Optimal design problems for two-phase conducting composites with weakly discontinuous objective functionals

Yury Grabovsky[*]
Department of Mathematics
Temple University
Philadelphia, PA 19122

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Abstract

We consider the relatively simple but non-trivial example of an optimal design problem with a weakly discontinuous objective functional. The objective functional is quadratic and was suggested by Tartar in his "Remarks on optimal design" paper. We analyze a problem of finding a layout of a conducting composite such that the fields in both phases provide least squares fit to a given field. The main result of the paper is an explicit formula for the relaxed optimal design problem, suitable for numerical solution. Further analysis of our explicit formula shows that the optimal layout is a rank one laminate locally, lending some support for Tartar's conjecture that the minimizing sequences always converge strongly.

1 Introduction

The bulk of the existing topology optimization literature focuses almost exclusively on the problems with weakly continuous objective functionals, such as energy, see e.g. a survey paper [30] and the books [4, 9] and references therein. The energy functional has the advantage that both the objective functional and the equilibrium equations of conductivity or elasticity can be handled by a single variational principle. Other weakly continuous functionals were sometimes considered [10, 13, 22, 23]. However the more general functionals are of great importance in applications. One such application is the design of materials optimal with respect to stress concentrations. The damage in a brittle material usually occurs at regions of

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high stress concentrations. Therefore, the natural objective functional here is the sup-norm of the stress field in the body. The problems of stress concentrations have been treated before by beautiful but somewhat ad hoc analytical methods [3, 8, 12, 15, 33, 34, 35] with the goal of describing the optimal geometry *analytically and explicitly*. Our objective is different. We are looking for the *relaxed* formulations of optimization problems, i.e. formulations that behave “nicer” with respect to the existence of solutions and numerical optimization routines. The sup-norm functional can be effectively approximated by the integral functionals

$$I = \int_{\Omega} J(\boldsymbol{x}, \boldsymbol{e}) d\boldsymbol{x}, \tag{1.1}$$

where $\boldsymbol{e}$ is the elastic strain or the electric field. Currently we are rather far away from studying (1.1) in general or even attacking the problem of stress concentrations via (1.1). However, see the work of Pedregal [27, 28] for a discussion of the relation of (1.1) to gradient Young measures and quasiconvexification problems.

In this paper we consider the simplest example of (1.1) suggested by Tartar [32]. It concerns the design of a two phase conductor with fields in each phase conforming as close as possible to the given fields [10, 21, 32].

$$I = \int_{\Omega} |\boldsymbol{e}(\boldsymbol{x}) - \boldsymbol{\eta}(\boldsymbol{x})|^2 d\boldsymbol{x} \quad \to \quad \min, \tag{1.2}$$

where $\boldsymbol{e}(\boldsymbol{x})$ is the electric field and $\boldsymbol{\eta}(\boldsymbol{x})$ is given.

The general problem (1.1) may or may not have a classical design solution if one desires a global optimum with no topological constraints. The failure of existence is linked to the fact that the minimizing sequence for (1.1) develops oscillations. The physical meaning of these oscillations is that the set of classical layouts is too narrow and the optimal layout must use composites as structural elements. In the case of weakly continuous objective functionals the passage to composites is sufficient to regularize the ill-posed problem [14, 16, 17]. In our case of weakly discontinuous objective functionals there is an additional difficulty. In order to determine the value of the objective functional for designs incorporating composites it is insufficient to know just the effective properties of the composite medium. The functional is sensitive to finer features of oscillations of the minimizing sequence, features that go beyond the effective properties. In the physics literature there has been some work on bounding the variance of the fluctuating fields in each phase of a two phase composite [1, 5, 7]. Very recently Lipton used the analytic method of Bergman [6] to obtain bounds on the variance of the fluctuating fields in the entire periodic composite [19, 20] and to solve an optimal design problem very similar to the one considered here [18].

In this paper after a brief review of the relation of the oscillating fields in a material to composites we compute the explicit relaxation of the functional (1.2) in section 4. The optimal design problem with the relaxed functional I^* should have at least one solution, every solution of the relaxed problem should be a weak limit of a minimizing sequence for the original problem and every weak limit of a minimizing sequence for I should be a minimizer for I^*. In addition, we must have $\min I^* = \inf I$.

2 Formulation of the problem

Consider a body $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) that is to be occupied by the two isotropic conductors α and β with $\alpha < \beta$. The electrostatic potential ϕ is given in Ω by

$$\begin{cases} \nabla \cdot a(\boldsymbol{x})\nabla\phi = f, \\ \phi|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $a(\boldsymbol{x})$ takes the values α and β :

$$a(\boldsymbol{x}) = \alpha\chi(\boldsymbol{x}) + \beta(1 - \chi(\boldsymbol{x})). \tag{2.2}$$

The source distribution f is given. The characteristic function $\chi(\boldsymbol{x})$ of the set where $a(\boldsymbol{x}) = \alpha$ plays the role of the control. Let

$$I(\chi) = \int_\Omega |\nabla\phi(\boldsymbol{x}) - \boldsymbol{\eta}(\boldsymbol{x})|^2 d\boldsymbol{x} \tag{2.3}$$

be the objective functional to be minimized, either over all possible characteristic functions χ, or only those with a prescribed volume average.

A very similar problem with the volume fraction constraint has been solved by Lipton [18]. His objective functional is the same as (2.3) with $\boldsymbol{\eta} = \boldsymbol{0}$. The field ϕ solves a pde in (2.1) with $f = 0$ but with affine Dirichlet boundary conditions.

If we impose no volume fraction constraints and if $\boldsymbol{\eta} = \boldsymbol{0}$ then Lipton (personal communication) has the following elementary but beautiful argument that shows that regardless of the value of f in the right hand side of (2.1) the pure conductor β gives the optimum.

Suppose ϕ_0 is the solution of (2.1) with $a(\boldsymbol{x}) = \beta$ and suppose ϕ is the solution of (2.1) with any other $a(\boldsymbol{x})$. Then integrating by parts $\|\nabla\phi_0\|^2$ we obtain

$$\|\nabla\phi_0\|^2 = -\frac{1}{\beta}\int_\Omega \phi_0 f d\boldsymbol{x} = -\frac{1}{\beta}\int_\Omega a(\boldsymbol{x})\nabla\phi \cdot \nabla\phi_0 d\boldsymbol{x} \leq \int_\Omega |\nabla\phi \cdot \nabla\phi_0| d\boldsymbol{x} \leq \|\nabla\phi\|\,\|\nabla\phi_0\|.$$

Thus $\|\nabla\phi_0\|^2 \leq \|\nabla\phi\|^2$, as claimed.

It is quite possible that the minimizer for (2.3) does not exist and the minimizing sequence develops oscillations. In this case there is a well known theory describing weak limits of such sequences. We review this theory for the reader's convenience.

3 H-convergence and G-closures

Let ϕ_ε be a minimizing sequence for I corresponding to the sequence of designs $\chi^\varepsilon(\boldsymbol{x})$. As in the finite dimensional optimization one would like to be able to extract a convergent subsequence. But in our infinite dimensional setting we have to be careful and specify precisely the type of convergence we are using. The appropriate convergence notion was introduced by Murat and Tartar [26]. They called it H-convergence (H stands for homogenization).

Definition 1 *A sequence of tensors* $\boldsymbol{a}^\varepsilon(\boldsymbol{x})$ *H-converges to the tensor* $\boldsymbol{a}^*(\boldsymbol{x})$ *if* ϕ_ε *solves*

$$\begin{cases} \nabla \cdot \boldsymbol{a}^\varepsilon(\boldsymbol{x}) \nabla \phi_\varepsilon = f, \\ \phi_\varepsilon|_{\partial\Omega} = 0, \end{cases} \tag{3.1}$$

and the following weak limit relations hold

$$\begin{array}{l} \nabla \phi_\varepsilon \ \rightharpoonup \ \nabla \phi_0, \\ \boldsymbol{a}^\varepsilon(\boldsymbol{x}) \nabla \phi_\varepsilon \ \rightharpoonup \ \boldsymbol{a}^*(\boldsymbol{x}) \nabla \phi_0, \end{array} \tag{3.2}$$

where ϕ_0 *solves the homogenized equilibrium equations*

$$\begin{cases} \nabla \cdot \boldsymbol{a}^*(\boldsymbol{x}) \nabla \phi_0 = f, \\ \phi_0|_{\partial\Omega} = 0. \end{cases} \tag{3.3}$$

The notion of H-convergence is perfectly suited for optimal design problems with constraints of the type (2.1) because it makes precise the sense in which equation (3.3) is a limit of the sequence of equations (3.1). Moreover, it possesses the compactness property we need in order to study the existence of solutions to optimal design problems. This property is given by the compactness theorem of Murat and Tartar [26].

