Exact relations for effective tensors of polycrystals. II: Applications to elasticity and piezoelectricity.

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Abstract

An important necessary condition for an exact relation for effective moduli of polycrystals to hold is stability of that relation under lamination. This requirement is so restrictive that it is possible (if not always feasible) to find all such relations explicitly. In order to do that one needs to combine the results developed in Part I of this paper and the representation theory of the rotation groups $SO(2)$ and $SO(3)$. More precisely, one needs to know all rotationally invariant subspaces of the space of material moduli. This paper presents an algorithm for getting all such subspaces. We illustrate the workings of the algorithm on the examples of 3-D elasticity, where we get all the exact relations and 2-D and 3-D piezoelectricity, where we get some (possibly all) of them.

1 Introduction.

Suppose that effective moduli of a polycrystal satisfy a set of equations, provided that the moduli of individual crystals satisfy them. The embedded local manifold described by such equations will be called an exact relation for effective moduli. The simplest example is the Keller-Dykhne-Mendelson [8, 14, 19] family of exact relations

$$\mathbb{M}_t = \{\sigma \in \mathrm{Sym}(\mathbf{R}^2) : \det \sigma = t\}.$$

There is a very extensive literature on the subject of microstructure independent relations. However, in all cases, the results are tied to a particular physical context. In [9], henceforth referred to as Part I, we have treated the whole range of physical problems in the unifying framework developed by Milton [22]. One disadvantage (which may be partially overcome, as we will show elsewhere) is that we obtain a complete characterization of exact relations for *laminates* of polycrystals only. Another minor disadvantage of our purely algebraic approach is the loss of physical interpretations of the exact relations we obtain.

The existing literature shows two major trends: the *constant field* approach that dates back to Hill [11] and Cribb [7] (see also [4] and references therein), and the *decoupling* approach that was

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initiated by Straley [23] (see also [5, 20, 21]). In its essence, the constant field approach is a "trivial" observation that the set

$$\Pi(u, v) = \{C : \; Cu = v\},$$

where u and v are given constant fields, is stable under homogenization. In its full generality, this observation was made by Lurie and Cherkaev [16][1].

The decoupling approach is based on transforming a coupled problem to a decoupled problem using similarity transformations for quadratic forms. The general treatment of the method for any number of coupled electrostatics problems is done by Milgrom and Shtrikman [20, 21]. We will show elsewhere that the former approach corresponds to rotationally invariant *ideals* while the latter may be placed in a broader class of exact relations corresponding to some special rotationally invariant *subalgebras* in the space of all linear transformations of physical fields.

The goal of this paper is to combine our general theory developed in Part I with the representation theory of the rotation groups $SO(2)$ and $SO(3)$ and to describe an algorithm that works in all cases where Part I applies. We illustrate the procedure by working out exact relations for 3-D elasticity and 2-D and 3-D piezoelectricity. Before we begin, let us briefly remind the reader of the abstract framework of Part I.

For each material property, there is an associated vector space $\mathcal{T}$ of field tensors. The physical fields in the body (or plate) have values in $\mathcal{T}$ at every point x. For example, in conductivity the electric and current fields are vectors, i.e. they take their values in $\mathcal{T} = \mathbf{R}^d$, $d = 2, 3$. In elasticity, the stress and strain are symmetric matrices, so in this case, $\mathcal{T}$ =Sym($\mathbf{R}^d$), is the linear space of symmetric operators on $\mathbf{R}^d$. It is important that $\mathcal{T}$ is a tensor space, as there is a special transformation rule that $\mathcal{T}$ has to obey under the change of coordinates. Let $R \in SO(d)$ be a rotation. If we rotate the body by R, then the components of the physical fields must be changed by a linear transformation. Let $E(x)$ be such a field in the original body. Then the field in the rotated body will be given by

$$E'(Rx) = \Theta(R)E(x), \tag{1.1}$$

where $\Theta(R)$ is an orthogonal operator on the vector space $\mathcal{T}$. The function $\Theta(R)$ depends on the rotation R in a very particular manner. For example, in the case of conductivity where $\mathcal{T} = \mathbf{R}^d$, we have $\Theta(R)\tau = R\tau$ for $\tau \in \mathbf{R}^d$. In the case of elasticity, $\mathcal{T} = \mathrm{Sym}(\mathbf{R}^d)$, and for any $\tau \in \mathrm{Sym}(\mathbf{R}^d)$, we have $\Theta(R)\tau = R\tau R^T$. The key here is that $\Theta(R)$ is a *representation* of $SO(d)$ on $\mathcal{T}$, i.e. $\Theta(R)$ preserves multiplication and inversion of the orthogonal matrices in $SO(d)$. Thus, $\Theta(R)$ is a function mapping the group $SO(d)$ into $SO(\mathcal{T})$ such that for any $R, Q \in SO(d)$,

$$\Theta(QR) = \Theta(Q)\Theta(R)$$

and

$$\Theta(Q^{-1}) = \Theta(Q)^{-1}.$$

Such a function is called a group homomorphism.

We now define the representation $g(R)$ of $SO(d)$ on the space of material moduli

$$\mathcal{Y} = \mathrm{Sym}(\mathcal{T}) \tag{1.2}$$

by the rule that for every $C \in \mathcal{Y}$,

$$g(R)C = \Theta(R)C[\Theta(R)]^T. \tag{1.3}$$

Obviously, any polycrystalline G-closure $\mathcal{G}$ must have rotational invariance: $C \in \mathcal{G}$ implies $g(R)C \in \mathcal{G}$. Our focus here is on the exact relations for effective moduli of polycrystals. An exact relation may be represented as a surface in the Euclidean space $\mathcal{Y}$ containing a G-closed set with

[1] See English translation in [17].

non-empty interior in the induced topology of the surface. In Part I, we have investigated surfaces containing sets closed under *lamination*, where it appeared more convenient to work with the new variables

$$S = (I - \frac{C}{c_0})^{-1}, \tag{1.4}$$

where c_0 is an arbitrary scalar constant chosen such that the inverse in (1.4) exists.

The main result of Part I was that the tangent space $\mathcal{L}$ at the isotropic tensor S_0 (in the variables S) is rotationally invariant, i.e. $L \in \mathcal{L}$ implies $g(R)L \in \mathcal{L}$ for every $R \in SO(d)$ and satisfies equations (3.13) of Part I. The knowledge of this subspace is sufficient to describe the exact relation manifold in the neighborhood of S_0. Let us remind the reader how to obtain explicit equations for such a manifold corresponding to $\mathcal{L}$.

Parametric equations

Let $\{L_1, \ldots, L_k\}$ be a basis for $\mathcal{L}$. Then

$$C(\lambda_1, \ldots, \lambda_k) = c_0 \left(I - \left\{ W_0^{-1} \left[W_0^{-1} + \sum_{i=1}^{k} \lambda_i L_i \right]^{-1} W_0^{-1} + \Gamma \right\}^{-1} \right), \tag{1.5}$$

where W_0 and Γ are constant tensors in $\mathcal{Y}$ defined in Part I. They depend on the physical setting, but not on the choice of the subspace $\mathcal{L}$.

Implicit equations

Let $\{K_1, \ldots, K_{N-k}\}$ be a basis for $\mathcal{L}^{\perp}$. Then

$$\left(\left[(I - \frac{C}{c_0})^{-1} - \Gamma \right]^{-1}, W_0^{-1} K_r W_0^{-1} \right) = \left(W_0^{-1}, K_r \right), \quad 1 \le r \le N - k. \tag{1.6}$$

It is easy to see that (1.5) and (1.6) are equivalent. Thus, instead of giving explicit equations for an exact relation, it is enough to specify only the subspace $\mathcal{L}$ that generates an exact relation via (1.5) or (1.6). It is in this form that we provide the answers in section 5. Below we briefly summarize the results.

In 3-D elasticity, we obtain 3 exact relations. One of them is the well-known result of Hill [12, 13] that a mixture of isotropic materials with constant shear modulus is isotropic and has the same shear modulus. The second one is a less known result of Hill [11], see also [2]. It says that the set of tensors having the 3x3 identity matrix as an eigenvector with fixed eigenvalue is a set stable under lamination (and homogenization). The third exact relation says that a rank-one tensor plus a null-Lagrangian is a conserved property under homogenization. This is a 3-D version of an analogous statement for 2-D [10].

In the context of piezoelectricity we have only searched for exact relations within a class of rotationally invariant subspaces that are “well-behaved” with respect to a “natural” decomposition of $\mathcal{Y}$. We conjecture that these are in fact all exact relations. In principle, it is possible to use our methods (and a Maple program) to find *all* exact relations. However, It is our hope that a better understanding of the algebraic properties of exact relations will lead to a less tedious and more illuminating way of checking our conjecture, as well as describing all exact relations in more complicated physical situations.

In 2-D piezoelectricity we have found 12 genuinely piezoelectric, essential exact relations[2]. The question of finding exact relations for piezoelectricity has been previously addressed by Y. Benveniste

[2]An exact relation is called essential if it cannot be obtained as an intersection of other exact relations.

[3, 4, 5]. However, he was looking for a different type of exact relations. He considered a 3-D crystal and a 2-D microgeometry, obtaining relations between in-plane and out-of-plane moduli. He used a "constant field" method that in our context would have yielded two truly piezoelectric exact relations (for both 2 and 3-D piezoelectricity). We identify them in the list of exact relations in section 5.

In 3-D piezoelectricity we have found only 4 genuinely piezoelectric, essential exact relations, two of which are of "constant field" type. We are not aware of any previous results in this setting.

2 Invariant subspaces of representations

We will first recall some general notions from the representation theory of compact groups (see for example [6]). Let G be a compact group, and let V be a continuous, finite dimensional real representation of G, i.e. V is a real, finite dimensional vector space together with a continuous homomorphism $\rho : G \to GL(V)$. (We will also call V a real G-module[3].) The representation V admits a G- invariant inner product, so that after an appropriate choice of basis for V, the image $\rho(G)$ will consist of orthogonal matrices. The representation V may be decomposed into an orthogonal direct sum of irreducible representations $V = \bigoplus_{i=1}^{k} V_i$. This decomposition is not in general unique. However, for any irreducible representation W, let $V(W)$ be the sum of all irreducible subrepresentations of V isomorphic to W. Equivalently, $V(W)$ is the sum of all the V_i's isomorphic to W. We then have the canonical decomposition of V into W-isotypic components $V = \bigoplus_{W \in \mathrm{Irr}(G,\mathbf{R})} V(W)$, where $\mathrm{Irr}(G, \mathbf{R})$ denotes the set of isomorphism classes of irreducible real representations of G.

We will also need to consider complex representations of G. Analogous statements hold in this situation, with obvious modifications. Note that in this case, explicit matrices for the representation will in general be unitary instead of orthogonal.

