Exact relations and links for fiber-reinforced composites with Hall effect

1 Composite materials

There is a mathematically rigorous (but non-elementary) theory of composite materials. It is not necessary for students participating in this REU program to know anything about. Therefore, here I will give only the intuitive picture of what I mean by a composite. In some sense, any object that consists of more than one material can be called a composite. However, when we speak about composites we usually mean that the constituent materials are mixed on a microscopic, invisible to the naked eye scale, while the object itself is on the human scale. The utility of composites comes from the fact that their properties (we call them effective properties) differ from the properties of their parent materials. Not only that, the effective properties of composite materials also depend on the microgeometry, or the geometric arrangement of constituent materials on the microscopic level. For example consider the following structures

Imagine that the gray material is hard, while the white material is soft. Then, the structure on the left will be hard for the vertical compression and will be soft for the horizontal compression. The structure on the right will feel soft in every direction. This example and many others that you may think of prove that it matters a lot how the component materials are arranged in a composite.

So, it is in some sense surprising that in some instances, when the constituent materials have a certain “sameness”, that “sameness” is preserved in a composite regardless of the microstructure. In this case we say that we have an exact relation for effective behavior of composites.
2 Conductivity and Hall effect tensors

If we place a three dimensional piece of a conductor in an electric field \( \mathbf{e} \in \mathbb{R}^3 \) we will generate a current field with density \( \mathbf{j} \in \mathbb{R}^3 \). The generalized Ohm’s law says that the relations between the electric field and the current field is the linear one given by

\[
\mathbf{j} = \mathbf{L} \mathbf{e},
\]

where \( \mathbf{L} \) is a symmetric and positive definite \( 3 \times 3 \) matrix. \( \mathbf{L} \) is called the conductivity tensor of the material. If in addition to the electric field we apply a static magnetic field \( \mathbf{h} \), then the relation (1) still holds but the matrix \( \mathbf{L} \) is no longer symmetric (it is still positive definite.) The deviation of the matrix \( \mathbf{L} \) from a symmetric matrix is called the Hall effect.

3 Fiber-reinforced composites

By fiber-reinforced composites we mean composites whose microstructure varies in a plane, for example, \( xy \)-plane, while there is no variation in the \( z \) direction. If we cut such a composite by a plane perpendicular to the \( z \)-axis we might see a microstructure similar to the one pictured on the right on page 1. If we cut such a composite by a plane parallel to the \( z \)-axis we might see a microstructure similar to the one pictured on the left on page 1.

4 Exact relations for the Hall effect

We say that we have an exact relation in the context of fiber-reinforced composites exhibiting Hall effect, if the effective tensor \( \mathbf{L}^* \) of a fiber-reinforced composite satisfies a system of equations

\[
F_1(\mathbf{L}) = 0, \ldots, F_n(\mathbf{L}) = 0
\]

whenever each of its constituents satisfies the same system of equations. Here, the magic is in the fact that even though the effective tensor \( \mathbf{L}^* \) of a composite does depend on the microstructure, it always satisfies the above system of equations.

To give you an example, consider composites made of materials that do not exhibit the Hall effect. It is intuitively clear that a composite will not exhibit the Hall effect either, no matter what the microstructure. That means that the system of equations \( L_{12} - L_{21} = 0, \ L_{13} - L_{31} = 0, \ L_{23} - L_{32} = 0 \) describes an exact relation. We can write this system in the matrix form as \( \mathbf{L}^T = \mathbf{L} \), where the superscript \( T \) denotes the matrix transpose. In the same fashion, we may write our general exact relation (2) in the vector form:

\[
\mathbf{F}(\mathbf{L}) = 0,
\]

where \( \mathbf{F}(\mathbf{L}) \) is a vector-valued function \( \mathbf{F}(\mathbf{L}) = (F_1(\mathbf{L}), \ldots, F_n(\mathbf{L})) \).