Theorem 1 *If for almost every* $\boldsymbol{x} \in \Omega$ *the sequence of symmetric matrices* $\boldsymbol{a}^\varepsilon(\boldsymbol{x})$ *satisfies*

$$\alpha \boldsymbol{I} \leq \boldsymbol{a}^\varepsilon(\boldsymbol{x}) \leq \beta \boldsymbol{I}$$

for some $\beta > \alpha > 0$ *in the sense of quadratic forms then there is an H-convergent subsequence* $\boldsymbol{a}^\varepsilon(\boldsymbol{x}) \xrightarrow{H} \boldsymbol{a}^*(\boldsymbol{x})$.

The most enlightening example of an H-convergent sequence of tensors $\boldsymbol{a}^\varepsilon$ is the case of locally periodic composites. Let $\chi^\varepsilon(\boldsymbol{x}) = \chi(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)$, where $\chi(\boldsymbol{x}, \boldsymbol{y})$ is periodic in the $\boldsymbol{y}$ variable with the parallelepiped of periods $Q = [0,1]^d$. Then $a(\boldsymbol{x}, \boldsymbol{x}/\epsilon)$ H-converges to $\boldsymbol{a}^*(\boldsymbol{x})$, defined by

$$\boldsymbol{a}^*(\boldsymbol{x}) = \int_Q a(\boldsymbol{x}, \boldsymbol{y}) \nabla \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{y} \tag{3.4}$$

in terms of the solution $\boldsymbol{\Phi} : \Omega \times Q \ \rightarrow \ \mathbb{R}^d$ of the so called cell problem:

$$\begin{cases} \nabla_{\boldsymbol{y}} \cdot a(\boldsymbol{x}, \boldsymbol{y}) \nabla_{\boldsymbol{y}} \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}) = 0, \quad \boldsymbol{y} \in Q \\ \displaystyle\int_Q \nabla_{\boldsymbol{y}} \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{y} = \boldsymbol{I}, \end{cases} \tag{3.5}$$

where $\nabla_{\boldsymbol{y}} \boldsymbol{\Phi}$ is assumed to be Q-periodic in $\boldsymbol{y}$. (We use an unusual convention $(\nabla \boldsymbol{\Phi})_{ij} = \dfrac{\partial \Phi_j}{\partial x_i}$.) In particular, for periodic media, the effective tensor $\boldsymbol{a}^*$ is constant. The explicit definition (3.4) prompted a natural question of computing G-closure sets. There are two types of them. The absolute G-closure of a set $\mathcal{S}$ of materials is the set of all effective tensors $\boldsymbol{a}^*$ of periodic composites made with materials from the set $\mathcal{S}$. The other type is the G_θ-closure—the set of all effective tensors $\boldsymbol{a}^*$ of periodic composites made with materials α and β taken in prescribed volume fraction θ.

The relevance of G and G_θ closures is brought out by the unpublished theorem of Kohn and Dal Maso (see [29] for the proof). The theorem says that $\boldsymbol{a}^*(\boldsymbol{x})$ is an H-limit of a sequence of $\boldsymbol{a}^\varepsilon(\boldsymbol{x})$, not necessarily locally periodic, if and only if $\boldsymbol{a}^*(\boldsymbol{x}) \in G_{\theta(\boldsymbol{x})}$ for almost every $\boldsymbol{x} \in \Omega$. In this paper we will be interested in more than just the effective conductivity tensor. Nevertheless, the effective tensors and the Kohn and Dal Maso theorem are going to play an important role in our analysis.

For two-phase conducting composites, where the G_θ-closure is explicitly known [24, 25, 31], the Kohn-Dal Maso theorem was proved directly by Tartar [31]. The set G_θ is described by two sets of bounds. The first set comprises the elementary, or Wiener, bounds [36]

$$h\boldsymbol{I} \leq \boldsymbol{a}^* \leq m\boldsymbol{I} \tag{3.6}$$

in the sense of quadratic forms, where m and h are arithmetic and harmonic means of α and β respectively:

$$m = \alpha\theta + \beta(1-\theta), \qquad h = (\theta/\alpha + (1-\theta)/\beta)^{-1}.$$

The second set contains more delicate trace bounds [24, 25, 31]

$$\operatorname{Tr}(\boldsymbol{a}^* - \alpha\boldsymbol{I})^{-1} \leq \frac{d}{m-\alpha} + \frac{\theta}{\alpha(1-\theta)}, \tag{3.7}$$

$$\operatorname{Tr}(\beta\boldsymbol{I} - \boldsymbol{a}^*)^{-1} \leq \frac{d}{\beta-m} - \frac{1-\theta}{\beta\theta}. \tag{3.8}$$

We would like to draw reader's attention to the fact that every $\boldsymbol{a}^*$ satisfying Wiener bounds (3.6) and achieving equality in the upper trace bound (3.8) can be realized as a composite with the uniform field in the phase of conductivity α *for any uniform applied field*. Conversely, any such composite will necessarily achieve equality in (3.8). One class of such composites is the multiple rank laminates [31], another is the confocal ellipsoid construction [31].

4 Relaxation

As we mentioned before, the knowledge of the effective tensor $\boldsymbol{a}^*$ is not sufficient. Not only our functional is weakly discontinuous, the limit $\lim_{\varepsilon \to 0} I(\chi^\varepsilon)$ can not be expressed in terms of the weak limit $\nabla\phi_0$ and the effective tensor $\boldsymbol{a}^*$. This means that the integrand in (2.3) "sees" finer features of oscillations of ϕ_ε than those that determine $\boldsymbol{a}^*(\boldsymbol{x})$.

Let ϕ_ε be a minimizing sequence for I corresponding to the sequence of designs $\chi^\varepsilon(\boldsymbol{x})$. We can restrict our attention to a subsequence labeled $\chi^\varepsilon(\boldsymbol{x})$ again, such that $\chi^\varepsilon(\boldsymbol{x}) \overset{*}{\rightharpoonup} \theta(\boldsymbol{x})$

and $\boldsymbol{a}^\varepsilon \overset{H}{\rightharpoonup} \boldsymbol{a}^*(\boldsymbol{x})$. Then ϕ_ε converges weakly in $W^{1,2}(\Omega)$ to ϕ_0 and ϕ_0 solves (3.3). We therefore obtain

$$\lim_{\varepsilon \to 0} I(\chi^\varepsilon) = \int_\Omega (|\boldsymbol{\eta}(\boldsymbol{x})|^2 - 2\boldsymbol{\eta}(\boldsymbol{x}) \cdot \nabla\phi_0(\boldsymbol{x}))d\boldsymbol{x} + \lim_{\varepsilon \to 0} \int_\Omega |\nabla\phi_\varepsilon|^2 d\boldsymbol{x}. \tag{4.1}$$

We have the following strategy for evaluating the limit in (4.1). We define

$$\mathcal{A}(\theta, \boldsymbol{a}^*) = \{\{\chi^\varepsilon(\boldsymbol{x}) :\ \varepsilon > 0\}|\ \boldsymbol{a}^\varepsilon \overset{H}{\rightharpoonup} \boldsymbol{a}^*(\boldsymbol{x}),\ \chi^\varepsilon(\boldsymbol{x}) \overset{*}{\rightharpoonup} \theta(\boldsymbol{x})\}$$

and then evaluate

$$J^*(\theta, \boldsymbol{a}^*) = \inf_{\chi^\varepsilon \in \mathcal{A}(\theta, \boldsymbol{a}^*)} \{\liminf_{\varepsilon \to 0} \int_\Omega |\nabla\phi_\varepsilon|^2 d\boldsymbol{x}\}. \tag{4.2}$$

It turns out that it is possible to find a good estimate for $J^*(\theta, \boldsymbol{a}^*)$.

Lemma **1**

$$J^*(\theta, \boldsymbol{a}^*) \geq \int_\Omega \left\{ -\frac{1}{\beta} f(\boldsymbol{x})\phi_0(\boldsymbol{x}) + \frac{|(\beta \boldsymbol{I} - \boldsymbol{a}^*(\boldsymbol{x}))\nabla\phi_0(\boldsymbol{x})|^2}{\beta(\beta - \alpha)\theta(\boldsymbol{x})} \right\}. \tag{4.3}$$

The equality is achieved for the effective tensors of composites with constant local field in the phase of conductivity α. To be more precise, the equality in (4.3) is achieved if and only if the Young measure $\nu_{\boldsymbol{x}}$ corresponding to the sequence $(\chi^\varepsilon(\boldsymbol{x}), \chi^\varepsilon(\boldsymbol{x})\nabla\phi_\varepsilon(\boldsymbol{x}))$ is supported on a ray

$$\mathcal{R}_{\boldsymbol{x}} = \{(\tau, \boldsymbol{y}) \in (0, +\infty) \times \mathbb{R}^d|\ \boldsymbol{y} = \tau \boldsymbol{v}_0(\boldsymbol{x})\}$$

for some $\boldsymbol{v}_0(\boldsymbol{x})$, for almost every $\boldsymbol{x} \in \Omega$.