We now turn to the classification of G-invariant subspaces of a representation. Let V be a representation of G over the field K, where K is the real or complex numbers, and let L be a subrepresentation of V. Again, L is the sum of its isotypic components with $L(W) \subseteq V(W)$. Thus, it suffices to determine all possible subrepresentations of $V(W) \cong W^{\oplus m}$ where $m = m_W$ is the multiplicity of W in V. Note that any such subrepresentation $L(W)$ is isomorphic to $W^{\oplus k}$ for some k, $0 \le k \le m$.

We will start by determining the irreducible subrepresentations of $W^{\oplus m}$. These are given by the images of nonzero elements of $\mathrm{Hom}_G(W, W^{\oplus m})$, the set of G-maps $W \to W^{\oplus m}$. Note that $d \in \mathrm{Hom}_G(W, W^{\oplus m})$ can be viewed as a row vector $(d_1, \ldots, d_m)$, where the d_i's are G-equivariant endomorphisms of W, i.e. elements of $D \stackrel{\mathrm{def}}{=} \mathrm{End}_G(W) = \mathrm{Hom}_G(W, W)$. We now use the fact that the endomorphism ring D has a particularly simple form. By Schur's lemma, the endomorphism ring D is a finite-dimensional division algebra over K. If $K = \mathbf{C}$, this implies that $D = \mathbf{C}$ and all endomorphisms of W are scalar multiples of the identity. However, if $K = \mathbf{R}$, D can be $\mathbf{R}$, $\mathbf{C}$, or the quaternions $\mathbf{H}$, partitioning the $\mathrm{Irr}(G, \mathbf{R})$ into representations of real, complex, and quaternionic type. Let $\{w_1, w_2, \ldots, w_n\}$ be a basis for W. Then any nonzero $d \in D^m$ gives rise to the irreducible subrepresentation of $W^{\oplus m}$ with basis $\{v_j = (d_1(w_j), \ldots, d_m(w_j)) \mid 1 \le j \le n\}$; moreover, another nonzero $d' \in D^m$ will give the same irreducible submodule if and only if d' and d are parallel. Thus, the irreducible representations of L isomorphic to W are in bijective correspondence with the set of one-dimensional subspaces of D^m, i.e. with $m-1$- dimensional projective space $\mathbf{P}^{m-1}(D)$.

More generally, there is a bijective correspondence between the subrepresentations of $W^{\oplus m}$ isomorphic to $W^{\oplus k}$ and the set of k-planes (through the origin) in D^m. Indeed, let $A \subseteq D^m$ be a k-dimensional subspace of D^m, say with basis $\{d^1, \ldots, d^k\}$. Then the corresponding submodule L_A has basis

$$\{v_j^l = (d_1^l(w_j), \ldots, d_m^l(w_j)) \mid 1 \le l \le k, 1 \le j \le n\}. \tag{2.7}$$

[3] There is no relation to the notion of G-closure.

As an immediate corollary, note that V has a finite number of invariant subspaces if and only if the multiplicity of each irreducible representation in V is at most one or equivalently, if each nontrivial isotypic component is irreducible.

In order to complete the description of the irreducible subspaces of V, it is still necessary to understand the action of the division algebra D on W in the cases where it is not scalar multiplication, i.e. when $K = \mathbf{R}$ and $D \neq \mathbf{R}$. It will be convenient to recharacterize our partition of $\mathrm{Irr}(G, \mathbf{R})$. Recall that for a complex representation M, the conjugate representation $\overline{M}$ is obtained by taking the complex conjugate of the homomorphism $G \to GL(M)$. It is irreducible if and only if M is. Let W be an irreducible real representation, and consider the complex representation $W_{\mathbf{C}} = W \otimes \mathbf{C}$. Note that $W_{\mathbf{C}}$ is always isomorphic to its conjugate. It is not necessarily irreducible. In fact, it can be shown that there are precisely three possibilities for the irreducible decomposition of $W_{\mathbf{C}}$ and that these correspond to real G-modules of real, complex, and quaternionic type respectively. The complexified representation may be irreducible, or it may split into two irreducibles: $W_{\mathbf{C}} \cong U \oplus \overline{U}$, where U is irreducible, but not self-conjugate, or $W_{\mathbf{C}} \cong U \oplus U$, where U is irreducible and self-conjugate. In particular, there is a bijective correspondence between irreducible real representations of real type and irreducible complex representations whose matrix coefficients can be taken to be real numbers.

Now suppose that W is of complex type, so that there exists an irreducible complex representation U, not isomorphic to its conjugate, such that $W_{\mathbf{C}} = U \oplus \overline{U}$. Choose a basis $\{u_1, \ldots, u_d\}$ of U and corresponding basis $\{\bar{u}_1, \ldots, \bar{u}_d\}$ of $\overline{U}$. This gives rise to a basis $\{(u_j + \bar{u}_j)/2, (\bar{u}_j - u_j)/2i \mid 1 \leq j \leq d\}$ for the real vector space W. The endomorphism of W determined by $\lambda \in \mathbf{C}$ is then induced by $u_j \mapsto \lambda u_j$ and $\bar{u}_j \mapsto \bar{\lambda} \bar{u}_j$. A similar analysis for real representations of quaternionic type is possible, though more complicated. However, since neither $SO(2)$ nor $SO(3)$ have any such representations, we will not supply the details.

The usefulness of the above theory is limited in practice by the necessity of finding an explicit decomposition of the given representation into irreducible components, which is a difficult problem in general. However, in our applications, we will be concerned with representations on spaces of symmetric matrices, spaces whose structure gives rise to certain simplifications. Let V be a real inner product space and make it into a real orthogonal G-module via the homomorphism $G \xrightarrow{\rho} O(V)$. We assume that the decomposition of V into irreducibles is known. Let $W = \mathrm{Sym}(V)$ be the set of symmetric linear operators on V, and define an action of G on W via $g \cdot A = \rho(g) A \rho(g)^t$. Note that W comes equipped with the G-invariant inner product $\langle A, B \rangle = \mathbf{Tr}(AB)$, where $\mathbf{Tr}(A)$ denotes the trace of A.

It will be convenient to identify W with $\mathrm{Sym}^2(V)$, the second symmetric power of V (or more concretely, homogeneous polynomials of degree two in elements of V). The group G acts naturally on $V \otimes V$ via $g \cdot (v_1 \otimes v_2) = \rho(g) v_1 \otimes \rho(g) v_2$, and this action descends to $\mathrm{Sym}^2(V)$. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of V, and let E_{ij} be the $n \times n$ matrix with 1 in the i, j entry and zeros elsewhere. It is then easy to check that the linear isomorphism $\mathrm{Sym}^2(V) \to W$ given by $e_i e_j \mapsto (E_{ij} + E_{ji})/2$ is G- equivariant, so that $W \cong \mathrm{Sym}^2(V)$.

Now let $V = \bigoplus_{i=1}^d V_i^{\oplus m_i}$ where the V_i's are pairwise nonisomorphic irreducible representations. Using the well-known fact that $\mathrm{Sym}^2(A \oplus B) \cong \mathrm{Sym}^2(A) \oplus (A \otimes B) \oplus \mathrm{Sym}^2(B)$ and induction, we obtain the G-module decomposition

$$\begin{aligned} W = \mathrm{Sym}^2(V) &\cong \bigoplus_{i=1}^d (\mathrm{Sym}^2(V_i)^{\oplus m_i} \oplus (V_i \otimes V_i)^{\oplus \frac{(m_i - 1)m_i}{2}}) \oplus \bigoplus_{i<j} (V_i \otimes V_j)^{\oplus m_i m_j} \\ &\cong \bigoplus_{i=1}^d (\mathrm{Sym}^2(V_i)^{\oplus \frac{m_i(m_i+1)}{2}} \oplus \Lambda^2(V_i)^{\oplus \frac{(m_i-1)m_i}{2}}) \oplus \bigoplus_{i<j} (V_i \otimes V_j)^{\oplus m_i m_j}, \end{aligned} \tag{2.8}$$

where $\Lambda^2(V_i)$ is the second exterior power of V_i with the natural G-action. The last isomorphism follows because $A \otimes A \cong \mathrm{Sym}^2(A) \oplus \Lambda^2(A)$. Viewing W as symmetric $\dim V \times \dim V$ matrices, this means that W decomposes into symmetric blocks along the diagonal and pairs of blocks which are the reflections of each other across the diagonal. Thus, if for any pair of irreducible representations V' and V'', we can decompose $\mathrm{Sym}^2(V')$, $\Lambda^2(V')$, and $V' \otimes V''$ into irreducible components, we can decompose W into irreducibles and thereby find the invariant subspaces of W.

3 Representation theory of $SO(2)$.

We now specialize the above analysis to the group $G = SO(2)$, i.e. the circle group $S^1 \subset \mathbf{C}$. First, we will recall the classification of irreducible real representations of S^1, and then we will show how to obtain an explicit decomposition of an arbitrary real representation into irreducibles.

The irreducible complex representations of S^1 are one- dimensional, since S^1 is abelian. In fact, they are given by one-dimensional vector spaces P_k, $k \in \mathbf{Z}$, with S^1-action $z \cdot p = z^k p$ for $z \in S^1$, $p \in P_k$. This follows from the fact that a continuous homomorphism $S^1 \cong \mathbf{R}/\mathbf{Z} \to \mathbf{C}^* = \mathbf{C} - \{0\}$ is a homomorphism $\mathbf{R} \to \mathbf{C}^*$ which is trivial on $\mathbf{Z}$. Note that $\overline{P_k} = P_{-k}$ and that the matrix coefficient for the self- conjugate representation P_0 is $1 \in \mathbf{R}$. Thus, $\mathrm{Irr}(S^1, \mathbf{R})$ consists of the trivial one-dimensional module M_0, which is of real type, and for each positive integer k, a two-dimensional representation M_k of complex type such that $(M_k)_\mathbf{C} = P_k \oplus P_{-k}$. Explicitly, for $k > 0$, let f_k and f_{-k} span P_k and P_{-k} respectively, and let M_k be the real vector space spanned by $e_{k,1} = (f_k + f_{-k})/2$ and $e_{k,2} = (f_{-k} - f_k)/2i$. If we set

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2), \tag{3.9}$$

it is immediate that S^1 acts on M_k with respect to the basis $\{e_{k,1}, e_{k,2}\}$ via the homomorphism $S^1 \to SO(2)$, $e^{i\theta} \mapsto R(k\theta)$.

