = 0$, then (6.27) becomes

$$-\sin(\mu_2 z) \left[\alpha_2 \frac{\sin \Delta\beta z}{\sinh \gamma z} + \alpha_1 \frac{\cos \Delta\beta z}{\cosh \gamma z} \right] = \alpha' k_r(\mu' z). \tag{6.28}$$

We note that as $z \to \infty$

$$\text{LHS} \sim -2e^{-\gamma z} \sin(\mu_2 z) \left[\alpha_2 \sin(\Delta\beta z) + \alpha_1 \cos(\Delta\beta z) \right],$$

comparing this with the asymptotics of RHS for r=1,2,3 we find that (6.28) is possible for non-trivial k if and only if $\Delta\beta = 0$ and $\alpha_1 = \alpha' = 0$. And k is given by (6.25) with $r = 2$.

$\square$

Corollary 18. Having three distinct modes $\lambda_1, \lambda_2 \in i\mathbb{R}$ and $\lambda_3 \notin i\mathbb{R}$ is impossible.

6.3 Item 1, $\gamma \neq 0$

The previous analysis shows that case IV is only possible when we have exactly three modes $\lambda_1, \lambda_2 \notin i\mathbb{R}$ and $\lambda_3 \in i\mathbb{R}$ with multiplicities 1, that is $d_j = 0$ for $j = 1, 2, 3$. Moreover, by Corollary 14 and Lemma 17 we conclude that $\lambda_1 = 2\gamma + 2i\beta$, $\lambda_2 = -2\gamma + 2i\beta$, $\lambda_3 = 2i\beta$ and $k(z)$ is given by (6.25). Invoking Remark 2 let us w.l.o.g. assume $\beta = 0$. Thus,

$$\lambda_1 = 2\gamma, \quad \lambda_2 = -2\gamma, \quad \lambda_3 = 0, \qquad \text{and} \quad k(z) = \frac{k_r(\mu z)}{\sinh \gamma z}, \quad r \in \{1, 2, 3\},$$

where k_r is defined in (6.20), moreover $\mathscr{b}(y) = \cosh(2\gamma y) - \cosh(2\gamma)$. Because of (6.2), $\mathscr{c}$ has the following form

$$\mathscr{c}(y) = (c_1 y + d_1)e^{\lambda_1 y} + (c_2 y + d_2)e^{\lambda_2 y} + (c_3 y + d_3)e^{\lambda_3 y} + c_4 e^{\tau y},$$

where τ is different from all λ_j's. Substituting these expressions into (6.1) and looking at linearly independent parts it is easy to conclude that $c_1 = c_2 = c_3 = c_4 = 0$, and $d_1 = \frac{\lambda_1^2 + 4\mu^2}{8}$, $d_2 = \frac{\lambda_2^2 + 4\mu^2}{8}$ if in the formula for k we have $r = 2$. When $r = 3$ in the expressions of d_1, d_2; μ should be replaced by $i\mu$ and when $r = 1$, in those formulas $\mu = 0$. This concludes item 1 of Theorem 2 in the case $\gamma \neq 0$.

6.4 Item 3

Finally we consider the case III, because of the boundary conditions one can find that $\lambda_2 - \lambda_1 = i\pi n$ with $0 \neq n \in \mathbb{Z}$, therefore $\lambda_1, \lambda_2 \in i\mathbb{R}$ (otherwise by Corollary 14 and Lemma 17 the difference $\lambda_2 - \lambda_1$ is real). Let us now take $\lambda_1 = 2i(\beta + \frac{\pi n}{4})$ and $\lambda_2 = 2i(\beta - \frac{\pi n}{4})$ with some $\beta \in \mathbb{R}$. In this case we find $\mathscr{b}(y) = e^{2i\beta y} \sin\left(\frac{\pi n(y-1)}{2}\right)$ and by (6.19)

$$k(z) = e^{i\beta z} \frac{\alpha_1 k_s(\mu_1 z) e^{i\pi n z/4} + \alpha_2 k_r(\mu_2 z) e^{-i\pi n z/4}}{\sin(\pi n z/2)}, \quad r, s \in \{1, 2, 3\}. \tag{6.29}$$