Proof: We have the following three identities

$$\nabla\phi_\varepsilon(\boldsymbol{x}) \ \rightharpoonup\ \nabla\phi_0(\boldsymbol{x}) \tag{4.4}$$

in $L^2(\Omega)$.

$$\boldsymbol{a}^\varepsilon(\boldsymbol{x})\nabla\phi_\varepsilon(\boldsymbol{x}) \ \rightharpoonup\ \boldsymbol{a}^*(\boldsymbol{x})\nabla\phi_0(\boldsymbol{x}) \tag{4.5}$$

in $L^2(\Omega)$.

$$a^\varepsilon(\boldsymbol{x})|\nabla\phi_\varepsilon(\boldsymbol{x})|^2 \ \rightharpoonup\ \boldsymbol{a}^*(\boldsymbol{x})\nabla\phi_0(\boldsymbol{x}) \cdot \nabla\phi_0(\boldsymbol{x}) \tag{4.6}$$

in the sense of distributions, or more precisely, in the sense of weak convergence of measures.

We will use them to estimate the integral in (4.2). Using equations (4.4) and (4.5) we obtain

$$\chi^\varepsilon(\boldsymbol{x})\nabla\phi_\varepsilon(\boldsymbol{x}) \ \rightharpoonup\ \frac{\beta\boldsymbol{I} - \boldsymbol{a}^*(\boldsymbol{x})}{\beta - \alpha}\nabla\phi_0(\boldsymbol{x}). \tag{4.7}$$

We can use the remaining relation (4.6) by representing $|\nabla\phi_\varepsilon|^2$ as follows

$$|\nabla\phi_\varepsilon(\boldsymbol{x})|^2 = \frac{\beta - \alpha}{\beta}\chi^\varepsilon(\boldsymbol{x})|\nabla\phi_\varepsilon(\boldsymbol{x})|^2 + \frac{1}{\beta}a^\varepsilon(\boldsymbol{x})|\nabla\phi_\varepsilon(\boldsymbol{x})|^2. \tag{4.8}$$

In order to estimate the first term in (4.8) we use the fact that the function

$$C(\tau, \boldsymbol{y}) = \begin{cases} |\boldsymbol{y}|^2/\tau, & \text{if } \tau > 0 \\ +\infty, & \text{if } \tau = 0,\ \boldsymbol{y} \neq \boldsymbol{0} \\ 0, & \text{if } \tau = 0,\ \boldsymbol{y} = \boldsymbol{0} \end{cases}$$

is convex and lower semicontinuous on $[0, +\infty) \times \mathbb{R}^d$ and that

$$\chi^\varepsilon(\boldsymbol{x})|\nabla\phi_\varepsilon(\boldsymbol{x})|^2 = C(\chi^\varepsilon(\boldsymbol{x}), \chi^\varepsilon(\boldsymbol{x})\nabla\phi_\varepsilon(\boldsymbol{x})).$$

Therefore, using (4.7) and weak lower semicontinuity of convex functions we obtain

$$\lim_{\varepsilon \to 0} \chi^\varepsilon(\boldsymbol{x})|\nabla\phi_\varepsilon(\boldsymbol{x})|^2 \geq \frac{|(\beta\boldsymbol{I} - \boldsymbol{a}^*(\boldsymbol{x}))\nabla\phi_0(\boldsymbol{x})|^2}{\theta(\boldsymbol{x})(\beta - \alpha)^2}, \tag{4.9}$$

where the limit is understood as a weak limit of measures. Now using (4.9) and (4.6) we can pass to the limit in (4.8) and obtain

$$\lim_{\varepsilon \to 0} |\nabla\phi_\varepsilon(\boldsymbol{x})|^2 \geq \frac{|(\beta\boldsymbol{I} - \boldsymbol{a}^*(\boldsymbol{x}))\nabla\phi_0(\boldsymbol{x})|^2}{\theta(\boldsymbol{x})\beta(\beta - \alpha)} + \frac{1}{\beta}\boldsymbol{a}^*(\boldsymbol{x})\nabla\phi_0(\boldsymbol{x}) \cdot \nabla\phi_0(\boldsymbol{x}),$$

Integration by parts and (3.3) gives us

$$\lim_{\varepsilon \to 0} \int_\Omega |\nabla\phi_\varepsilon(\boldsymbol{x})|^2 d\boldsymbol{x} \geq \int_\Omega \left\{ \frac{|(\beta\boldsymbol{I} - \boldsymbol{a}^*(\boldsymbol{x}))\nabla\phi_0(\boldsymbol{x})|^2}{\theta(\boldsymbol{x})\beta(\beta - \alpha)} - \frac{1}{\beta} f(\boldsymbol{x})\phi_0(\boldsymbol{x}) \right\} d\boldsymbol{x}. \tag{4.10}$$

Let us examine when an equality holds in (4.9). The language of Young measures [2] is the appropriate way to formulate the answer rigorously. Let $\nu_{\boldsymbol{x}}(\tau, \boldsymbol{y})$ be the Young measure corresponding to the weakly convergent sequence $(\chi^\varepsilon(\boldsymbol{x}), \chi^\varepsilon(\boldsymbol{x})\nabla\phi_\varepsilon(\boldsymbol{x}))$. Then

$$\lim_{\varepsilon \to 0} \chi^\varepsilon(\boldsymbol{x})|\nabla\phi_\varepsilon(\boldsymbol{x})|^2 = \int_{\mathcal{D}} C(\tau, \boldsymbol{y}) d\nu_{\boldsymbol{x}}(\tau, \boldsymbol{y}),$$

where $\mathcal{D} = [0, +\infty) \times \mathbb{R}^d$. The inequality (4.9) can be written as

$$\int_{\mathcal{D}} C(\tau, \boldsymbol{y}) d\nu_{\boldsymbol{x}}(\tau, \boldsymbol{y}) \geq C(\overline{\tau}(\boldsymbol{x}), \overline{\boldsymbol{y}}(\boldsymbol{x})),$$

where $(\overline{\tau}(\boldsymbol{x}), \overline{\boldsymbol{y}}(\boldsymbol{x}))$ is the first moment of the Young measure $\nu_{\boldsymbol{x}}$, or, in other words, the weak limit of the sequence $(\chi^\varepsilon(\boldsymbol{x}), \chi^\varepsilon(\boldsymbol{x})\nabla\phi_\varepsilon(\boldsymbol{x}))$:

$$(\overline{\tau}(\boldsymbol{x}), \overline{\boldsymbol{y}}(\boldsymbol{x})) = (\theta(\boldsymbol{x}), \frac{\beta\boldsymbol{I} - \boldsymbol{a}^*(\boldsymbol{x})}{\beta - \alpha}\nabla\phi_0(\boldsymbol{x})).$$

The inequality (4.9) becomes equality whenever

$$\int_{\mathcal{D}} C(\tau, \boldsymbol{y}) d\nu_{\boldsymbol{x}}(\tau, \boldsymbol{y}) = C(\overline{\tau}(\boldsymbol{x}), \overline{\boldsymbol{y}}(\boldsymbol{x})).$$

An easy computation will show that

$$\int_{\mathcal{D}} C(\tau, \boldsymbol{y}) d\nu_{\boldsymbol{x}}(\tau, \boldsymbol{y}) - C(\overline{\tau}, \overline{\boldsymbol{y}}) = \frac{1}{2\overline{\tau}} \int_{\mathcal{D}} \int_{\mathcal{D}} \tau\tau' \left| \frac{\boldsymbol{y}}{\tau} - \frac{\boldsymbol{y}'}{\tau'} \right|^2 d\nu_{\boldsymbol{x}}(\tau, \boldsymbol{y}) d\nu_{\boldsymbol{x}}(\tau', \boldsymbol{y}').$$

Therefore, the equality in (4.9) holds if and only if the Young measure $\nu_{\boldsymbol{x}}$ is supported on the ray

$$\mathcal{R}_{\boldsymbol{x}} = \{(\tau, \boldsymbol{y}) \in \mathcal{D} |\ \boldsymbol{y} = \tau \boldsymbol{v}_0(\boldsymbol{x})\}$$

for each fixed $\boldsymbol{x} \in \Omega$. Physically, this means that an optimal composite placed at the point $\boldsymbol{x} \in \Omega$ must have a constant field in the phase α:

$$\chi^{\varepsilon}(\boldsymbol{x}) \nabla \phi_{\varepsilon}(\boldsymbol{x}) \approx \chi^{\varepsilon}(\boldsymbol{x}) \boldsymbol{v}_0. \tag{4.11}$$

In particular, the weak limits of both sides of (4.11) are equal. Thus we obtain

$$\boldsymbol{v}_0(\boldsymbol{x}) = \frac{\beta \boldsymbol{I} - \boldsymbol{a}^*(\boldsymbol{x})}{\theta(\boldsymbol{x})(\beta - \alpha)} \nabla \phi_0(\boldsymbol{x}).$$

∎

We remark here that Lipton and Velo [21] have computed the limit in (4.2) with the restriction that the sequence ϕ_ε comes from a locally layered microstructure at every point $\boldsymbol{x} \in \Omega$. Such a microstructure has the constant field property required to attain equality in (4.3). Therefore, the limit in (4.2) is uniquely determined by $\boldsymbol{a}^*(\boldsymbol{x})$ and equals to the right hand side of (4.3). In general we do not know if equality in (4.3) holds for any pair $(\boldsymbol{a}^*(\boldsymbol{x}), \theta(\boldsymbol{x}))$. Nevertheless, we are going to show that for the minimizing sequences for (2.3) the pair $(\boldsymbol{a}^*(\boldsymbol{x}), \theta(\boldsymbol{x}))$ always achieves equality in (3.8) and therefore, the equality in (4.3) indeed holds.