Let V be a real representation of $SO(2)$. The M_0 isotypic component $V(0) \stackrel{\text{def}}{=} V(M_0)$ is just the set of fixed points in V, say of dimension k_0. To find $V(k) \stackrel{\text{def}}{=} V(M_k)$ for $k > 0$, consider the complexification $V_\mathbf{C}$. Choose a basis $\{f_j^k \mid 1 \le j \le j_k\}$ for the P_k isotypic component of $V_\mathbf{C}$, i.e. diagonalize the action of S^1 and let $\{f_j^k\}$ be a basis for the eigenspace of the eigenvalue $e^{ik\theta}$. (Such a simultaneous eigenvalue is called a weight of P_k, and the isotypic component of $V_\mathbf{C}(P_k)$ is the corresponding weight space.) The $-k$ weight space has basis $\{\bar{f}_j^k\}$. It follows easily that $V(k)$ is isomorphic to j_k copies of M_k via $V(k) = \bigoplus_{j=1}^{j_k}(\mathbf{R}e_{1,j}^k \oplus \mathbf{R}e_{2,j}^k)$, where $e_{1,j}^k = (f_j^k + \bar{f}_j^k)/2$ and $e_{2,j}^k = (\bar{f}_j^k - f_j^k)/2i$. Thus, $V \cong \bigoplus_{k=0}^{\infty} M_k^{\oplus j_k}$; of course, $j_k = 0$ for all but finitely many k. The invariant subspaces of V are just direct sums of invariant subspaces of the $V(k)$'s, which are given by (2.7). In particular, the irreducible subrepresentations isomorphic to M_0 are the one-dimensional subspaces of $V(0)$ while the irreducible subspaces isomorphic to M_k for $k > 0$ are

$$W_\mathbf{z} = \mathbf{R}(\sum_{j=1}^{j_k}(x_j e_{1,j}^k + y_j e_{2,j}^k)) \oplus \mathbf{R}(\sum_{j=1}^{j_k}(x_j e_{2,j}^k - y_j e_{1,j}^k)), \tag{3.10}$$

where $\mathbf{z} = (\mathbf{x}+i\mathbf{y})$ a nonzero element of $\mathbf{C}^{j_k}$ determined up to homothety, i.e an element of $P^{j_k-1}(\mathbf{C})$.

In order to use (2.8) to find the irreducible components of $W = \mathrm{Sym}^2(V)$, it remains to decompose $\mathrm{Sym}^2(M_k)$, $\Lambda^2(M_k)$, and $M_k \otimes M_l$ for $k < l$. This can be done easily using the above methods, noting that complexification is compatible with tensor products and symmetric and alternating powers. It is immediate that $\mathrm{Sym}^2(M_0) \cong M_0$, $\Lambda^2(M_0) = \{0\}$, and $M_0 \otimes M_l \cong M_l$ for $l > 0$, so we assume that $k \ge 1$. Also, $\Lambda^2(M_k) \cong M_0$, since it is a one-dimensional real representation.

Let $\{e_{k,1}, e_{k,2}\}$ be the standard basis for M_k, so that $f_k = e_1^k - ie_2^k$ and $f_{-k} = e_1^k + ie_2^k$ respectively span the k and $-k$ weight spaces of $(M_k)_\mathbf{C}$. Then $(\mathrm{Sym}^2(M_k))_\mathbf{C}$ has one-dimensional weight spaces with weights $2k$, $-2k$, and 0 spanned by f_k^2, f_{-k}^2, and $f_k f_{-k} = e_{k,1}^2 + e_{k,2}^2$. Thus, $\mathrm{Sym}^2(M_k) \cong$

$M_{2k} \oplus M_0$; the M_0 component has basis $\{e_{k,1}^2 + e_{k,2}^2\}$ and the M_{2k} component has standard basis $\{e_{k,1}^2 - e_{k,2}^2, 2e_{k,1}e_{k,2}\}$. Finally, $(M_k \otimes M_l)_{\mathbf{C}}$ has one-dimensional weight spaces with weights $l+k$, $-(l+k)$, $l-k$, and $-(l-k)$ spanned by $f_k \otimes f_l$, $f_{-k} \otimes f_{-l}$, $f_{-k} \otimes f_l$, and $f_k \otimes f_{-l}$ respectively. This gives rise to the decomposition

$$\begin{aligned} M_k \otimes M_l &= (\mathbf{R}(e_{k,1} \otimes e_{l,1} - e_{k,2} \otimes e_{l,2}) \oplus \mathbf{R}(e_{k,1} \otimes e_{l,2} + e_{k,2} \otimes e_{l,1}) \\ &\quad \oplus (\mathbf{R}(e_{k,1} \otimes e_{l,1} + e_{k,2} \otimes e_{l,2}) \oplus \mathbf{R}(e_{k,1} \otimes e_{l,2} - e_{k,2} \otimes e_{l,1}) \\ &\cong M_{l+k} \oplus M_{l-k}. \end{aligned} \tag{3.11}$$

We summarize this as a theorem:

THEOREM 1 *For $k \geq 1$, let M_k be the real irreducible representation of $SO(2)$ whose complexification has weights $\pm k$, and let M_0 be the trivial representation. Then if $l > k \geq 1$, $Sym^2(M_k) \cong M_{2k} \oplus M_0$, $\Lambda^2(M_k) \cong M_0$, and $M_k \otimes M_l \cong M_{l+k} \oplus M_{l-k}$. Moreover, $Sym^2(M_0) \cong M_0$, $\Lambda^2(M_0) = \{0\}$, and $M_0 \otimes M_l \cong M_l$.*

4 Representation theory of $SO(3)$.

We now turn to the group $G = SO(3)$. Again, we begin by recalling the classification of the irreducible representations of $SO(3)$.

The simplest description of these irreducible representations makes use of the fact that the group $SU(2)$ is the universal cover of $SO(3)$ via a double cover $SU(2) \xrightarrow{\pi} SO(3)$. To define the map π, we view $SU(2)$ as the unit quaternions using the isomorphism

$$\rho: \quad a_1 + a_2 i + b_1 j + b_2 k \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \tag{4.12}$$

where $a = a_1 + ia_2$, $b = b_1 + ib_2$, and $|a|^2 + |b|^2 = 1$. The unit quaternions act orthogonally on the pure quaternions $\mathbf{R}i \oplus \mathbf{R}j \oplus \mathbf{R}k \cong \mathbf{R}^3$ by conjugation, so we have $\rho(\pi(g)x) = g\rho(x)g^{-1}$ for $g \in SU(2)$ and $x \in \mathbf{R}^3$. The kernel of π is $\{\pm I\}$. Since any representation of $SO(3)$ lifts to the universal cover $SU(2)$ while a representation of $SU(2)$ factors through $SO(3)$ if and only if it is trivial on the kernel of π, we see that an irreducible representation of $SO(3)$ is just an irreducible representation of $SU(2)$ on which $-I$ acts as the identity.

Let V_1 be the standard representation of $SU(2)$ given by $\mathbf{C}^2$ with the natural action. The complex irreducible representations of $SU(2)$ are just $V_k = \mathrm{Sym}^k(V_1)$ for $k \geq 0$, where $\mathrm{Sym}^k(V_1)$ denotes the k-th symmetric power of V_1. Note that V_k has dimension $k+1$. In order to decompose representations into irreducible components, we will need a more intrinsic characterization of the V_k's. Consider the subgroup

$$T = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \cong S^1 \tag{4.13}$$

in $SU(2)$. This subgroup is maximal with respect to the property of being connected and abelian and is called a maximal torus of $SU(2)$. Any representation of $SU(2)$ is a fortiori a T-module and hence decomposes into weight spaces. It is easy to see that if e_1 and e_2 are the standard basis vectors for V_1, then $e_1^{k-j}e_2^j$ is a weight vector of weight $k-2j$ for $0 \leq j \leq k$. Thus, V_k is the unique irreducible representation with highest weight k, and moreover, V_k splits into one-dimensional weight spaces with weights $k, k-2, \ldots, -k+2, -k$.

Since $-I$ acts as scalar multiplication by $(-1)^k$ on V_k, the complex irreducible representations of $SO(3)$ are just V_{2k} for $k \geq 0$. In particular, the weight spaces for odd weights of representations of $SO(3)$ are zero. This should come as no surprise, considering that the image of $\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)$ under the homomorphism π is

$$R(2\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix}. \tag{4.14}$$

The matrices for V_{2k} can be chosen to have real coefficients, so the real irreducible representations of $SO(3)$ are all of real type and are given by modules W_k such that $(W_k)_{\mathbf{C}} \cong V_{2k}$.

An elementary (though perhaps not the most straightforward) approach to verifying that the V_{2k} are complexifications of real representations involves an entirely different description of the representations in terms of harmonic polynomials. Note that $SO(3)$ acts on the space P_k of real homogeneous polynomials in three variables of degree k via $g \cdot p(x) = p(xg)$ for $g \in SO(3)$, $x \in \mathbf{R}^3$; moreover, this action commutes with the Laplace operator $\Delta : P_k \to P_{k-2}$. Thus, the kernel W_k of this linear map, i.e. the subspace of harmonic polynomials of degree k, is a subrepresentation. It is easy to check that $\dim(W_k) = 2k+1$ and that $(x_2 + ix_3)^k \in (W_k)_{\mathbf{C}}$ is a weight vector of weight $2k$. This implies that $(W_k)_{\mathbf{C}} \cong V_{2k}$ as desired. (The irreducible representations V_{2k+1} of $SU(2)$ correspond to real irreducible representations of quaternionic type whose complexifications are isomorphic to $V_{2k+1} \oplus V_{2k+1}$.)

We can use the torus T to obtain an inductive procedure for decomposing a real $SO(3)$-module V into irreducible components. Let d_j be the dimension of the $2j$-weight space of $V_{\mathbf{C}}$ for each $j \in \mathbf{Z}$. Since $\dim V < \infty$ (and assuming that $V \neq \{0\}$), $V_{\mathbf{C}}$ has a highest weight $2l \geq 0$, so that $d_l \geq 1$ and $d_j = 0$ for $j > l$. Then $V \cong W_l \oplus V'$, where the orthogonal complement V' is a real representation of strictly lower dimension with $d'_j = d_j - 1$ for $|j| \leq l$ and $d'_j = 0$ otherwise. Now find a highest weight for $V'_{\mathbf{C}}$ and continue until the orthogonal complement is $\{0\}$.

In order to find bases for the irreducible components of V, we will pass to the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$. It is elementary that a representation of a Lie group G gives rise to a representation of its Lie algebra $\mathfrak{g}$ by differentiating the group action; moreover, one recovers the original representation of G by exponentiating the Lie algebra action. If G is simply connected, we obtain a canonical bijective correspondence between the representations of G and $\mathfrak{g}$ in this way. Thus, a decomposition of V (and thereby $V_{\mathbf{C}}$) into irreducible $SO(3)$-modules is also a decomposition into irreducible $\mathfrak{so}(3)$-modules. In fact, since a representation of $\mathfrak{so}(3)$ on a complex vector space extends uniquely by linearity to a representation of the complexified Lie algebra $\mathfrak{so}(3)_{\mathbf{C}} = \mathfrak{so}(3) \otimes \mathbf{C}$, decompositions of $V_{\mathbf{C}}$ into $SO(3)$ and $\mathfrak{so}(3)_{\mathbf{C}}$ irreducibles are the same.