From (6.2), $\mathscr{c}$ has the form

$$\mathscr{c}(y) = (c_1 y + d_1)e^{\lambda_1 y} + (c_2 y + d_2)e^{\lambda_2 y} + c_3 e^{\tau y},$$

with $\tau \neq \lambda_j$, note that also $\tau = \frac{2k'(0)}{k(0)} \in i\mathbb{R}$. The denominator of k has zeros at $z = \frac{2m}{n}$ for $m \in \mathbb{Z}$, since we want k to be smooth in $[-2, 2]$, we need

$$(-1)^m \alpha_1 k_s\left(\tfrac{2\mu_1 m}{n}\right) + \alpha_2 k_r\left(\tfrac{2\mu_2 m}{n}\right) = 0, \qquad \forall m \in \mathbb{Z} \quad \text{s.t. } \tfrac{m}{n} \in [-1, 1]. \tag{6.30}$$

1. $r = s = 3$, if $n \neq \pm 1$, then (6.30) must hold for $m = 1, 2$, one can easily see that this leads to a contradiction. Therefore $n = \pm 1$, in which case (6.30) implies $\alpha_1 \sinh(2\mu_1) = \alpha_2 \sinh(2\mu_2)$. To find c, we substitute these expressions into (6.1) and look at the coefficients of linearly independent parts, which must vanish. In particular the coefficient of $e^{\tau y}$ gives

$$c_3 \left\{ \alpha_2 \sinh(\mu_2 z) \left[e^{-\frac{\lambda_2 - 2\tau}{2} z} - e^{\frac{\lambda_2}{2} z} \right] + \alpha_1 \sinh(\mu_1 z) \left[e^{-\frac{\lambda_1 - 2\tau}{2} z} - e^{\frac{\lambda_1}{2} z} \right] \right\} = 0.$$

The four exponentials in square brackets are linearly independent, moreover their exponents are purely imaginary, while μ_1, μ_2 are real, hence all the terms are linearly independent, therefore our conclusion is that $c_3 = 0$, otherwise $k = 0$. Using similar arguments and looking at coefficients of $ye^{\lambda_j y}, e^{\lambda_j y}$ we find $c_1 = c_2 = 0$ and

$$d_1 = -\frac{ie^{-\frac{i\pi}{2}}}{8}[\lambda_1^2 - 4\mu_2^2], \qquad d_2 = \frac{ie^{\frac{i\pi}{2}}}{8}[\lambda_2^2 - 4\mu_1^2] \tag{6.31}$$

2. $s = 1, r = 3$, we can absorb μ_1 into α_1 and relabel μ_2 by μ, as in 1 we see $n = \pm 1$ and $2\alpha_1 = \alpha_2 \sinh(2\mu)$. Then one can find $c_1 = c_2 = c_3 = 0$ and (6.31) holds with $\mu_2 = 0$ and $\mu_1 = \mu$.

3. $r = s = 1$, absorb μ_j into α_j, again $n = \pm 1$ and $\alpha_1 = \alpha_2$, in which case (up to a real multiplicative constant) $k(z) = e^{i\beta z} \frac{z}{\sin(\pi z/4)}$, then we can conclude $c_1 = c_2 = 0$, $\tau = 2i\beta$ and (6.31) holds with $\mu_1 = \mu_2 = 0$.

4. $s = 1, r = 2$, absorb μ_1 into α_1. If $n = \pm 1$ we get $2\alpha_1 = \alpha_2 \sin(2\mu_2)$, and following the strategy described in 1 we find $c_1 = c_2 = c_3 = 0$, and (6.31) holds with $\mu_1 = 0$ and μ_2 replaced by $i\mu_2$. If $|n| > 1$, then (6.30) holds for at least $m = 1, 2$. It is easy to see that these two equations imply $\alpha_1 = 0$ and $\sin\left(\frac{2\mu_2}{n}\right) = 0$. But in that case (6.30) holds for any $m \in \mathbb{Z}$. So $\mu_2 = \frac{\pi n l}{2}$ for some $l \in \mathbb{Z}$, hence we see that k is a trigonometric polynomial, and therefore is trivial.