Theorem **2** *Let*

$$I_0 = \inf_{\chi} I(\chi)$$

and let

$$I^* = \inf_{\theta(\boldsymbol{x})} \inf_{\boldsymbol{a}^* \in G_{\theta(\boldsymbol{x})}} \int_{\Omega} \left\{ |\boldsymbol{\eta}(\boldsymbol{x})|^2 - 2\boldsymbol{\eta}(\boldsymbol{x}) \cdot \nabla\phi(\boldsymbol{x}) - \frac{1}{\beta} f(\boldsymbol{x})\phi(\boldsymbol{x}) + \frac{|(\beta - \boldsymbol{a}^*(\boldsymbol{x}))\nabla\phi(\boldsymbol{x})|^2}{\beta(\beta - \alpha)\theta(\boldsymbol{x})} \right\} d\boldsymbol{x} \tag{4.12}$$

where $\phi(\boldsymbol{x})$ solves

$$\begin{cases} \nabla \cdot \boldsymbol{a}^*(\boldsymbol{x}) \nabla \phi = f, \\ \phi|_{\partial\Omega} = 0. \end{cases} \tag{4.13}$$

Then $I_0 = I^*$.

PROOF: Our idea is to show that the optimal value of $\boldsymbol{a}^*$ can be chosen to achieve equality in the upper trace bound (3.8) for almost every $\boldsymbol{x} \in \Omega$. For such values of $\boldsymbol{a}^*$ the equality in (4.3) in achieved and the Theorem 2 would be proved.

For the purposes of convenience let us denote

$$T_1(\phi) = |\boldsymbol{\eta}(\boldsymbol{x})|^2 - 2\boldsymbol{\eta}(\boldsymbol{x}) \cdot \nabla\phi(\boldsymbol{x}) - \frac{1}{\beta} f(\boldsymbol{x})\phi(\boldsymbol{x})$$

and

$$T_2(\theta, \boldsymbol{a}^*, \nabla\phi) = \frac{|(\beta - \boldsymbol{a}^*(\boldsymbol{x}))\nabla\phi(\boldsymbol{x})|^2}{\beta(\beta-\alpha)\theta(\boldsymbol{x})}.$$

Our idea is to include the PDE constraint (4.13) into (4.12) as a Lagrange multiplier and carry out the minimization over $\boldsymbol{a}^* \in G_{\theta(\boldsymbol{x})}$ explicitly. Let $\lambda \in W_0^{1,2}(\Omega)$ be our Lagrange multiplier. Then the problem (4.12), (4.13) can be stated as

$$I^* = \inf_{\theta(\boldsymbol{x})} \inf_{\boldsymbol{a}^*} \inf_{\phi} \sup_{\lambda} \int_{\Omega} \Big\{ T_1(\phi) + T_2(\theta, \boldsymbol{a}^*, \nabla\phi) + 2(\boldsymbol{a}^*(\boldsymbol{x})\nabla\phi, \nabla\lambda) + 2f(\boldsymbol{x})\lambda(\boldsymbol{x}) \Big\} d\boldsymbol{x}, \quad (4.14)$$

where both ϕ and λ may vary over the whole space $W_0^{1,2}(\Omega)$. Now, observe that the augmented integrand in (4.14) is a convex function of $\boldsymbol{a}^*$ and an affine (an therefore, concave) function of λ. We would like to interchange the order of sup in λ and inf in $\boldsymbol{a}^*$. In order to do this we apply a general result from convex analysis [11, Proposition 2.3]. We need to check several additional conditions beyond convex/concave structure of the functional. The set where λ varies is the whole space $W_0^{1,2}$, which is a reflexive Banach space. The set

$$ad(\theta) = \{\boldsymbol{a}^*(\boldsymbol{x}) \in L^2(\Omega; \mathrm{Sym}(\mathbb{R}^d))| \ \boldsymbol{a}^*(\boldsymbol{x}) \in G_{\theta(\boldsymbol{x})} \ \forall \boldsymbol{x} \in \Omega\},$$

where $\boldsymbol{a}^*(\boldsymbol{x})$ varies can be thought of as a closed subset of $L^2(\Omega; \mathrm{Sym}(\mathbb{R}^d))$—a reflexive Banach space. It is remarkable that for all $\theta \in [0,1]$ the set G_θ is convex. As a consequence the set $ad(\theta)$ is also convex for any measurable function $\theta(\boldsymbol{x}) : \Omega \to [0,1]$ The convexity of G_θ sets is a special feature of the G-closure of two conductors. In general G-closure sets do not have to be convex, like for the case of a polycrystal. Finally, we need that the infimum in $\boldsymbol{a}^*$ of the supremum in λ be achieved. Supremum over λ in (4.14) is a functional $\mathcal{F}(\theta, \phi, \boldsymbol{a}^*)$. It is a convex lower semi-continuous functional of $\boldsymbol{a}^*$ as a supremum of such functionals. Therefore,

$$\min_{\boldsymbol{a}^* \in ad(\theta)} \mathcal{F}(\theta, \phi, \boldsymbol{a}^*)$$

is attained. Thus all conditions of [11, Proposition 2.3] are satisfied and we can interchange the inf over $\boldsymbol{a}^*$ with the sup over λ in (4.14). We obtain

$$I^* = \inf_{\phi} \inf_{\theta(\boldsymbol{x})} \sup_{\lambda} \inf_{\boldsymbol{a}^*} \int_{\Omega} \Big(T_1(\phi) + T_2(\theta, \boldsymbol{a}^*, \nabla\phi) + 2f(\boldsymbol{x})\lambda(\boldsymbol{x}) + 2(\boldsymbol{a}^*(\boldsymbol{x})\nabla\phi, \nabla\lambda \Big) d\boldsymbol{x}. \quad (4.15)$$

Hence, we need to carry out a finite-dimensional optimization problem:

$$K^* = \min_{\boldsymbol{a}^* \in G_\theta} \left\{ 2(\boldsymbol{a}^*\boldsymbol{v}, \boldsymbol{u}) + \frac{|\beta\boldsymbol{v} - \boldsymbol{a}^*\boldsymbol{v}|^2}{\beta(\beta-\alpha)\theta} \right\} \quad (4.16)$$

Here $\boldsymbol{v}$ stands for $\nabla\phi$ and $\boldsymbol{u}$ stands for $\nabla\lambda$. We may rewrite (4.16) as

$$K^* = \frac{(\tilde{K}^*)^2}{\beta(\beta-\alpha)\theta} + 2\beta\boldsymbol{u}\cdot\boldsymbol{v} - \beta(\beta-\alpha)\theta|\boldsymbol{u}|^2,$$

where

$$\tilde{K}^* = \min_{\boldsymbol{a}^*\in G_\theta} |\boldsymbol{a}^*\boldsymbol{v} - \boldsymbol{p}| \tag{4.17}$$

and

$$\boldsymbol{p} = \beta\boldsymbol{v} - \beta(\beta-\alpha)\theta\boldsymbol{u}.$$

This is a geometric problem of finding the distance between a given vector $\boldsymbol{p}$ and the set $G_\theta\boldsymbol{v}$. This problem requires some effort. We present the solution in two steps. First, for given $\boldsymbol{a}^* \in G_\theta$ we compute the set

$$\mathcal{O}(\boldsymbol{a}^*, \boldsymbol{v}) = \{\boldsymbol{R}\boldsymbol{a}^*\boldsymbol{R}^t\boldsymbol{v}\,;\boldsymbol{R}\in SO(d)\}. \tag{4.18}$$