This perspective is useful because $\mathfrak{so}(3)_{\mathbf{C}}$ is isomorphic to the 2×2 traceless matrices $\mathfrak{sl}(2, \mathbf{C})$, and there is a simple, direct method for writing down explicit bases of irreducible $\mathfrak{sl}(2, \mathbf{C})$- invariant subspaces. We take

$$\tilde{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tilde{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \tilde{Y} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{4.15}$$

as basis vectors for $\mathfrak{sl}(2, \mathbf{C})$. Note that $[\tilde{H}, \tilde{X}] = 2\tilde{X}$, $[\tilde{H}, \tilde{Y}] = -2\tilde{Y}$, and $[\tilde{X}, \tilde{Y}] = -\tilde{H}$, with the other commutators vanishing. The Lie algebra $\mathfrak{so}(3)$ of 3×3 skew-symmetric matrices has as basis the infinitesimal rotations around the coordinate axes:

$$Z_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad Z_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.16}$$

Setting $H = -2iZ_1$, $X = Z_2 - iZ_3$, and $Y = Z_2 + iZ_3$, we see that H,X, and Y satisfy the commutation relations for $\mathfrak{sl}(2, \mathbf{C})$ above and hence define an isomorphism $\mathfrak{so}(3)_{\mathbf{C}} \cong \mathfrak{sl}(2, \mathbf{C})$.

Let w be a k-weight vector in a complex representation of $SU(2)$ (and thereby $\mathfrak{su}(2)_{\mathbf{C}} \cong \mathfrak{so}(3)_{\mathbf{C}} \cong \mathfrak{sl}(2, \mathbf{C})$). Then differentiating the equation $\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right) \cdot w = e^{ik\theta} w$ shows that $i\tilde{H} = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) \in \mathfrak{su}(2) \subset \mathfrak{sl}(2, \mathbf{C})$ acts on w via $i\tilde{H}w = ikw$. Accordingly, $\tilde{H}w = kw$ and so the k-weight space of W is just the eigenspace of $\tilde{H}$ with eigenvalue k. Furthermore, the commutation relations show that $\tilde{X}w$ has weight $k+2$ and $\tilde{Y}w$ has weight $k-2$; as a result, X and Y are called raising and lowering operators respectively. In fact, a complex irreducible $SU(2)$-module of highest weight l has a basis of weight vectors $\{w, \tilde{Y}w, \ldots, \tilde{Y}^l w\}$ where w is a highest weight vector or equivalently, a nonzero vector annihilated by $\tilde{X}$. (These statements are easily verified by direct calculation on $\mathrm{Sym}^k(V_1)$, noting that $A \cdot (v_1 \cdots v_k) = \sum_{j=1}^k v_1 \cdots A v_j \cdots v_k$ for $A \in \mathfrak{sl}(2, \mathbf{C})$, $v_1, \ldots, v_k \in V_1$.)

We now return to the representation V of $SO(3)$. The action of Z_j, $1 \leq j \leq 3$, on $v \in V$ is determined by differentiating $\exp(tZ_j) \cdot v$ at $t = 0$, and linearity gives the action of H, X, and Y. Since $V_{\mathbf{C}}$ has highest weight $2l$, an irreducible subrepresentation isomorphic to V_{2l} is obtained by choosing a highest weight vector w. Although a basis can be given as above, a more convenient choice makes use of the fact that this irreducible submodule is a complexification of a real irreducible submodule. This implies that it is closed under complex conjugation, so $Y^{2l-j}w$ and $\overline{Y^j w}$ are both $2j-2l$ weight vectors in the submodule for $0 \leq j \leq l$. Thus, we have a basis of weight vectors $\{Y^j w, \overline{Y^j w} \mid 0 \leq j \leq l-1\} \cup \{Y^l w\}$. A real subrepresentation isomorphic to W_l has basis vectors $(Y^j w + \overline{Y^j w})/2$ and $(\overline{Y^j w} - Y^j w)/2i$ for $0 \leq j \leq l-1$ as well as the real or the imaginary part of Y^l, the choice being arbitrary if they are both nonzero. We continue recursively, starting with a weight vector $w' \in V_{\mathbf{C}}$ of highest weight subject to being linearly independent to the weight vectors already used to split off irreducible summands.

In fact, using the Lie algebra approach, we can avoid the recursion altogether. The kernel of X is H-invariant, and so breaks up into weight spaces $\mathrm{Ker}(X)_{2k}$. Let $\{w_{k,1}, \ldots, w_{k,b_k}\}$ be a basis for $\mathrm{Ker}(X)_{2k}$. The $w_{k,m}$'s will be highest weight vectors for irreducible subrepresentations isomorphic to V_{2k} into which $V_{\mathbf{C}}$ splits. Accordingly, we have $V \cong \bigoplus_{k=0}^l W_k^{\oplus b_k}$, where bases for the summands are obtained by the above procedure. Note that the number of summands is just the dimension of $\mathrm{Ker}(X)$.

The invariant subspaces of V are now given by (2.7), where we are in the simple case that the endomorphism rings of the irreducible representations are all just the real numbers. Alternatively, an $SO(3)$-submodule is obtained by choosing subspaces $L_k \subseteq \mathrm{Ker}(X)_{2k}$. The subrepresentation will have $\dim(L_k)$ components isomorphic to W_k. In particular, an irreducible invariant subspace isomorphic to W_k is generated by a highest weight vector $a_1 w_{k,1} + \cdots + a_{b_k} w_{k,b_k}$ for $(a_1, \ldots, a_{b_k}) \in \mathbf{R}^{b_k}$.

It only remains to find the decompositions of $\mathrm{Sym}^2(W_k)$, $\Lambda^2(W_k)$, and $W_k \otimes W_l$ for $k < l$. Let $\{w_j^k \mid -k \leq j \leq k\}$ be a basis of weight vectors for $(W_k)_{\mathbf{C}} = V_{2k}$ such that w_j^k has weight $2j$. It is easy to see that for $0 \leq j \leq 2k$, the $4k-2j$ weight space of $\mathrm{Sym}^2(V_{2k})$ has basis $\{w_{k-p}^k w_{k-j+p}^k \mid 0 \leq p \leq [j/2]\}$, so its dimension is $[j/2]+1$. This implies that $\mathrm{Sym}^2(V_{2k}) \cong \bigoplus_{j=0}^k V_{4j}$, so that $\mathrm{Sym}^2(W_k) \cong \bigoplus_{j=0}^k W_{2j}$. A similar argument shows that $\Lambda^2(W_k) \cong \bigoplus_{j=1}^k W_{2j-1}$. Finally, the $2(l+k-j)$ weight space of $V_{2k} \otimes V_{2l}$ has dimension $j+1$ with basis $\{w_{k-p}^k \otimes w_{l-(j-p)}^l \mid 0 \leq p \leq j\}$ for $0 \leq j \leq 2k$ and dimension $2k+1$ with basis $\{w_{k-p}^k \otimes w_{l-(j-p)}^l \mid 0 \leq p \leq 2k\}$ for $2k \leq j \leq l+k$. Thus, we get the Clebsch-Gordon formula $W_k \otimes W_l \cong \bigoplus_{j=0}^{2k} W_{l+k-j}$. This proves the following theorem:

THEOREM **2** *Let W_k be the real irreducible representation of $SO(3)$ of dimension $2k+1$. We then have Sym$^2(W_k) \cong \bigoplus_{j=0}^k W_{2j}$ and $\Lambda^2(W_k) \cong \bigoplus_{j=1}^k W_{2j-1}$. For $l > k$, we also have $W_k \otimes W_l \cong \bigoplus_{j=0}^{2k} W_{l+k-j}$.*

5 Applications

5.1 $SO(2)$: 2-D elasticity and piezoelectricity.

Throughout this section, we will use the notation that vectors denoted by f's and v's with subscripts will be weight vectors for the complexification of a representation with weight given by the index. Vectors of weight zero will be taken to be in the underlying real representation. If v_k is a weight vector with positive weight k, v_{-k} will be the complex conjugate of v_k. Corresponding real bases will be denoted by the preceding letter. Thus, the basis for the real representation whose complexification is spanned by v_k and v_{-k}, where $k > 0$ (resp. v_0) is $\{u_k, u_{-k}\}$ with $u_k = (v_k + v_{-k})/2$, and $u_{-k} = (v_{-k} - v_k)/2i$ (resp. $\{u_0\}$ where $u_0 = v_0$). In order to maintain consistency with this convention, we will denote the standard basis for $\mathbf{R}^2$ by $\{e_1, e_{-1}\}$. The irreducible representations of weight k will be denoted by a capital Latin letter with a subscript k. Different letters will denote representations coming from different physics. Our (arbitrary) convention is to use N for conductivity, K for elasticity and L for piezoelectric cross moduli. The prime will distinguish between isomorphic representations appearing within the same physics. We are still using the letter M to denote the abstract isomorphism class for a given representation. We recall that for two vectors $\{a, b\} \subset \mathcal{T} \otimes \mathbf{C}$, we identify symmetric tensors and degree two homogeneous polynomials via

$$ab = \frac{1}{2}(a \otimes b + b \otimes a). \tag{5.17}$$

$\mathbf{Sym}^2(\mathbf{Sym}^2(\mathbf{R}^2))$—two dimensional elasticity.

The standard representation of $SO(2)$ on $\mathbf{R}^2$ is isomorphic to M_1, with weight vectors $f_1 = e_1 - ie_{-1}$ and $f_{-1} = e_1 + ie_{-1}$. Consequently, $\mathrm{Sym}^2(\mathbf{R}^2) = N_2 \oplus N_0$. Then N_0 is spanned by $\hat{u} = f_1 f_{-1} = e_1^2 + e_{-1}^2$. Weight vectors for N_2 are given by $v_2 = f_1^2$ and $v_{-2} = f_{-1}^2$, with corresponding real basis $u_2 = e_1^2 - e_{-1}^2$ and $u_{-2} = 2e_1 e_{-1}$. This is the same basis we have used in Part I (see also [1, 15, 18]).

Theorem 1 and the decomposition formula (2.8) for symmetric squares gives

$$\begin{aligned}\mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^2)) &= \mathrm{Sym}^2(N_2 \oplus N_0)\\ &= \mathrm{Sym}^2(N_2) \oplus (N_2 \otimes N_0) \oplus \mathrm{Sym}^2(N_0)\\ &= (K_4 \oplus K_0) \oplus K_2 \oplus K_0'.\end{aligned} \tag{5.18}$$

Clearly, $\mathrm{Sym}^2(N_0) = K_0'$ is spanned by $\hat{u}^2$ and $K_2 = N_2 \otimes N_0$ has weight vectors $\hat{u}v_2$ and $\hat{u}v_{-2}$ with corresponding real basis $\{\hat{u}u_2, \hat{u}u_{-2}\}$. Moreover, the line K_0 of $SO(2)$-fixed points in $\mathrm{Sym}^2(N_2)$ is spanned by $v_2 v_{-2} = u_2^2 + u_{-2}^2$ while the irreducible summand $K_4 \cong M_4$ has weight vectors v_2^2 and v_{-2}^2, with real basis $\{u_2^2 - u_{-2}^2, u_2 u_{-2}\}$.