During 2002 and 2003 Sumer REU programs the students have identified all such systems (there are a lot). This year the focus will be on the new kind of exact relations called
“links”. Suppose now that we build our composite out of \( r \) materials \( L_1, \ldots, L_r \). Suppose that another set of \( r \) materials \( N_1, \ldots, N_r \) is related to our original materials by the same system of equations \( F(L, N) = 0 \). We call such a relation a “link”, if the effective tensor \( N^* \) of composite made by replacing materials \( L_j \) by the materials \( N_j \) for each \( j = 1, \ldots, r \), is related to the effective tensor \( L^* \) of the original composite by the same system of equations: \( F(L^*, N^*) = 0 \). Here we use the same vector notation that we used in replacing equations (2) by the equation (3).

In order to understand how we go about finding the links we need to review how we compute exact relations.

5 Computing exact relations

There are two steps. On the first step we need to compute all subspaces \( \Pi \) of \( \text{End}(\mathbb{R}^3) \)—the space of \( 3 \times 3 \) matrices, that satisfy the equation

\[
KAK \in \Pi
\]

for all \( K \in \Pi \) and all matrices \( A \) from the subspace \( \mathcal{A} \):

\[
\mathcal{A} = \left\{ \begin{bmatrix} s & t & 0 \\ t & -s & 0 \\ 0 & 0 & 0 \end{bmatrix} : t \in \mathbb{R}, \ s \in \mathbb{R} \right\}
\]

Once this is done one needs to compute the image of the subspace \( \Pi \) under the following transformation \( \Pi \ni K \mapsto L \):

\[
L = L_0 - U_0(I - KM)^{-1}KV_0,
\]

where \( I \) is a \( 3 \times 3 \) identity matrix, \( L_0, U_0 \) and \( V_0 \) are constant matrices (I will not dwell on what they are here.) and

\[
M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

The first step has been completed by Summer 2002 REU team, while the second step has been completed by Summer 2003 REU team.

6 Computing the “links”

For technical reasons all links come in two different flavors. Let us call them “global links” and “local links”. (The names are a little arbitrary but at the moment I cannot do any better.) Both global and local links come from two pairs of solutions to the equation (4).
Each pair \((\Pi_1, \Pi_2)\) has to be an algebra-ideal pair. It means that \(\Pi_1\) is a subspace in \(\Pi_2\) and in addition, for every \(K_1 \in \Pi_1\) and every \(K_2 \in \Pi_2\) and every \(A \in \mathcal{A}\)

\[ K_1AK_2 + K_2AK_1 \in \Pi_1. \]  

(6)

For example, the pairs \(\{(0), \Pi\}\) and \((\Pi, \Pi)\) have this property. The subspace \(\Pi_1\) is called an ideal in \(\Pi_2\). Another, less obvious example of an algebra-ideal pair is the derived ideal.

Let \(\Pi^2\) denote the subspace of \(\Pi\) given by

\[ \Pi^2 = \text{Span}\{KAK : K \in \Pi, \ A \in \mathcal{A}\}. \]

Then \(\Pi^2\) is an ideal in \(\Pi\), called the derived ideal. Moreover, any subspace \(\mathcal{K}\) of \(\Pi\) that contains the derived ideal is also an ideal in \(\Pi\). Two such pairs \((\mathcal{K}_1, \Pi_1)\) and \((\mathcal{K}_2, \Pi_2)\) generate a global link provided the co-dimension of \(\mathcal{K}_1\) in \(\Pi_1\) is the same as the co-dimension of \(\mathcal{K}_2\) in \(\Pi_2\). It is quite easy to identify all such pairs of algebra-ideal pairs.