Lemma **2** *Suppose $d = 2$. Let $a_1 \geq a_2$ be the eigenvalues of $\boldsymbol{a}^*$. Let $c = (a_1 + a_2)/2$ and $r = (a_1 - a_2)/2$. Then $\mathcal{O}(\boldsymbol{a}^*, \boldsymbol{v})$ is a circle centered at $c\boldsymbol{v}$ with radius $r|\boldsymbol{v}|$.*

Suppose $d = 3$. Let $a_1 \geq a_2 \geq a_3$ be the eigenvalues of $\boldsymbol{a}^*$. Define

$$c_1 = (a_2 + a_3)/2, \qquad c_2 = (a_1 + a_3)/2, \qquad c_3 = (a_1 + a_2)/2$$

and

$$r_1 = (a_2 - a_3)/2, \qquad r_2 = (a_1 - a_3)/2, \qquad r_3 = (a_1 - a_2)/2.$$

Then $\mathcal{O}(\boldsymbol{a}^*, \boldsymbol{v})$ consists of all points of the closed ball centered at $c_2\boldsymbol{v}$ of radius $r_2|\boldsymbol{v}|$, that are exterior to the open balls with centers at $c_1\boldsymbol{v}$ and $c_3\boldsymbol{v}$ with radii $r_1|\boldsymbol{v}|$ and $r_3|\boldsymbol{v}|$ respectively.

It is very curious to note that the "shadow" that the three dimensional group $SO(3)$ casts onto $\mathbb{R}^3$ has cavities.

Proof: We prove a more interesting three dimensional case. The reader may prove the easier two dimensional case either by the method we use for $d = 3$ or by explicitly computing the vector $\boldsymbol{R}_\gamma\boldsymbol{a}^*\boldsymbol{R}_\gamma^t\boldsymbol{v}$ where $\boldsymbol{R}_\gamma$ is a rotation matrix through the angle γ.

Let $\mathcal{S}$ be the set described in the Lemma for $d = 3$. And let $\mathcal{O}$ denote the set $\mathcal{O}(\boldsymbol{a}^*, \boldsymbol{v})$. We first show that $\mathcal{O} \subset \mathcal{S}$. Then we will show that $\partial\mathcal{O} \subset \partial\mathcal{S}$. We then conclude that $\mathcal{O} = \mathcal{S}$.

To prove that $\mathcal{O} \subset \mathcal{S}$ we observe that for $\boldsymbol{a} \in \mathcal{O}$ the matrix $\boldsymbol{a} - c_3\boldsymbol{I}$ has eigenvalues $\lambda_1 = r_3$, $\lambda_2 = -r_3$ and $\lambda_3 = a_3 - (a_1 + a_2)/2$. We easily check that $|\lambda_3| \geq \lambda_1 = r_3$ and therefore, for any vector $\boldsymbol{v} \in \mathbb{R}^3$ we have

$$|\boldsymbol{a}\boldsymbol{v} - c_3\boldsymbol{v}| \geq r_3|\boldsymbol{v}|$$

Similarly, we obtain that

$$|\boldsymbol{a}\boldsymbol{v} - c_1\boldsymbol{v}| \geq r_1|\boldsymbol{v}|$$

and that

$$|\boldsymbol{av} - c_2\boldsymbol{v}| \leq r_2|\boldsymbol{v}|.$$

We deduce that $\mathcal{O} \subset \mathcal{S}$.

Now we need to get some information about the boundary of the set $\mathcal{O}$. Let $\boldsymbol{a}^*$ and $\boldsymbol{v}$ be fixed and $\boldsymbol{f} : SO(3) \to \mathbb{R}^3$ be given by

$$\boldsymbol{f}(\boldsymbol{R}) = \boldsymbol{R}\boldsymbol{a}^*\boldsymbol{R}^t\boldsymbol{v}.$$

The function $\boldsymbol{f}$ is a C^∞ map between smooth manifolds. We also know that $\mathcal{O} = \boldsymbol{f}(SO(3))$ is a compact and path-connected subset of $\mathcal{S}$. Let $\boldsymbol{R}_0$ be fixed and $\boldsymbol{f}_* : T_{\boldsymbol{R}_0}SO(3) \to \mathbb{R}^3$ is a linear map between the three dimensional tangent spaces of the two manifolds. If the map $\boldsymbol{f}_*$ is non-singular then the inverse function theorem guarantees that $\boldsymbol{f}(\boldsymbol{R}_0)$ is an interior point of $\mathcal{O}$. Thus

$$\partial\mathcal{O} \subset \{\boldsymbol{f}(\boldsymbol{R}_0)|\ \boldsymbol{f}_* \text{ is singular}\}.$$

We will show now that $\partial\mathcal{S}$ coincides with the set of critical values of the map $\boldsymbol{f}$. Let us find an explicit expression for $\boldsymbol{f}_*$ at $\boldsymbol{R}_0$. For that purpose we think of $SO(3)$ as a submanifold in the space of 3 by 3 matrices. The tangent space $T_{\boldsymbol{R}_0}SO(3)$ can then be identified with the Lie algebra $so(3) = T_{\boldsymbol{I}}(SO(3))$ via the relation $T_{\boldsymbol{R}_0} = so(3)\boldsymbol{R}_0$. The Lie algebra $so(3)$ is the space of all skew-symmetric matrices $\boldsymbol{L}$ that can be identified with $\mathbb{R}^3$ by the cross product map $\pi : \mathbb{R}^3 \to so(3)$ as follows

$$\pi(\boldsymbol{x})\boldsymbol{y} = \boldsymbol{x} \times \boldsymbol{y}. \tag{4.19}$$

Now, we easily compute $\boldsymbol{f}_*$:

$$\boldsymbol{f}_*\boldsymbol{x} = [\pi(\boldsymbol{x}), \boldsymbol{a}]\boldsymbol{v}, \tag{4.20}$$

where $[,]$ denotes the commutator of two matrices and $\boldsymbol{a} = \boldsymbol{R}_0\boldsymbol{a}^*\boldsymbol{R}_0^t$. We can rewrite (4.20) using (4.19) as follows:

$$\boldsymbol{f}_*\boldsymbol{x} = (\boldsymbol{a}\pi(\boldsymbol{v}) - \pi(\boldsymbol{av}))\boldsymbol{x}.$$

We see now that the value $\boldsymbol{av}$ is critical if and only if $\det(\boldsymbol{a}\pi(\boldsymbol{v}) - \pi(\boldsymbol{av})) = 0$. In order to evaluate this determinant we use the following decomposition formula valid in three dimensions only:

$$\det(\boldsymbol{A} + \boldsymbol{B}) = \det(\boldsymbol{A}) + (\operatorname{adj}(\boldsymbol{A}), \boldsymbol{B}) + (\boldsymbol{A}, \operatorname{adj}(\boldsymbol{B})) + \det(\boldsymbol{B}),$$

where $(\boldsymbol{A}, \boldsymbol{B}) = \operatorname{Tr}(\boldsymbol{AB}^t)$ and $\operatorname{adj}(\boldsymbol{A})$ can be defined as the gradient of the determinant:

$$(\operatorname{adj}(\boldsymbol{A}), \boldsymbol{B}) = \lim_{\epsilon \to 0}(\det(\boldsymbol{A} + \epsilon\boldsymbol{B}) - \det(\boldsymbol{A}))/\epsilon.$$

(There are probably as many different definitions of adj as those of det.) The adjoint adj enjoys many pleasant properties some of which we are going to use. The reader can easily check that

$$\operatorname{adj}(\boldsymbol{AB}) = \operatorname{adj}(\boldsymbol{A})\operatorname{adj}(\boldsymbol{B})$$

and that

$$\mathrm{adj}(\pi(\boldsymbol{x})) = \boldsymbol{x} \otimes \boldsymbol{x}.$$

With these properties and the observation that $\det(\pi(\boldsymbol{x})) = 0$ for any $\boldsymbol{x} \in \mathbb{R}^3$ we easily compute

$$\det(\boldsymbol{a}\pi(\boldsymbol{v}) - \pi(\boldsymbol{a}\boldsymbol{v})) = -\mathrm{adj}(\boldsymbol{a})\boldsymbol{v} \cdot \pi(\boldsymbol{a}\boldsymbol{v})\boldsymbol{v} + \boldsymbol{a}\pi(\boldsymbol{v})\boldsymbol{a}\boldsymbol{v} \cdot \boldsymbol{a}\boldsymbol{v}.$$

In order to simplify the above expression further we need to recall the triple product $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ of vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$. We define

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{x} \cdot (\boldsymbol{y} \times \boldsymbol{z}) = -(\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{z}) = -(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}).$$