Since the irreducible representation M_0 appears with multiplicity two in $\mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^2))$ and has real type, the lines of fixed points are parametrized by the real projective line, i.e. by the circle. Accordingly, these lines are given by

$$F_\phi = \mathbf{R}(\cos(\phi)\hat{u}^2 + \sin(\phi)(u_2^2 + u_{-2}^2)). \tag{5.19}$$

Now let A be an invariant subspace of $\mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^2))$. Then $A = A_4 \oplus A_2 \oplus A_0$ where A_4 is K_4 or $\{0\}$, A_2 is K_2 or $\{0\}$, and A_0 is $K_0' \oplus K_0$, F_ϕ, or $\{0\}$.

In order to obtain matrix representation of the basis vectors, one should replace each basis vector above, which is given by a quadratic form in three variables $\hat{u}$, u_2 and u_{-2}, by the corresponding 3x3 symmetric matrix. These matrices are written in the orthonormal basis $\{\hat{u}, u_2, u_{-2}\}$ given above as quadratic forms in two variables e_1 and e_{-1} (see Appendix).

We have already studied the question of exact relations for 2-D elasticity in Part I. We will need the above analysis presently when we consider 2-D piezoelectricity.

$\mathbf{Sym}^2(\mathbf{Sym}^2(\mathbf{R}^2) \oplus \mathbf{R}^2)$—two dimensional piezoelectricity.

Again, we start by applying theorem 1 and the decomposition formula (2.8) for symmetric squares to $\mathcal{Y} = \mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^2) \oplus \mathbf{R}^2)$, giving

$$\begin{aligned}\mathcal{Y} &\cong \mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^2)) \oplus (\mathrm{Sym}^2(\mathbf{R}^2) \otimes \mathbf{R}^2) \oplus \mathrm{Sym}^2(\mathbf{R}^2) \\ &= ((K_4 \oplus K_0) \oplus K_2 \oplus K_0') \oplus ((N_2 \oplus N_0) \otimes \mathbf{R}^2) \oplus (N_2 \oplus N_0) \\ &= ((K_4 \oplus K_0) \oplus K_2 \oplus K_0') \oplus ((L_3 \oplus L_1) \oplus L_1')) \oplus (N_2 \oplus N_0).\end{aligned} \tag{5.20}$$

We have already dealt with the first and last summands, so it only remains to consider the cross-term.

It is clear that the submodule $L_1' = N_0 \otimes \mathbf{R}^2 \cong M_1$ has weight vectors $\hat{u}f_1$ and $\hat{u}f_{-1}$ and real basis $\{\hat{u}e_1, \hat{u}e_{-1}\}$. Let L_j denote the subrepresentations coming from $N_2 \otimes \mathbf{R}^2$ isomorphic to M_j. Then L_3 has weight vectors $v_2 f_1$ and $v_{-2} f_{-1}$ with real basis $\{u_2 e_1 - u_{-2} e_{-1}, u_2 e_{-1} + u_{-2} e_1\}$, and L_1 has weight vectors $v_2 f_{-1}$ and $v_{-2} f_1$ with real basis $\{u_2 e_1 + u_{-2} e_{-1}, -u_2 e_{-1} + u_{-2} e_1\}$.

Since M_1 and M_2 are representations of complex type appearing with multiplicity two in $\mathcal{Y}$, the irreducible submodules in these isomorphism classes are parametrized by $\mathbf{P}^1(\mathbf{C})$. Given $[p, q] \in \mathbf{P}^1(\mathbf{C})$ with $p = p_1 + ip_2$ and $q = q_1 + iq_2$, we have the corresponding subrepresentation $N_{[p,q]} \cong M_1$ with weight vectors $pv_2 f_{-1} + q\hat{u}f_1$ and $\bar{p}v_{-2} f_1 + \bar{q}\hat{u}f_{-1}$. Therefore, $N_{[p,q]}$ has real basis

$$\begin{aligned}\{&p_1(u_2 e_1 + u_{-2} e_{-1}) + p_2(-u_2 e_{-1} + u_{-2} e_1) + q_1 \hat{u} e_1 + q_2 \hat{u} e_{-1}, \\ &- p_2(u_2 e_1 + u_{-2} e_{-1}) + p_1(-u_2 e_{-1} + u_{-2} e_1) - q_2 \hat{u} e_1 + q_1 \hat{u} e_{-1}\}.\end{aligned} \tag{5.21}$$

Similarly, for $[x = x_1 + ix_2, y = y_1 + iy_2] \in \mathbf{P}^1(\mathbf{C})$, we have the submodule $K_{[x,y]} \cong M_2$ with real basis

$$\begin{aligned}\{&x_1 \hat{u} u_2 + x_2 \hat{u} u_{-2} + y_1(e_1^2 - e_{-1}^2) + y_2(2e_1 e_{-1}), \\ &- x_2 \hat{u} u_2 + x_1 \hat{u} u_{-2} - y_2(e_1^2 - e_{-1}^2) + y_1(2e_1 e_{-1})\}.\end{aligned} \tag{5.22}$$

Since M_0 has real type and has multiplicity three in $\mathcal{Y}$, the trivial irreducible submodules are parametrized by $\mathbf{P}^2(\mathbf{R})$. Given $a = [a_1, a_2, a_3] \in \mathbf{P}^2(\mathbf{R})$, the corresponding submodule P_a has basis $\{a_1 \hat{u}^2 + a_2(u_2^2 + u_{-2}^2) + a_3(e_1^2 + e_{-1}^2)\}$. The submodules isomorphic to $M_0 \oplus M_0$ are parametrized by two-dimensional subspaces of $\mathbf{R}^3$. If $Z \subset \mathbf{R}^3$ is such a subspace with basis $\{a, b\}$, the associated submodule is $Q_Z = P_a \oplus P_b$.

We can now classify the invariant subspaces of $\mathcal{Y} = \mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^2) \oplus \mathbf{R}^2)$. Let B be an invariant subspace. Then $B = B_4 \oplus B_3 \oplus B_2 \oplus B_1 \oplus B_0$ where B_4 is K_4 or $\{0\}$, B_3 is L_3 or $\{0\}$, B_2 is $K_2 \oplus N_2$, $K_{[x,y]}$, or $\{0\}$, B_1 is $L_1 \oplus L_1'$, $N_{[p,q]}$, or $\{0\}$, and B_0 is $K_0 \oplus K_0' \oplus N_0$, Q_Z, P_a or $\{0\}$.

In order to obtain matrix representation of the basis vectors one should replace each basis vector above, which is given by a quadratic form in 5 variables $\hat{u}$, u_2, u_{-2}, e_1 and e_{-1} by the corresponding 5x5 symmetric matrix (see Appendix for a combined list of bases).

The existence of reducible isotypic classes (an irreducible representation appearing with multiplicity greater than one) prevents a fully automated search over all possible invariant subspaces B. The large number of possibilities to examine (142) makes a manual search (examining Maple output by eye) infeasible. Therefore, as a compromise, we make a fully automated search over those invariant subspaces which appear as partial sums in the canonical decomposition (5.20). Since the number of summands in (5.20) is 9, we have $2^9 - 2 = 510$ possibilities to examine.

The intuition behind our choice is that the basis vectors of such submodules of $\mathrm{Sym}(\mathcal{T})$, viewed as linear operators on $\mathcal{T}$, have "smallest possible" rank. Thus, the chances that (3.13) of Part I will be satisfied are maximized. This heuristic reasoning is supported by our results for 2 and 3-D elasticity, where we were able to examine *all* possibilities. There the exact relations were generated only by the invariant subspaces coming from the "canonical" splitting. For piezoelectricity, we did obtain several exact relations for both 2 and 3-D, but it is theoretically possible that some others were missed.

Before we begin describing our results, note that the automated search must return certain "uninteresting" exact relations. These are of two types: "trivial" and "dull". The "trivial" exact relations are exact relations for the uncoupled problem. If we have no piezoelectric effect and assume that our crystal is isotropic as a conductor, then our program will find the exact relations for elasticity. The "dull" exact relations are those obtained as an intersection of other exact relations. If we throw out all the "uninteresting" exact relations, then we obtain a list of "essential" ones. In the case at hand, we have a total of 46 exact relations. However, 27 of these are "trivial" and 7 more are "dull", leaving 12 essential exact relations. Of these, only 7 contain strictly positive definite material tensors, the remaining 5 consisting entirely of degenerate tensors. We shall thus distinguish between *general* and *degenerate* essential exact relations. Unfortunately, at present we don't know the physical interpretations of the exact relations corresponding to the rotationally invariant subspaces of the space of piezoelectric tensors listed below.

General essential exact relations.

- $\mathcal{L} = K_2 \oplus K_0' \oplus L_1'$.
- $\mathcal{L} = K_4 \oplus L_1 \oplus N_2$.
- $\mathcal{L} = K_4 \oplus L_3 \oplus N_2$.
- $\mathcal{L} = K_0' \oplus L_1' \oplus N_2 \oplus N_0$.
- $\mathcal{L} = K_2 \oplus K_0' \oplus L_1' \oplus N_2$.
- $\mathcal{L} = K_2 \oplus K_0' \oplus L_1' \oplus N_2 \oplus N_0$.
- $\mathcal{L} = K_4 \oplus K_0 \oplus L_3 \oplus L_1 \oplus N_2 \oplus N_0$.

Here, the fourth and the seventh exact relations correspond to a "constant field" class of exact relations.

Degenerate essential exact relations.

To obtain these, we need to use formula (1.5) or (1.6) with $(S_0)_{11} = 1$, where S_0 is an isotropic tensor in the variables S lying on the exact relation surface and entering the formula for W_0:

$$W_0 = (S_0 - \Gamma)^{-1}.$$

Here is the list of invariant subspaces generating the exact relations:

- $\mathcal{L} = K_4 \oplus K_0 \oplus L_1 \oplus N_2$.
- $\mathcal{L} = K_4 \oplus K_0 \oplus L_3 \oplus N_2$.
- $\mathcal{L} = K_4 \oplus K_0 \oplus L_1 \oplus L_3 \oplus N_2$.
- $\mathcal{L} = K_4 \oplus K_2 \oplus K_0 \oplus L_1 \oplus L_3$.
- $\mathcal{L} = K_4 \oplus K_2 \oplus K_0 \oplus L_1 \oplus L_3 \oplus N_2 \oplus N_0$.

See Appendix for the list of basis vectors for each of the above subspaces.

5.2 $SO(3)$: 3-D elasticity and piezoelectricity.