5. $s = 3, r = 2$, again if $|n| > 1$ we get $\alpha_1 = 0$ and $\sin\left(\frac{2\mu_2}{n}\right) = 0$, which again implies k is trivial. So $n = \pm 1$, and we find $\alpha_1 \sinh(2\mu_1) = \alpha_2 \sin(2\mu_2)$

6. $s = r = 2$, as we saw in Lemma 12 if $n \neq \pm 1$, then k is trivial. So $n = \pm 1$ and $\alpha_1 \sin(2\mu_1) = \alpha_2 \sin(2\mu_2)$, one of α_j is nonzero, assume it is α_2. When $\sin(2\mu_1) = 0$, then $\sin(2\mu_2) = 0$ and again k is a trigonometric polynomial. So $\sin(2\mu_1) \neq 0$ and also $\sin(2\mu_2) \neq 0$, again because of the same reason. We then find $c_1 = c_2 = 0$, (6.31) holds with μ_j replaced by $i\mu_j$ for $j = 1, 2$. Finally the relation for $e^{\tau y}$ reads

$$c_3 \left\{ \tilde{\alpha}_1 \sin(\mu_1 z) \left[e^{(\tau - \frac{\lambda_1}{2})z} - e^{\frac{\lambda_1}{2} z} \right] + \tilde{\alpha}_2 \sin(\mu_2 z) \left[e^{(\tau - \frac{\lambda_2}{2})z} - e^{\frac{\lambda_2}{2} z} \right] \right\} = 0,$$

where $\tilde{\alpha}_j = \sin(2\mu_j) \neq 0$, $\lambda_1 - \lambda_2 = \frac{i\pi}{2}$. Now $c_3 = 0$ or the function in curly brackets (denote it by $f(z)$) vanishes, looking at the asymptotics $f(iz)$ as $z \to \infty$, and also at $f'(0), f''(0), f^{(4)}(0)$ we can find that $f = 0$ if and only if $\mu_2 = \mu_1 \pm \frac{\pi}{2}$ (which implies $\tilde{\alpha}_1 = -\tilde{\alpha}_2$) and $\tau = 2i(\beta - \frac{\pi}{4} \pm \mu_1)$.

Choosing $\beta = 0$ (cf. Remark 2) we conclude item 3 of Theorem 2.

7 $L_2 = -L_1$

Assume the setting of Theorem 4, recall that $\mathscr{b} := \mathscr{b}_1$ and $\mathscr{c} := \mathscr{c}_1$. Now (R) reads

$$\begin{aligned} \mathscr{b}(y)k''(-z) + \mathscr{b}(y+z)k''(z) + \mathscr{b}'(y)k'(-z) + \mathscr{b}'(y+z)k'(z) + \\ + \mathscr{c}(y)k(-z) + \mathscr{c}(y+z)k(z) = 0. \end{aligned} \tag{7.1}$$

The analysis in the beginning of Section 4 shows that (in the case $L_2 = -L_1$) $\mathscr{b}(y)$ solves second order, linear homogeneous ODE with constant coefficients, and because of the boundary conditions it must be of the form

$$\mathscr{b}(y) = b_1 e^{\lambda_1 y} + b_2 e^{\lambda_2 y}, \quad \mathscr{c}(y) = c_1 e^{\lambda_1 y} + c_2 e^{\lambda_2 y} + c_0, \qquad \lambda_1 \neq \lambda_2,$$

where $\mathscr{c}$ is of the same form as $\mathscr{b}$ because it satisfies $\mathscr{c}' = -\frac{k_1}{k_0}\mathscr{b}' - \frac{k_2}{k_0}\mathscr{b}$. Clearly both b_j are different from zero, and from boundary conditions

$$\lambda_1 - \lambda_2 = \pi i n, \qquad n \in \mathbb{Z}. \tag{7.2}$$

With these formulas, (7.1) becomes a linear combination of functions $e^{\lambda_j y}$ with coefficients depending on z, hence each coefficient must vanish. Let us concentrate on the coefficient of $e^{\lambda_1 y}$, making the change of variables $k(z) = \kappa(z)e^{-\lambda_1 z/2}$ we rewrite it as

$$\kappa_+''(z) - \mu^2 \kappa_+(z) = 0, \qquad \mu = \sqrt{\tfrac{\lambda_1^2}{4} - \tfrac{c_1}{b_1}},$$

where κ_+ is the even part of κ, because it is an even function we get

$$\kappa_+(z) = \alpha \cosh(\mu z).$$

The symmetry of k implies

$$e^{-\overline{\lambda_1} z/2}\left(\overline{\kappa_+(z)} + \overline{\kappa_-(z)}\right) = e^{\lambda_1 z/2}\left(\kappa_+(z) - \kappa_-(z)\right).$$

If $\lambda_1 = 2i\beta$ with $\beta \in \mathbb{R}$, then κ_- is an arbitrary odd and purely imaginary function. Moreover, κ_+ must be real valued, hence

$$k(z) = e^{-i\beta z}\left(\kappa_-(z) + \begin{cases} \alpha\cosh(\mu z) \\ \alpha\cos(\mu z) \end{cases}\right), \tag{7.3}$$

where $\alpha, \mu \in \mathbb{R}$.