In terms of the triple product we compute

$$\det(\boldsymbol{f}_*) = (\mathrm{adj}(\boldsymbol{a})\boldsymbol{v} + \boldsymbol{a}^2\boldsymbol{v}, \boldsymbol{v}, \boldsymbol{a}\boldsymbol{v}).$$

The triple product $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is zero if and only if vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ lie in the same plane. Therefore, $\boldsymbol{a}\boldsymbol{v}$ is a critical value for $\boldsymbol{f}$ if and only if vectors $\boldsymbol{v}$, $\boldsymbol{a}\boldsymbol{v}$ and $\mathrm{adj}(\boldsymbol{a})\boldsymbol{v} + \boldsymbol{a}^2\boldsymbol{v}$ lie in the same plane. Now we show that for any symmetric matrix $\boldsymbol{a}$ and vector $\boldsymbol{v}$ the vectors $\boldsymbol{v}$, $\boldsymbol{a}\boldsymbol{v}$ and $\mathrm{adj}(\boldsymbol{a})\boldsymbol{v} - \boldsymbol{a}^2\boldsymbol{v}$ always lie in the same plane. Cayley-Hamilton theorem says that any matrix annihilates its characteristic polynomial. For any invertible symmetric matrix $\boldsymbol{a}$ we can multiply the Cayley-Hamilton identity by $\boldsymbol{a}^{-1}$ and obtain

$$\boldsymbol{a}^2 - t\boldsymbol{a} + j\boldsymbol{I} - \mathrm{adj}(\boldsymbol{a}) = 0, \tag{4.21}$$

where $t = \mathrm{Tr}\,\boldsymbol{a}$ and $2j = t^2 - \mathrm{Tr}\,\boldsymbol{a}^2$ and where we used the formula $\boldsymbol{a}^{-1} = \mathrm{adj}(\boldsymbol{a})/\det(\boldsymbol{a})$. Now applying (4.21) to a vector $\boldsymbol{v}$ we obtain that the vectors $\boldsymbol{v}$, $\boldsymbol{a}\boldsymbol{v}$ and $\mathrm{adj}(\boldsymbol{a})\boldsymbol{v} - \boldsymbol{a}^2\boldsymbol{v}$ always lie in the same plane for an invertible symmetric matrix $\boldsymbol{a}$. The same will be true for any symmetric matrix $\boldsymbol{a}$ because invertible matrices are dense in the set of all matrices and because being in the same plane is a closed condition. Thus vectors $\boldsymbol{v}$, $\boldsymbol{a}\boldsymbol{v}$ and $\mathrm{adj}(\boldsymbol{a})\boldsymbol{v}+\boldsymbol{a}^2\boldsymbol{v}$ lie in the same plane if and only if vectors $\boldsymbol{v}$, $\boldsymbol{a}\boldsymbol{v}$ and $\boldsymbol{a}^2\boldsymbol{v}$ do. In other words $\boldsymbol{a}\boldsymbol{v}$ is a critical value for $\boldsymbol{f}$ whenever $\boldsymbol{v}$ lies in a proper invariant subspace for $\boldsymbol{a}$. Suppose $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ are eigenvectors of $\boldsymbol{a}$. And suppose $\boldsymbol{v} = v_1\boldsymbol{u}_1 + v_2\boldsymbol{u}_2$. Then

$$\boldsymbol{a}\boldsymbol{v} - c_3\boldsymbol{v} = \frac{1}{2}(a_1 - a_2)(v_1\boldsymbol{u}_2 - v_2\boldsymbol{u}_2).$$

Thus $|\boldsymbol{a}\boldsymbol{v} - c_3\boldsymbol{v}| = r_3|\boldsymbol{v}|$ and $\boldsymbol{a}\boldsymbol{v} \in \partial\mathcal{S}$. Similarly, the same conclusion is reached if $\boldsymbol{v}$ is in the span of $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ or $\boldsymbol{u}_1$ and $\boldsymbol{u}_3$.

Now we show that every vector $\boldsymbol{q} \in \partial\mathcal{S}$ is a critical value for $\boldsymbol{f}$. Suppose $r_3 \neq 0$. (If $r_3 = 0$ then this part of the boundary of $\mathcal{S}$ disappears and we need to consider only spheres with non-vanishing radius.) Let $\boldsymbol{q}$ be on the boundary of the ball centered at $c_3\boldsymbol{v}$ with radius $r_3|\boldsymbol{v}|$. Let $\boldsymbol{p} = (\boldsymbol{q} - c_3\boldsymbol{v})/r_3$. Then $|\boldsymbol{p}| = |\boldsymbol{v}|$. Define $\boldsymbol{u}_1 = (\boldsymbol{p} + \boldsymbol{v})/2$ and $\boldsymbol{u}_2 = (\boldsymbol{v} - \boldsymbol{p})/2$. Observe that vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are orthogonal. Therefore, we can find a rotation $\boldsymbol{R}$ such that $\boldsymbol{a} = \boldsymbol{R}\boldsymbol{a}^*\boldsymbol{R}^t$ has eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. We have $\boldsymbol{v} = \boldsymbol{u}_1 + \boldsymbol{u}_2$ lies in the proper invariant subspace of $\boldsymbol{a}$ and $\boldsymbol{a}\boldsymbol{v} - c_3\boldsymbol{v} = r_3(\boldsymbol{u}_1 - \boldsymbol{u}_2) = r_3\boldsymbol{p}$, from which it follows that

$\boldsymbol{av} = \boldsymbol{q}$. Thus $\boldsymbol{q}$ is a critical value for $\boldsymbol{f}$. So, the set of all critical values for $\boldsymbol{f}$ coincides with $\partial\mathcal{S}$. It might happen that either $\boldsymbol{u}_1$ or $\boldsymbol{u}_2$ is zero. The argument above works in this case too, except the zero vector should not be *called* an eigenvector of $\boldsymbol{a}$.

Now we show that $\mathcal{S} = \mathcal{O}$. If the matrix $\boldsymbol{a}^*$ is a multiple of the identity then both $\mathcal{S} = \mathcal{O} = \{\boldsymbol{a}^*\boldsymbol{v}\}$. If the matrix $\boldsymbol{a}^*$ has a double eigenvalue then $\mathcal{S} = \partial\mathcal{S}$ and $\partial\mathcal{S} \subset \mathcal{O}$ because $\partial\mathcal{S}$ is the set of critical values of $\boldsymbol{f}$ as shown in the previous paragraph. Thus $\mathcal{S} = \mathcal{O}$. Therefore we may suppose that $\boldsymbol{a}^*$ has three distinct eigenvalues resulting in the set $\mathcal{O}$ having a non-empty interior, since it contains the images of all regular points of $\boldsymbol{f}$. Suppose there is a point $\boldsymbol{x}_0 \in \mathcal{S}$ such that $\boldsymbol{x}_0 \notin \mathcal{O}$. Pick another point $\boldsymbol{x}_1 \in \mathcal{O}$ such that $\boldsymbol{x}_1 \notin \partial\mathcal{S}$. For example we can choose any point $\boldsymbol{x}_1$ in the interior of $\mathcal{O}$. Now connect the points $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ by a continuous path $\gamma(t)$ such that $\gamma(0) = \boldsymbol{x}_0$ and $\gamma(1) = \boldsymbol{x}_1$ and such that $\gamma(t)$ lies in the interior of $\mathcal{S}$ for all $t > 0$. Now let

$$t_0 = \inf\{t \in [0,1] : \gamma(t) \in \mathcal{O}\}.$$

Observe that $\gamma(t_0) \in \mathcal{O}$ because $\mathcal{O}$ is closed. Therefore, $t_0 > 0$, since $\gamma(0) = \boldsymbol{x}_0 \notin \mathcal{O}$. Also $\gamma(t_0) \in \partial\mathcal{O} \subset \partial\mathcal{S}$ because $\gamma(t_0)$ can not be an interior point of $\mathcal{O}$ by definition of t_0 and because a path connected set $\mathcal{O}$ does not have isolated points. But $\gamma(t)$ does not intersect the boundary of $\mathcal{S}$ for all $t > 0$. Contradiction. Thus $\mathcal{O} = \mathcal{S}$. ∎

Using Lemma 2 we can compute K^* in (4.16) and show that the minimum can be achieved at matrices $\boldsymbol{a}^*$ satisfying (3.8) with equality.