In this section, vectors denoted by f's or v's with subscripts will be weight vectors for the complexification of an irreducible representation with weight given by twice the index. We will use the Lie algebra approach described at the end of section four to find highest weight vectors for submodules. Thus, in order to find a highest weight vector for a submodule isomorphic to W_k, we will find a vector of weight $2k$ which is annihilated by $X = Z_2 - iZ_3 \in \mathfrak{so}(3)_{\mathbf{C}}$, using the notation of (4.16). We will choose an appropriate multiple of the highest weight vector so that the corresponding vector of weight zero will be in the underlying real representation. If v_k is a highest weight vector, v_j for $j \geq 0$ will be obtained by taking a convenient scalar multiple of $Y^{k-j}v_k$ (using the same scalar for the same weight vector in an isomorphic representation) while v_{-j} for $j > 0$ will be the complex conjugate of v_j (see Appendix). Corresponding real bases will be denoted by the preceding letter. Thus, for a basis of weight vectors $\{v_{\pm j} \mid 0 \leq j \leq k\}$, the basis for the real representation is $\{u_{\pm j} \mid 0 \leq j \leq k\}$ with $u_0 = v_0$, $u_j = (v_j + v_{-j})/2$, and $u_{-j} = (v_{-j} - v_j)/2i$. We will only deviate from this convention in order to retain $\{e_1, e_2, e_3\}$ as the standard basis for $\mathbf{R}^3$. As in the previous section, we will use the letter N for conductivity, K for elasticity, and L for piezoelectric cross moduli. The prime will distinguish between isomorphic representations appearing within the same physics. We will use the letter W to denote the abstract isomorphism class for a given representation.

$\mathbf{Sym}^2(\mathbf{Sym}^2(\mathbf{R}^3))$—three dimensional elasticity.

The standard representation of $SO(3)$ on $\mathbf{R}^3$ is isomorphic to W_1, with weight vectors $f_0 = e_1$, $f_1 = e_2 - ie_3$, and $f_{-1} = e_2 + ie_3$. Note that $e_2 = (f_1 + f_{-1})/2$ and $e_3 = (f_{-1} - f_1)/2i$. Accordingly, we have $\mathrm{Sym}^2(\mathbf{R}^3) = N_2 \oplus N_0$. N_0 will be spanned by a linear combination of zero weight vectors e_1^2 and $f_1 f_{-1}$ annihilated by X. Since $X \cdot (e_1^2 + f_1 f_{-1}) = 0$, N_0 is spanned by $\hat{u} = e_1^2 + f_1 f_{-1} = e_1^2 + e_2^2 + e_3^3$. The monomial f_1^2 is a highest weight vector for N_2, giving the basis of weight vectors and corresponding real basis of N_2:

$$\begin{aligned} v_2 &= f_1^2, & u_2 &= e_2^2 - e_3^2, \\ v_{-2} &= f_{-1}^2, & u_{-2} &= 2e_2e_3, \\ v_1 &= f_0 f_1, & u_1 &= e_1 e_2, \\ v_{-1} &= f_0 f_{-1}, & u_{-1} &= e_1 e_3, \\ v_0 &= 2f_0^2 - f_1 f_{-1}, & u_0 &= 2e_1^2 - e_2^2 - e_3^2. \end{aligned} \tag{5.23}$$

Using (2.8) and theorem 2, we have

$$\begin{aligned} \mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^3)) &= \mathrm{Sym}^2(N_2 \oplus N_0) \\ &= \mathrm{Sym}^2(N_2) \oplus (N_2 \otimes N_0) \oplus \mathrm{Sym}^2(N_0) \\ &= (K_4 \oplus K_2 \oplus K_0) \oplus K_2' \oplus K_0'. \end{aligned} \tag{5.24}$$

Note that $\mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^3))$ is 21-dimensional, and the problem has now been reduced to decomposing summands of dimensions 1, 5, and 15, appearing on the second line of (5.24).

It is immediate that $K_0' = \mathrm{Sym}^2(N_0)$ is spanned by $\hat{u}^2$ and that $K_2' = N_2 \otimes N_0$ has weight vectors $\{\hat{u}v_{\pm j} \mid 0 \leq j \leq 2\}$ with corresponding real basis $\{\hat{u}u_{\pm j} \mid 0 \leq j \leq 2\}$. It remains to find the irreducible summands K_j isomorphic to W_j in $\mathrm{Sym}^2(W_2)$. The line of fixed points K_0 will be spanned by a linear combination of zero weight vectors v_0^2, $v_1 v_{-1}$ and $v_2 v_{-2}$ which is annihilated by X. We easily find that $X \cdot \alpha = 0$ for

$$\alpha = v_0^2 + 12 v_1 v_{-1} + 3 v_2 v_{-2} = u_0^2 + 12(u_1^2 + u_{-1}^2) + 3(u_2^2 + u_{-2}^2).$$

Similarly in order to find the highest weight vector in K_2 we need to find a linear combination of the weight four vectors v_1^2 and $v_0 v_2$ which is annihilated by X. We obtain $X \cdot (3v_1^2 - v_0 v_2) = 0$, so the basis of weight vectors and corresponding real basis of K_2 are:

$$
\begin{aligned}
&3v_1^2 - v_0v_2, && \mu_2 = 3(u_1^2 - u_{-1}^2) - u_0u_2,\\
&3v_{-1}^2 - v_0v_{-2}, && \mu_{-2} = 6u_1u_{-1} - u_0u_{-2},\\
&\frac{3}{2}v_{-1}v_2 + \frac{1}{2}v_0v_1, && \mu_1 = \frac{3}{2}(u_1u_2 + u_{-1}u_{-2}) + \frac{1}{2}u_0u_1,\\
&\frac{3}{2}v_1v_{-2} + \frac{1}{2}v_0v_{-1}, && \mu_{-1} = \frac{3}{2}(u_1u_{-2} - u_{-1}u_2) + \frac{1}{2}u_0u_{-1},\\
&3v_{-1}v_1 + \frac{1}{2}v_0^2 - \frac{3}{2}v_{-2}v_2, && \mu_0 = 3(u_1^2 + u_{-1}^2) + \frac{1}{2}u_0^2 - \frac{3}{2}(u_2^2 + u_{-2}^2).
\end{aligned}
\tag{5.25}
$$

Finally, the highest weight vector v_2^2 generates K_4, giving bases:

$$
\begin{aligned}
&v_2^2, && u_2^2 - u_{-2}^2,\\
&v_{-2}^2, && 2u_2u_{-2},\\
&v_1v_2, && u_1u_2 - u_{-1}u_{-2},\\
&v_{-1}v_{-2}, && u_1u_{-2} + u_{-1}u_2,\\
&4v_1^2 + v_0v_2, && 4(u_1^2 - u_{-1}^2) + u_0u_2,\\
&4v_{-1}^2 + v_0v_{-2}, && 8u_1u_{-1} + u_0u_{-2},\\
&v_2v_{-1} - 2v_1v_0, && u_1u_2 + u_{-1}u_{-2} - 2u_1u_0,\\
&v_1v_{-2} - 2v_{-1}v_0, && u_1u_{-2} - u_{-1}u_2 - 2u_{-1}u_0,\\
&v_2v_{-2} - 16v_1v_{-1} + 2v_0^2, && u_2^2 + u_{-2}^2 - 16(u_1^2 + u_{-1}^2) + 2u_0^2.
\end{aligned}
\tag{5.26}
$$

Since W_0 and W_2 appear twice as summands of $\mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^3))$, the irreducible subrepresentations of these types are parametrized by $\mathbf{P}^1(\mathbf{R}) \cong S^1$. Consequently, the lines of fixed points in $\mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^3))$ are just $M_\gamma = \mathbf{R}(\cos(\gamma)\hat{u}^2 + \sin(\gamma)\alpha)$. Similarly, the irreducible submodules isomorphic to W_2 are the subspaces N_δ with bases $\{\cos(\delta)\hat{u}u_{\pm j} + \sin(\delta)\mu_{\pm j} \mid 0 \le j \le 2\}$.

We can now list the invariant subspaces of $\mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^3))$. Let A be an invariant subspace. Then $A = A_4 \oplus A_2 \oplus A_0$ where A_4 is K_4 or $\{0\}$, A_2 is $K_2 \oplus K_2'$, N_δ, or $\{0\}$, and A_0 is $K_0 \oplus K_0'$, M_γ, or $\{0\}$.

We should warn the reader here that the basis $\{\hat{u}\} \cup \{u_{\pm j} \mid 0 \le j \le 2\}$ of $\mathrm{Sym}^2(\mathbf{R}^3)$ is orthogonal, but not orthonormal. Therefore, the corresponding matrix representations of the basis vectors for $\mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^3))$ *cannot* be obtained by taking matrices corresponding to quadratic forms listed above. One has to make the following substitutions:

$$
\hat{u} = \sqrt{3}\hat{w}, \quad u_0 = \sqrt{6}w_0, \quad u_1 = \frac{1}{\sqrt{2}}w_1, \quad u_{-1} = \frac{1}{\sqrt{2}}w_{-1}, \quad u_2 = \sqrt{2}w_2, \quad u_{-2} = \sqrt{2}w_{-2} \tag{5.27}
$$

in the above quadratic forms. The resulting quadratic forms in 6 variables $\hat{w}$ and $w_{\pm j}$, $0 \le j \le 2$ correspond to their matrix representations. See Appendix for a complete list.

Here, we have just 16 different possibilities to examine by hand (introducing free parameters γ and δ for the reducible isotypic classes). We have found only 3 exact relations. All of them are general essential exact relations and are generated by invariant subspaces that are the direct sums of the irreducible summands appearing in the decomposition (5.24).

1. $\mathcal{L} = K_0'$.
   This is the exact relation discovered by Hill [12, 13]. It says that a mixture of isotropic materials with a common shear modulus will be isotropic with the same shear modulus.

2. $\mathcal{L} = K_2' \oplus K_0'$.
   This exact relation says that a mixture of materials whose Hooke's laws can be represented as

a rank one tensor plus a fixed null-Lagrangian has the same property. The equation

$$C(x) = c(x) \otimes c(x) - \mu T$$

implies

$$C^* = c^* \otimes c^* - \mu T,$$

where the constant μ is such that $C(x)$ is strictly positive definite. The null-Lagrangian T is a tensor whose quadratic form is given by the sum of three principal 2x2 minors of a 3x3 symmetric matrix ξ:

$$(T\xi, \xi) = J_2(\xi) = \sum_{i<j} (\xi_{ii}\xi_{jj} - \xi_{ij}\xi_{ji}) = \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1,$$

where ξ_1, ξ_2 and ξ_3 are the eigenvalues of ξ. The above quadratic form is called the second orthogonal invariant of ξ. It also appears as one of the coefficients of the characteristic polynomial of ξ. This exact relation is a generalization of the one obtained in [10] for 2-D elasticity.

3. $\mathcal{L} = K_4 \oplus K_2 \oplus K_0$.
   This exact relation says that a mixture of crystals isotropic under hydrostatic loading and sharing the same value of the bulk modulus is isotropic under hydrostatic loading and has the same bulk modulus. This exact relation is due to Hill [11], see also [2].

The exact relations 1 and 3 above can be obtained as "constant field" exact relations (see Introduction).

$\mathbf{Sym}^2(\mathbf{Sym}^2(\mathbf{R}^3) \oplus \mathbf{R}^3)$—three dimensional piezoelectricity.