If $\lambda_1 = 2\gamma + 2i\beta$ with $\gamma \neq 0$, then (recalling that k is smooth at 0), with $\kappa_0 \in \mathbb{R}$

$$k(z) = \alpha e^{-i\beta z}\frac{e^{\gamma z}\cosh(\mu z) - e^{-\gamma z}\cosh(\overline{\mu} z)}{\sinh(2\gamma z)}.$$

Now k should come from two distinct modes λ_1, λ_2, and from (7.2) we see that $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 =: 2\gamma$, so if $\gamma \neq 0$ we must have

$$\alpha_1 e^{-i\beta_1 z}\left(e^{\gamma z}\cosh(\mu z)-e^{-\gamma z}\cosh(\overline{\mu}z)\right)=\alpha_2 e^{-i\beta_2 z}\left(e^{\gamma z}\cosh(\nu z)-e^{-\gamma z}\cosh(\overline{\nu}z)\right),$$

which implies $\beta_1=\beta_2$, leading to a contradiction. Indeed, the function on LHS (denoted by $f(z)$) determines β_1, because with $\mu=\mu_1+i\mu_2$

$$f(iz)=\kappa_0 e^{\beta_1 z}\left[ie^{\mu_2 z}\sin\left((\gamma-\mu_1)z\right)+e^{-\mu_2 z}\cos\left((\gamma+\mu_1)z\right)\right].$$

Assume $\mu_2>0$, then $f(iz)\sim\kappa_0 e^{(\beta_1+\mu_2)z}\sin\left((\gamma-\mu_1)z\right)$ as $z\to+\infty$, hence $\beta_1+\mu_2$ is determined by f, but by looking at the asymptotics as $z\to-\infty$ we see that also $\beta_1-\mu_2$ is determined, hence so is β_1. The case $\mu_2\le 0$ is done analogously.

Thus $\lambda_j=2i\beta_j\in i\mathbb{R}$ and k is given by (7.3), then κ_- is determined and we can find

$$k(z)=\frac{\alpha_1 k_s'(\mu_1 z)e^{i\beta_1 z}+\alpha_2 k_r'(\mu_2 z)e^{i\beta_2 z}}{i\sin(\beta_1-\beta_2)z},\qquad r,s\in\{1,2,3\},\tag{7.4}$$

where all the constants are real, and k_r' is the derivative of function k_r defined in (6.20). Moreover because k is smooth at 0, we must have $\alpha_2=-\alpha_1$. The denominator of the above function vanishes at $z=\frac{2m}{n}$ with $m\in\mathbb{Z}$, since k is smooth in $[-2,2]$ we should require

$$(-1)^m k_s'\left(\tfrac{2\mu_1 m}{n}\right)-k_2'\left(\tfrac{2\mu_2 m}{n}\right)=0,\qquad \forall m\in\mathbb{Z},\ \text{s.t. } \tfrac{m}{n}\in[-1,1].$$

Because $n\neq 0$, this condition should hold at least for $m=1$. One can easily check that this implies that the functions given by (7.4) are either zero, or trigonometric polynomials, and therefore: trivial.

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References

[1] Y. Grabovsky and N. Hovsepyan, *On the commutation properties of finite convolution and differential operators I: commutation*, preprint

[2] F. A. Grunbaum, *Differential operators commuting with convolution integral operators*, Journal of Mathematical Analysis and Applications, Volume 91, Issue 1, 1983, pg. 80-93

[3] J. A. Morrison, *On the commutation of finite integral operators, with difference kernels, and linear self-adjoint differential operators*, Notices Amer. Math. Soc. 9, 119 (1962).

[4] H. Widom, *Asymptotic behavior of the eigenvalues of certain integral equations II*, Arch. Rat. Mech. Anal. 17 (1964) 215–229

[5] P. E. Wright, *Finite symmetric convolution operators and singular symmetric differential operators*, Journal of Mathematical Analysis and Applications, Volume 148, Issue 2, 1990, pg. 538-547