LEMMA **3**

$$\tilde{K}^* = \left(\left|\frac{m+h}{2}\boldsymbol{v} - \boldsymbol{p}\right| - \frac{m-h}{2}|\boldsymbol{v}|\right)_+, \tag{4.22}$$

where $(a)_+ = \max(a, 0)$. *Moreover, the minimum in (4.17) is attained at* $\boldsymbol{a}^*$ *satisfying (3.8) with equality.*

PROOF: The set $G_\theta\boldsymbol{v} = \{\boldsymbol{a}^*\boldsymbol{v} |\, \boldsymbol{a}^* \in G_\theta\}$ can be represented as the union

$$G_\theta\boldsymbol{v} = \bigcup_{\boldsymbol{a}^*\in G_\theta} \mathcal{O}(\boldsymbol{a}^*, \boldsymbol{v})$$

of sets (4.18). Each of these sets is the circle $S(c\boldsymbol{v}, r|\boldsymbol{v}|)$ for $d = 2$ or contains the sphere $S(c_2\boldsymbol{v}, r_2|\boldsymbol{v}|)$, if $d = 3$. The centers and the radii of the circle or the sphere are described in Lemma 2. Our first observation is that the sphere corresponding to $\boldsymbol{a}^*$ with eigenvalues (m, m, h) contains spheres corresponding to any $\boldsymbol{a}^*$ with eigenvalues $m \geq a_1 \geq a_2 \geq a_3 \geq h$. Let $\boldsymbol{c}_0 = (m+h)\boldsymbol{v}/2$, $\boldsymbol{c}' = (a_1 + a_3)\boldsymbol{v}/2$, $r_0 = (m-h)|\boldsymbol{v}|/2$ and $r' = (a_1 - a_2)|\boldsymbol{v}|/2$. Then we easily check that

$$|\boldsymbol{c}_0 - \boldsymbol{c}'| = \left|\frac{m-a_1}{2} - \frac{a_3-h}{2}\right| |\boldsymbol{v}| \leq \left(\frac{m-a_1}{2} + \frac{a_3-h}{2}\right) |\boldsymbol{v}| = r_0 - r'.$$

Geometrically, this means that the sphere $S(\boldsymbol{c}_0, r_0)$ contains the sphere $S(\boldsymbol{c}', r')$. Now, we show that every point in the interior of the largest sphere $S(\boldsymbol{c}_0, r_0)$ belongs to an outer sphere of some $\boldsymbol{a}^* \in G_\theta$ achieving equality in (3.8). Suppose $\boldsymbol{p}$ is in the interior of the sphere $S(\boldsymbol{c}_0, r_0)$. Take an isotropic $\boldsymbol{a}^* = a_0 \boldsymbol{I} \in G_\theta$ that achieves equality in (3.8). The set $\mathcal{O}(\boldsymbol{a}^*, \boldsymbol{v})$ corresponding to this matrix degenerates into a single point $a_0 \boldsymbol{v}$. If $\boldsymbol{p} = a_0 \boldsymbol{v}$ then our assertion is true. If not then $\boldsymbol{p}$ is certainly outside of this degenerate sphere. Now let us connect the point $\boldsymbol{a}^* = a_0 \boldsymbol{I}$ with point $\boldsymbol{a}_0^*$ with eigenvalues (m, m, h) by a smooth curve entirely lying on the lower trace bound. As we move along the curve the outer sphere will continuously deform from the single point $a_0 \boldsymbol{v}$ to the largest sphere $S(\boldsymbol{c}_0, r_0)$. Along the way there will be a point on that curve such that the corresponding outer sphere will pass through the point $\boldsymbol{p}$. Thus, $\tilde{K}^*$ is zero, if $\boldsymbol{p}$ is inside the sphere $S(\boldsymbol{c}_0, r_0)$ and $\tilde{K}^* = |\boldsymbol{p} - \boldsymbol{c}_0| - r_0$ if $\boldsymbol{p}$ is outside of $S(\boldsymbol{c}_0, r_0)$. Lemma 3 is proved. ∎

In Lemma 3 we showed that the optimal value of $\boldsymbol{a}^*$ can always be taken such that equality in Lemma 1 is achieved. This finishes the proof of the Theorem 2. ∎

5 The optimal design is a rank-1 laminate

In the case of no resource constraints of $\theta(\boldsymbol{x})$ our analysis shows more. If $\boldsymbol{p}$ is outside of the sphere $S(\boldsymbol{c}_0, r_0)$ then the optimal $\boldsymbol{a}$ must have eigenvalues (m, m, h) corresponding to the rank-1 laminate. We continue our evaluation of I^* and show that this is indeed always the case.

Theorem 3 *The optimal composite must be a rank-1 laminate at every point in the domain.*

A strong suggestion that a theorem like this might be true was made in the paper of Tartar [32], where he showed that for $\boldsymbol{\eta}(\boldsymbol{x})$ in a dense G_δ subset of L^2 the minimizing sequence for the original optimal design problem converges strongly. On that dense G_δ set Tartar's theorem says more than we prove. Namely, it says that the optimal microstructure is a rank-1 laminate at every point in the material, where $\nabla\phi$ is *parallel to the layers*. At present we still can not prove that last part.

Proof: In order to prove the theorem we are going to show that the minimum in (4.17) is attained at $\boldsymbol{a}^*$ with eigenvalues (m, m, h). According to our analysis this is the case if and only if

$$\left|\frac{m+h}{2}\boldsymbol{v} - \boldsymbol{p}\right| \geq \frac{m-h}{2}|\boldsymbol{v}|. \tag{5.1}$$

Now we return to the computation of I^* in (4.15). Using Lemma 3 we compute

$$I^* = \inf_{\phi} \inf_{\theta(\boldsymbol{x})} \sup_{\lambda} \int_\Omega (L(\phi, \lambda) + Q(\theta, \nabla\phi, \nabla\lambda)) d\boldsymbol{x}, \tag{5.2}$$

where

$$Q(\theta, \nabla\phi, \nabla\lambda) = \frac{(N(\theta, \nabla\phi, \nabla\lambda))_+^2}{\beta(\beta-\alpha)\theta(\boldsymbol{x})} - \beta(\beta-\alpha)\theta(\boldsymbol{x})|\nabla\lambda(\boldsymbol{x})|^2,$$

$$N(\theta, \nabla\phi, \nabla\lambda) = \left|(\frac{m+h}{2} - \beta)\nabla\phi(\boldsymbol{x}) + \beta(\beta-\alpha)\theta(\boldsymbol{x})\nabla\lambda(\boldsymbol{x})\right| - \frac{m-h}{2}|\nabla\phi(\boldsymbol{x})| \qquad (5.3)$$

and

$$L(\phi, \lambda) = |\boldsymbol{\eta}(\boldsymbol{x})|^2 - 2\boldsymbol{\eta}(\boldsymbol{x}) \cdot \nabla\phi(\boldsymbol{x}) - \frac{1}{\beta} f(\boldsymbol{x})\phi(\boldsymbol{x}) + 2f(\boldsymbol{x})\lambda(\boldsymbol{x}) + 2\beta\nabla\lambda(\boldsymbol{x}) \cdot \nabla\phi(\boldsymbol{x}).$$

We will show that the optimal value for the field $\theta(\boldsymbol{x})$ in (5.2) is such that $N(\theta, \nabla\phi, \nabla\lambda) > 0$, meaning that the minimum in (4.17) is attained at $\boldsymbol{a}^*$ with eigenvalues (m, m, h), corresponding to the rank-1 laminate structure. We achieve this goal by justifying the interchange of sup in λ and inf in θ and then showing that for any value of the fields $\nabla\phi$ and $\nabla\lambda$ the minimum of $Q(\theta, \nabla\phi, \nabla\lambda)$ is attained at $\theta^*(\boldsymbol{x})$ that makes $N(\theta^*, \nabla\phi, \nabla\lambda)$ strictly positive.

Observe that the functional $\int(L+Q)$ is concave in λ because it was obtained as a minimum of linear functions in λ. Surprisingly, the function Q turns out to be convex in θ.

Lemma 4 *The function $F(\theta) = Q(\theta, \boldsymbol{v}, \boldsymbol{u})$ is convex in θ on $[0,1]$ for any choice of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.*

We remark that convexity of Q in θ does not follow from any general principle because the sets G_θ are dependent on θ and the set of pairs

$$\{(\boldsymbol{a}^*, \theta) |\, 0 \le \theta \le 1,\ \boldsymbol{a}^* \in G_\theta\}$$

is *not* convex.

Proof: Substituting the values of m and h into the formula for F we get

$$F(\theta) = \frac{\theta}{\beta(\beta-\alpha)} \left\{ \left(\left| \frac{\beta \boldsymbol{v}'}{\beta\theta + \alpha(1-\theta)} - (\boldsymbol{u}' - \boldsymbol{v}') \right| - \frac{(\beta-\alpha)(1-\theta)}{\beta\theta + \alpha(1-\theta)} |\boldsymbol{v}'| \right)_+^2 - |\boldsymbol{u}'|^2 \right\}, \qquad (5.4)$$

where $\boldsymbol{u}' = \beta(\beta-\alpha)\boldsymbol{u}$ and $\boldsymbol{v}' = (\beta-\alpha)\boldsymbol{v}/2$.