As in the previous example, we use (2.8) and theorem 2 to split the 45- dimensional space $\mathcal{Y} = \mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^3) \oplus \mathbf{R}^3)$ into more manageable pieces. We have

$$\begin{aligned} \mathcal{Y} &= \mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^3)) \oplus (\mathrm{Sym}^2(\mathbf{R}^3) \otimes \mathbf{R}^3) \oplus \mathrm{Sym}^2(\mathbf{R}^3) \\ &= ((K_4 \oplus K_2 \oplus K_0) \oplus K_2' \oplus K_0') \oplus ((N_2 \otimes \mathbf{R}^3) \oplus (N_0 \otimes \mathbf{R}^3)) \oplus (N_0 \oplus N_2) \\ &= ((K_4 \oplus K_2 \oplus K_0) \oplus K_2' \oplus K_0') \oplus ((L_3 \oplus L_2 \oplus L_1) \oplus L_1') \oplus (N_0 \oplus N_2). \end{aligned} \tag{5.28}$$

We have already analyzed the first and last summands in the first line of (5.28), so it only remains to consider the submodule isomorphic to $\mathrm{Sym}^2(\mathbf{R}^3) \otimes \mathbf{R}^3$. It is obvious that the subrepresentation $L_1' \cong W_1$ coming from the $N_0 \otimes \mathbf{R}^3$ term has basis $\{\hat{u}e_1, \hat{u}e_2, \hat{u}e_3\}$ with corresponding weight vectors $\hat{u}f_{\pm 1}$ and $\hat{u}f_0$. Let L_j be the irreducible submodule isomorphic to W_j in the $N_2 \otimes \mathbf{R}^3$ term. The highest weight vector in L_1 is a linear combination of the three vectors of weight two v_1f_0, v_0f_1 and v_2f_{-1}, which is annihilated by the raising operator X. Since X annihilates $6v_1f_0 + 3v_2f_{-1} - v_0f_1$, the basis of weight vectors and corresponding real basis of L_1 are:

$$\begin{aligned} &6v_1f_0 + 3v_2f_{-1} - v_0f_1, && \tau_1 = 6u_1e_1 + 3u_2e_2 + 3u_{-2}e_3 - u_0e_2, \\ &6v_{-1}f_0 + 3v_{-2}f_1 - v_0f_{-1}, && \tau_{-1} = 6u_{-1}e_1 - 3u_2e_3 + 3u_{-2}e_2 - u_0e_3, \\ &3v_1f_{-1} + 3v_{-1}f_1 + 2v_0f_0, && \tau_0 = 6u_1e_2 + 6u_{-1}e_3 + 2u_0e_1. \end{aligned} \tag{5.29}$$

The highest weight vector in L_2 is a linear combination of the two vectors of weight four v_1f_1 and v_2f_0 which is annihilated by X. We find that $v_2f_0 - v_1f_1$ is killed by X, so it generates L_2, giving the bases:

$$\begin{aligned}
&\frac{1}{i}(v_2 f_0 - v_1 f_1), && \nu_2 = u_{-2}e_1 + u_{-1}e_2 + u_1 e_3,\\
&-\frac{1}{i}(v_{-2}f_0 - v_{-1}f_{-1}), && \nu_{-2} = u_2 e_1 - u_1 e_2 + u_{-1}e_3,\\
&\frac{1}{4i}(2v_1 f_0 - v_2 f_{-1} - v_0 f_1), && \nu_1 = \frac{1}{4}(u_{-2}e_2 + u_0 e_3 - 2u_{-1}e_1 - u_2 e_3),\\
&-\frac{1}{4i}(2v_{-1}f_0 - v_{-2}f_1 - v_0 f_{-1}), && \nu_{-1} = \frac{1}{4}(2u_1 e_1 - u_{-2}e_3 - u_0 e_2 - u_2 e_2),\\
&\frac{3}{2i}(v_{-1}f_1 - v_1 f_{-1}), && \nu_0 = 3(u_{-1}e_2 - u_1 e_3).
\end{aligned} \tag{5.30}$$

Here we chose a $1/i$ multiple of the highest weight vector in order to make the zero weight vector real, as was the case for all other subrepresentations.

Finally, the submodule L_3 is generated by the highest weight vector $v_2 f_1$, and we obtain the bases:

$$\begin{aligned}
&v_2 f_1, && u_2 e_2 - u_{-2}e_3,\\
&v_{-2}f_{-1}, && u_2 e_3 + u_{-2}e_2,\\
&2v_1 f_1 + v_2 f_0, && 2u_1 e_2 - 2u_{-1}e_3 + u_2 e_1,\\
&2v_{-1}f_{-1} + v_{-2}f_0, && 2u_1 e_3 + 2u_{-1}e_2 + u_{-2}e_1,\\
&2v_0 f_1 + 8v_1 f_0 - v_2 f_{-1}, && 2u_0 e_2 + 8u_1 e_1 - u_2 e_2 - u_{-2}e_3,\\
&2v_0 f_{-1} + 8v_{-1}f_0 - v_{-2}f_1, && 2u_0 e_3 + 8u_{-1}e_1 + u_2 e_3 - u_{-2}e_2,\\
&v_{-1}f_1 + v_1 f_{-1} - v_0 f_0, && 2u_1 e_2 + 2u_{-1}e_3 - u_0 e_1.
\end{aligned} \tag{5.31}$$

Since W_1 appears in $\mathcal{Y}$ with multiplicity two, the irreducible subspaces isomorphic to W_1 are parametrized by a circle, given by K_β with basis

$$\{\cos(\beta)\hat{u}e_2 + \sin(\beta)\tau_1, \cos(\beta)\hat{u}e_3 + \sin(\beta)\tau_{-1}, \cos(\beta)\hat{u}e_1 + \sin(\beta)\tau_0\}. \tag{5.32}$$

The trivial representation W_0 has real type and has multiplicity three in $\mathcal{Y}$, so the trivial irreducible submodules are parametrized by $\mathbf{P}^2(\mathbf{R})$. Given $c = [c_1, c_2, c_3] \in \mathbf{P}^2(\mathbf{R})$, the corresponding submodule R_c has basis $\{c_1\hat{u}^2 + c_2\alpha + c_3\hat{u}\}$, where the last $\hat{u} = e_1^2 + e_2^2 + e_3^3$ is understood as a quadratic form in $\{e_1, e_2, e_3\}$, while the first $\hat{u}$ must be regarded as a variable in a quadratic form. The submodules isomorphic to $W_0 \oplus W_0$ are parametrized by two-dimensional subspaces of $\mathbf{R}^3$. If $Z \subset \mathbf{R}^3$ is such a subspace with basis $\{c, d\}$, the associated submodule is $S_Z = R_c \oplus R_d$.

The situation is more complicated for W_2, which has multiplicity four in $\mathcal{Y}$. The irreducible subrepresentations of this type are parametrized by $\mathbf{P}^3(\mathbf{R})$; for $a = [a_1, a_2, a_3, a_4] \in \mathbf{P}^3(\mathbf{R})$, the corresponding submodule P_a has basis $\{a_1\hat{u}u_{\pm j} + a_2\mu_{\pm j} + a_3\nu_{\pm j} + a_4 u_{\pm j} \mid 0 \le j \le 2\}$, where in the last term $u_{\pm j}$ are quadratic forms in $\{e_1, e_2, e_3\}$ given by (5.23), while in the first term $u_{\pm j}$ are to be understood as variables in a quadratic form. The submodules isomorphic to $W_2 \oplus W_2$ are parametrized by two-dimensional subspaces of $\mathbf{R}^4$; the submodule associated to $Z_2 \subset \mathbf{R}^4$ with basis $\{a, b\}$ is $Q_{Z_2} = P_a \oplus P_b$. Similarly the submodules isomorphic to $W_2 \oplus W_2 \oplus W_2$ are parametrized by three-dimensional subspaces of $\mathbf{R}^4$; the submodule associated to $Z_3 \subset \mathbf{R}^4$ with basis $\{a, b, c\}$ is $Q_{Z_3} = P_a \oplus P_b \oplus P_c$.

Finally, let B be an invariant subspace of $\mathcal{Y} = \mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^3) \oplus \mathbf{R}^3)$. Then $B = B_4 \oplus B_3 \oplus B_2 \oplus B_1 \oplus B_0$ where B_4 is K_4 or $\{0\}$, B_3 is L_3 or $\{0\}$, B_2 is $K_2 \oplus K_2' \oplus L_2 \oplus M_2$, Q_{Z_3}, Q_{Z_2}, P_a, or $\{0\}$, B_1 is $L_1 \oplus L_1'$, K_β, or $\{0\}$, and B_0 is $K_0 \oplus K_0' \oplus M_0$, S_Z, R_c, or $\{0\}$.

Again, we note that vectors $\hat{u}$, $u_{\pm j}$, $0 \le j \le 2$, and e_k, $1 \le k \le 3$, forming the basis of $\mathcal{T}$ are orthogonal but not orthonormal. Therefore the matrices of quadratic forms above are *not* the correct basis vectors for appropriate subspaces. In order to obtain correct matrices, one needs to make the substitutions (5.27) first. The matrices of quadratic forms in the variables $\hat{w}$ and $w_{\pm j}$, $0 \leq j \leq 2$, and e_k, $1 \leq k \leq 3$, will be the correct matrices (see Appendix).

The number of different possibilities to consider (318) and the complexity of each one precludes an exhaustive search. Therefore, we use the same strategy as for 2-D piezoelectricity: we consider the "canonical" splitting of the isotypic classes given by (5.28). Thus, we have $2^{11} - 2 = 2046$ possibilities to examine. The search produces 14 general exact relations and no degenerate ones. Only 5 of them are nontrivial. Among those 5 there are 4 essential exact relations which we list below:

- $\mathcal{L} = K_2' \oplus K_0' \oplus L_1'$.
- $\mathcal{L} = K_0' \oplus L_1' \oplus N_2 \oplus N_0$.
- $\mathcal{L} = K_2' \oplus K_0' \oplus L_1' \oplus N_2 \oplus N_0$.
- $\mathcal{L} = K_4 \oplus K_2 \oplus K_0 \oplus L_3 \oplus L_2 \oplus L_1 \oplus N_2 \oplus N_0$.

The second and fourth relations are of "constant field" type. We still don't know the physical interpretation of the remaining two exact relations but we hope to get it soon.

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6 Appendix

Here we list bases for various subspaces appearing in the paper. Usually, in papers on effective properties of composites, the bases for tensor spaces are given by symmetric matrices of appropriate dimension. In our situation, such a representation would take many pages filled with matrices. Instead, we use the quadratic form notation. The matrix $(a_{ij})_{i,j}^n$ is given by its quadratic form $\sum_{i,j=1}^n a_{ij}x_ix_j$ in variables $\{x_1,\ldots,x_n\}$.

We use the following system of notation. Different Latin letters are assigned to submodules coming from different physics. We use the letter K for submodules coming from elasticity, the letter N for submodules coming from conductivity, and the letter L for piezoelectric cross moduli. The Arabic subscripts identify the maximal weight of the complexified subrepresentation for $SO(2)$-modules and half the maximal weight for $SO(3)$-modules. We use primes to distinguish subspaces with the same maximal weight, but having different algebraic origins (i.e. coming from different terms of (2.8) via theorems 1 and 2). We took special care in listing the bases of the spaces below so that isomorphic submodules are spanned by isomorphic bases with the same order of basis vectors throughout.