Local links are much more numerous and harder to identify. Local links are generated by pairs \(\{(I_1, \Pi_1), (I_2, \Pi_2)\}\) of algebra-ideal pairs such that the factor-algebras \(\Pi_1/I_1\) and \(\Pi_2/I_2\) are isomorphic. What that means is that there is a linear bijective map \(\Phi : \Pi_1/I_1 \rightarrow \Pi_2/I_2\) such that for every \(K \in \Pi_1\) and every \(A \in \mathcal{A}\)

\[ \Phi(KAK) = \Phi(K)A\Phi(K), \]

where \(K\) denotes the equivalence class in \(\Pi_1/I_1\) containing \(K\). It is important to mention here that different maps \(\Phi\) result in different links. Let us also make a simple remark that any global link is also a local link, but not vice versa. One particularly easy to understand example is when \(I_1 = I_2 = \{0\}\). In that case \(\Pi_1/I_1\) and \(\Pi_2/I_2\) are just \(\Pi_1\) and \(\Pi_2\) respectively. We then need to characterize all bijective linear maps \(\Phi : \Pi_1 \rightarrow \Pi_2\) such that for every \(K \in \Pi_1\) and every \(A \in \mathcal{A}\)

\[ \Phi(KAK) = \Phi(K)A\Phi(K). \]

We can identify some of these maps \(\Phi\) by asking which of them have the required property for all \(K \in \text{End}(\mathbb{R}^3)\), i.e. which of them have the required property for all \(3 \times 3\) matrices \(K\). The answer is, all maps \(\Phi\) of the form \(\Phi(K) = BKC\), where

\[
B = \begin{bmatrix}
\cos \alpha & -\sin \alpha & 0 \\
\sin \alpha & \cos \alpha & 0 \\
b_1 & b_2 & b_3
\end{bmatrix}, \quad C = \begin{bmatrix}
\cos \alpha & -\sin \alpha & c_1 \\
\sin \alpha & \cos \alpha & c_2 \\
0 & 0 & c_3
\end{bmatrix}.
\]

Summer 2003 REU team has identified all algebra-ideal pairs and has partially understood the structure of factor algebras \(\Pi_1/I_1\). It has also found all pairs of algebra-ideal pairs generating global links.

The first task of Summer 2004 REU team will be to find all pairs of isomorphic factor-algebras. The second step, as in the section on exact relations would be to compute the links \(F(L, N) = 0\) in terms of physical variables \(L\).
Now we remark that links can be thought of as exact relations that live in a space of twice the dimension. In the above description both local and global links are generated by a pair of algebra-ideal pairs \( (\mathcal{I}_1, \Pi_1) \) and \( (\mathcal{I}_2, \Pi_2) \) and by a bijective map \( \Phi \) between the factor-algebras. We can then represent the link as a subspace \( \hat{\Pi} \) of \( V = \text{End}(\mathbb{R}^3 \oplus \mathbb{R}^3) \):

\[
\hat{\Pi} = \left\{ \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} : K_1 \in \Pi_1, \ K_2 \in \Phi(K_1) \right\}.
\]

The subspace \( \hat{\Pi} \) satisfies the same kind of equation that the exact relations \( \Pi \) satisfy:

\[
\hat{K} \hat{A} \hat{K} \in \hat{\Pi}
\]

for every \( \hat{K} \in \hat{\Pi} \) and every \( \hat{A} \in \hat{\mathcal{A}} \), where

\[
\hat{\mathcal{A}} = \left\{ \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : A_1 \in \mathcal{A}, \ A_2 \in \mathcal{A} \right\}
\]

for global links and

\[
\hat{\mathcal{A}} = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \mathcal{A} \right\}
\]

for local links.

The second task that Summer 2004 REU team will address is the computation of the pairs \( (L, N) \) using formulas similar to (5):

\[
L = L_0 - U_1 (I - K_1 M)^{-1} K_1 V_1,
\]

\[
N = N_0 - U_2 (I - K_2 M)^{-1} K_2 V_2,
\]

where

\[
\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \in \hat{\Pi}.
\]

The parameter matrices \( L_0, N_0, U_1, U_2, V_1, V_2 \) are all related, but I will not go into details here. The problem here is to compute the dependence between the matrices \( L \) and \( N \) induced by the map \( \Phi \) relating \( K_1 \) and \( K_2 \).

If you are not familiar with some of the mathematical terminology used in this short description, do not worry. I will explain what is necessary during the first week of the program. The goal of this note is to give you a more or less concrete idea of what this REU program is about.