In order to prove convexity we will first eliminate as many parameters from F as possible. It will be convenient to denote

$$\boldsymbol{e} = \boldsymbol{v}'/|\boldsymbol{v}'|, \quad \boldsymbol{w} = \boldsymbol{u}'/|\boldsymbol{v}'| - \boldsymbol{e}, \quad \nu = \alpha/\beta, \quad \xi = \theta + \nu(1-\theta). \qquad (5.5)$$

We can also discard the part of F that is linear in θ (or ξ). Thus we need to study convexity of the function

$$H(\xi) = \frac{\xi - \nu}{\xi^2}(g(\xi))_+^2,$$

where

$$g(\xi) = |\xi\boldsymbol{w} - \boldsymbol{e}| + \xi - 1.$$

If θ is between 0 an 1 then ξ is between ν and 1. We need to prove that $H(\xi)$ is convex in ξ. In the special case when $\boldsymbol{w} = \boldsymbol{e}$ we find that H is identically zero and therefore convex.

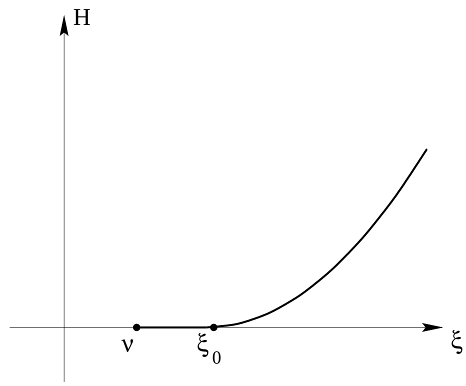

Figure 1: The graph of $H(\xi)$ near ξ_0.

From now on we assume that $\boldsymbol{w} \neq \boldsymbol{e}$. First we show that there exists $\xi_0 \in [\nu, 1)$ such that $H(\xi) = 0$ for all $\xi \in [\nu, \xi_0]$ and $H(\xi) > 0$ for all $\xi \in (\xi_0, 1]$. If $|\boldsymbol{w}| \leq 1$ then

$$g(\xi) \geq 1 - \xi|\boldsymbol{w}| + \xi - 1 = \xi(1 - |\boldsymbol{w}|) \geq 0.$$

If $g(\xi) = 0$ even for one particular value of $\xi \in [\nu, 1]$ then $\boldsymbol{w} = \boldsymbol{e}$, the case we ruled out. Thus $g(\xi) > 0$ for all ξ. In that case $\xi_0 = \nu$. If $|\boldsymbol{w}| > 1$ then $g(\xi) \leq 0$ is equivalent to

$$\xi \leq \frac{2(w_1 - 1)}{|\boldsymbol{w}|^2 - 1},$$

where $w_1 = \boldsymbol{w} \cdot \boldsymbol{e}$. If

$$\frac{2(w_1 - 1)}{|\boldsymbol{w}|^2 - 1} \geq \nu$$

then

$$\xi_0 = \frac{2(w_1 - 1)}{|\boldsymbol{w}|^2 - 1} < 1,$$

and $H(\xi)$ behaves as claimed. If

$$\frac{2(w_1 - 1)}{|\boldsymbol{w}|^2 - 1} < \nu$$

then $g(\xi) > 0$ for all ξ and $\xi_0 = \nu$. If $\xi_0 \in (\nu, 1)$ then $g(\xi_0) = 0$ and $g(\xi)$ is smooth around ξ_0. Therefore, the function $H(\xi)$ behaves on the interval $[\nu, \xi_0 + \epsilon]$ as shown in Figure 1 Thus, in order to prove that $H(\xi)$ is convex it is sufficient to establish convexity of $H(\xi)$ only on the interval $(\xi_0, 1)$. For $\xi \in (\xi_0, 1)$ we have $g(\xi) > 0$ and

$$H(\xi) = \frac{\xi - \nu}{\xi^2} g(\xi)^2. \tag{5.6}$$

We differentiate (5.6) twice keeping g unevaluated and then factor out $2(\xi - \nu)/\xi^2$:

$$H''(\xi) = 2\frac{\xi - \nu}{\xi^2}(gg'' + \psi(\xi)),$$

where

$$\psi(\xi) = g'^2 + \frac{2(2\nu - \xi)}{\xi(\xi - \nu)} gg' + \frac{\xi - 3\nu}{\xi^2(\xi - \nu)} g^2.$$

Obviously, $g(\xi)$ is convex and therefore $gg'' \geq 0$ on $(\xi_0, 1)$. We are going to show that $\psi(\xi)$ is also non-negative. Observe that $\psi(\xi)$ factors:

$$\psi(\xi) = (g' - \frac{1}{\xi} g)(g' + \frac{3\nu - \xi}{\xi(\xi - \nu)} g).$$

Observe also that

$$(g' + \frac{3\nu - \xi}{\xi(\xi - \nu)} g) - (g' - \frac{1}{\xi} g) = \frac{2\nu}{\xi(\xi - \nu)} g > 0 \tag{5.7}$$

Finally, an easy computation involving explicit differentiation of g in ξ yields:

$$g' - \frac{1}{\xi} g = \frac{\xi w_1 - 1 + |\xi \boldsymbol{w} - \boldsymbol{e}|}{\xi |\xi \boldsymbol{w} - \boldsymbol{e}|}.$$

But

$$|\xi w_1 - 1| = |(\xi \boldsymbol{w} - \boldsymbol{e}) \cdot \boldsymbol{e}| \leq |\xi \boldsymbol{w} - \boldsymbol{e}|.$$

Thus

$$g' - \frac{1}{\xi} g \geq 0$$

and in view of (5.7), $\psi(\xi) \geq 0$. Therefore, $H''(\xi) \geq 0$ and the lemma is proved. ∎

Now we can finish the proof of Theorem 3. Lemma 4 allows us to justify the interchange of sup in λ and inf in θ in (5.2). The supremum over λ in (5.2) is a convex lower semi-continuous functional in $\theta(\boldsymbol{x})$ that varies in a closed, convex and bounded subset of $L^2(\Omega)$. Therefore, infimum over θ in (5.2) is attained. Thus, we can apply the min-max theorem [11, Proposition 2.3] once again and write that

$$I^* = \inf_{\phi} \sup_{\lambda} \int_{\Omega} (L(\phi, \lambda) + Q'(\nabla\phi, \nabla\lambda)) d\boldsymbol{x}, \tag{5.8}$$

where

$$Q'(\boldsymbol{v}, \boldsymbol{u}) = \min_{\theta \in [0,1]} Q(\theta, \boldsymbol{v}, \boldsymbol{u}). \tag{5.9}$$

In order to find some information about where the minimum in Q' is achieved we study the function $F(\theta)$ given by (5.4). This function differs from $H(\xi)$ by a constant multiple and by an extra linear term $-\theta|\boldsymbol{u}'|^2/\beta(\beta - \alpha)$. If ξ_0 from the proof of Lemma 4 is equal to ν and $g(\xi_0) > 0$ then no matter where the minimum of F is achieved we will always have $N > 0$, where N is defined by (5.3). In the remaining cases there exists $\theta_0 \in [0, 1]$ corresponding to ξ_0 via (5.5) such that $F(\theta)$ is linear and decreasing on $[0, \xi_0]$ and its derivative at $\theta = \theta_0$ is $-|\boldsymbol{u}'|^2/\beta(\beta - \alpha) < 0$. Therefore, the minimum in (5.9) is always achieved at $\theta^* > \theta_0$. But for all $\theta > \theta_0$ we have that $N > 0$. Thus if $\theta^*(\boldsymbol{x})$ is the optimal volume fraction for variational problem (5.2) then $N(\theta^*, \nabla\phi, \nabla\lambda) > 0$. Going back to our analysis in Lemma 3 we conclude that the minimum in (4.12) is always achieved at $\boldsymbol{a}^*$ corresponding to a rank-1 laminate. ∎

We remark that the proof of Theorem 3 holds for the optimal design problems with resource constraints up to (5.9). If in (5.9) we replace minimization of $Q(\theta, \boldsymbol{v}, \boldsymbol{u})$ with

$$\inf_{\langle\theta\rangle=\gamma} \int_{\Omega} Q(\theta(\boldsymbol{x}), \nabla\phi(\boldsymbol{x}), \nabla\lambda(\boldsymbol{x}))d\boldsymbol{x},$$

then we cannot conclude that the minimizer $\theta^*(\boldsymbol{x}) > \theta_0$ for all $\boldsymbol{x} \in \Omega$. Nevertheless, Tartar [32] showed that for $\boldsymbol{\eta}$ in a dense G_δ subset of L^2 the optimal microstructure at every point in a composite must be laminar for a problem with or without resource constraints.

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