6.1 $SO(2)$: Invariant subspaces.

Let $\{e_1, e_{-1}\}$ be the standard basis of $\mathbf{R}^2$. Then we let

$$\hat{u} = \frac{1}{\sqrt{2}}(e_1^2 + e_{-1}^2), \quad u_2 = \frac{1}{\sqrt{2}}(e_1^2 - e_{-1}^2), \quad u_{-2} = \sqrt{2}e_1 e_{-1}. \tag{6.33}$$

This is the usual basis of the space of 2x2 symmetric matrices (see for example [1, 15, 18]) given by their quadratic forms in 2 variables $\{e_1, e_{-1}\}$. The bases of the irreducible subrepresentations of

$$\mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^2)) = (K_4 \oplus K_0) \oplus K_2 \oplus K_0' \tag{6.34}$$

will be given as quadratic forms in 3 variables $\{\hat{u}, u_2, u_{-2}\}$, replacing the list of six 3x3 symmetric matrices:

$$\begin{array}{rcl} K_4 & = & \mathrm{Span}\{u_2^2 - u_{-2}^2,\ u_2 u_{-2}\} \\ K_2 & = & \mathrm{Span}\{\hat{u}u_2,\ \hat{u}u_{-2}\} \\ K_0 & = & \mathrm{Span}\{u_2^2 + u_{-2}^2\} \\ K_0' & = & \mathrm{Span}\{\hat{u}^2\}. \end{array} \tag{6.35}$$

The bases of the irreducible submodules

$$\mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^2) \oplus \mathbf{R}^2) = ((K_4 \oplus K_0) \oplus K_2 \oplus K_0') \oplus ((L_3 \oplus L_1) \oplus L_1')) \oplus (N_2 \oplus N_0) \tag{6.36}$$

will be given as quadratic forms in 5 variables $\{\hat{u}, u_2, u_{-2}, e_1, e_{-1}\}$. The bases for subspaces K_4, K_2, K_0 and K_0' are already given in (6.35).

$$\begin{array}{rcl} L_3 & = & \mathrm{Span}\{u_2 e_1 - u_{-2}e_{-1},\ u_2 e_{-1} + u_{-2}e_1\} \\ L_1 & = & \mathrm{Span}\{u_2 e_1 + u_{-2}e_{-1},\ u_{-2}e_1 - u_2 e_{-1}\} \\ L_1' & = & \mathrm{Span}\{\hat{u}e_1,\ \hat{u}e_{-1}\} \\ N_2 & = & \mathrm{Span}\{e_1^2 - e_{-1}^2,\ 2e_1 e_{-1}\} \\ N_0 & = & \mathrm{Span}\{e_1^2 + e_{-1}^2\}. \end{array} \tag{6.37}$$

6.2 $SO(3)$: Invariant subspaces.

In this paper, we use the following scheme for producing bases of irreducible submodules from their highest weight vectors

$$\begin{array}{llllll} W_1: & v_1 & v_0 = -1/(2i)Y(v_1) & & & \\ W_2: & v_2 & v_1 = -1/(4i)Y(v_2) & v_0 = -1/iY(v_1) & & \\ W_3: & v_3 & v_2 = -1/(2i)Y(v_3) & v_1 = -1/iY(v_2) & v_0 = 1/(12i)Y(v_1) & \\ W_4: & v_4 & v_3 = -1/(8i)Y(v_4) & v_2 = -1/iY(v_3) & v_1 = 1/(6i)Y(v_2) & v_0 = 1/iY(v_1). \end{array} \tag{6.38}$$

We also define $v_{-k} = \bar{v}_k$. We choose a multiple of the highest weight vector which produces real v_0 according to the scheme (6.38).

Now we list bases of the $SO(3)$-invariant subspaces. Let $\{e_1, e_2, e_3\}$ be the standard basis in $\mathbf{R}^3$. Let

$$\begin{array}{rclrcl} \hat{w} & = & \frac{1}{\sqrt{3}}(e_1^2 + e_2^2 + e_3^2), & w_0 & = & \frac{1}{\sqrt{6}}(2e_1^2 - e_2^2 - e_3^2) \\ w_1 & = & \sqrt{2}e_1 e_2, & w_{-1} & = & \sqrt{2}e_1 e_3 \\ w_2 & = & \frac{1}{\sqrt{2}}(e_2^2 - e_3^2), & w_{-2} & = & \sqrt{2}e_2 e_3 \end{array} \tag{6.39}$$

be the orthonormal basis of the space of 3x3 symmetric matrices. Then the irreducible subspaces in the decomposition

$$\mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^3)) = (K_4 \oplus K_2 \oplus K_0) \oplus K_2' \oplus K_0' \tag{6.40}$$

have the following bases:

$$\begin{array}{rcl} K_0' &=& \mathrm{Span}\{\hat{w}^2\} \\ K_0 &=& \mathrm{Span}\{w_0^2 + w_1^2 + {w_{-1}}^2 + {w_2}^2 + {w_{-2}}^2\} \\ K_2' &=& \mathrm{Span}\{2\sqrt{3}\hat{w}\, w_0,\ \hat{w}\, w_1,\ \hat{w}\, w_{-1},\ 2\hat{w}\, w_2,\ 2\hat{w}\, w_{-2}\} \end{array} \tag{6.41}$$

The subspace K_2 has the basis

$$\begin{array}{rcl} \mu_0 &=& -6\, {w_2}^2 - 6\, {w_{-2}}^2 + 3\, {w_1}^2 + 3\, {w_{-1}}^2 + 6\, {w_0}^2, \\ \mu_1 &=& 3\, w_1\, w_2 + 3\, w_{-1}\, w_{-2} + \sqrt{3}\, w_0\, w_1, \\ \mu_{-1} &=& 3\, w_1\, w_{-2} - 3\, w_{-1}\, w_2 + \sqrt{3}\, w_0\, w_{-1}, \\ \mu_2 &=& 3\, {w_1}^2 - 3\, {w_{-1}}^2 - 4\sqrt{3}\, w_0\, w_2, \\ \mu_{-2} &=& 6\, w_1\, w_{-1} - 4\sqrt{3}\, w_0\, w_{-2}. \end{array} \tag{6.42}$$

The subspace K_4 has the basis

$$\begin{array}{l} 2\, {w_2}^2 - 2\, {w_{-2}}^2, \\ 4\, w_2\, w_{-2}, \\ w_1\, w_2 - w_{-1}\, w_{-2}, \\ w_1\, w_{-2} + w_{-1}\, w_2, \\ 2\, {w_1}^2 - 2\, {w_{-1}}^2 + 2\sqrt{3}\, w_0\, w_2, \\ 4\, w_1\, w_{-1} + 2\sqrt{3}\, w_0\, w_{-2}, \\ w_1\, w_2 + w_{-1}\, w_{-2} - 2\sqrt{3}\, w_0\, w_1, \\ w_1\, w_{-2} - w_{-1}\, w_2 - 2\sqrt{3}\, w_0\, w_{-1}, \\ 2\, {w_2}^2 + 2\, {w_{-2}}^2 - 8\, {w_1}^2 - 8\, {w_{-1}}^2 + 12\, {w_0}^2. \end{array} \tag{6.43}$$

These bases are given in terms of quadratic forms in 6 variables $\{\hat{w}, w_2, w_0, w_{-1}, w_{-2}, w_1\}$.

The bases of the irreducible subrepresentations of

$$\mathrm{Sym}^2(\mathrm{Sym}^2(\mathbf{R}^3) \oplus \mathbf{R}^3) = ((K_4 \oplus K_2 \oplus K_0) \oplus K_2' \oplus K_0') \oplus ((L_3 \oplus L_2 \oplus L_1) \oplus L_1') \oplus (N_0 \oplus N_2). \tag{6.44}$$

will be given as quadratic forms in 9 variables $\{\hat{w}, w_2, w_0, w_{-1}, w_{-2}, w_1, e_1, e_2, e_3\}$. The bases for subspaces K_4, K_2, K_2', K_0 and K_0' are already given in (6.41), (6.42) and (6.43).

$$\begin{array}{rcl} L_1' &=& \mathrm{Span}\{\hat{w}e_1,\ \hat{w}e_2,\ \hat{w}e_3\} \\ N_0 &=& \mathrm{Span}\{e_1^2 + e_2^2 + e_3^2\} \\ N_2 &=& \mathrm{Span}\{2e_1^2 - e_2^2 - e_3^2,\ e_1e_2,\ e_1e_3,\ e_2^2 - e_3^2,\ 2e_2e_3\}. \end{array} \tag{6.45}$$

The subspace L_1 has the basis

$$\begin{array}{rcl} \tau_0 &=& 3\, w_1\, e_2 + 3\, w_{-1}\, e_3 + 2\sqrt{3}\, w_0\, e_1, \\ \tau_1 &=& 3\, w_1\, e_1 + 3\, w_2\, e_2 + 3\, w_{-2}\, e_3 - \sqrt{3}\, w_0\, e_2, \\ \tau_{-1} &=& 3\, w_{-1}\, e_1 - 3\, w_2\, e_3 + 3\, w_{-2}\, e_2 - \sqrt{3}\, w_0\, e_3. \end{array} \tag{6.46}$$

The subspace L_2 has the basis

$$\begin{array}{rcl} \nu_0 &=& 6\, w_{-1}\, e_2 - 6\, w_1\, e_3, \\ \nu_1 &=& w_{-2}\, e_2 + \sqrt{3}\, w_0\, e_3 - w_{-1}\, e_1 - w_2\, e_3, \\ \nu_{-1} &=& -w_{-2}\, e_3 - \sqrt{3}\, w_0\, e_2 + w_1\, e_1 - w_2\, e_2, \\ \nu_2 &=& 2\, w_{-1}\, e_2 + 2\, w_1\, e_3 + 4\, w_{-2}\, e_1, \\ \nu_{-2} &=& 2\, w_{-1}\, e_3 + 4\, w_2\, e_1 - 2\, w_1\, e_2. \end{array} \tag{6.47}$$

The subspace L_3 has the basis

$$\begin{aligned}
&w_2\,e_2 - w_{-2}\,e_3,\\
&w_2\,e_3 + w_{-2}\,e_2,\\
&w_1\,e_2 - w_{-1}\,e_3 + w_2\,e_1,\\
&w_1\,e_3 + w_{-1}\,e_2 + w_{-2}\,e_1,\\
&2\sqrt{3}\,w_0\,e_2 + 4\,w_1\,e_1 - w_2\,e_2 - w_{-2}\,e_3,\\
&2\sqrt{3}\,w_0\,e_3 + 4\,w_{-1}\,e_1 + w_2\,e_3 - w_{-2}\,e_2,\\
&w_1\,e_2 + w_{-1}\,e_3 - \sqrt{3}\,w_0\,e_1.
\end{aligned} \tag{6.48}$$