Exact relations for effective tensors of polycrystals.
I:
Necessary conditions.

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Abstract

The set of all effective moduli of a polycrystal usually has a nonempty interior. When it does not, we say that there is an exact relation for effective moduli. This can indeed happen as evidenced by recent results [4, 10, 12] on polycrystals. In this paper we describe a general method for finding such relations for effective moduli of *laminates*. The method is applicable to any physical setting that can be put into the Hilbert space framework developed by Milton [13]. The idea is to use the W-function of Milton [13] that transforms lamination formula into a convex combination. The method reduces the problem of finding exact relations to a problem from representation theory of $SO(d)$ ($d = 2$ or 3) corresponding to a particular physical setting. When this last problem is solved there is a finite amount of calculation required to be done in order to answer the question completely. At present, each candidate relation has to be examined separately in order to confirm the stability under homogenization. We apply our general theory to the settings of conductivity and two-dimensional elasticity.

1 Introduction.

Recent years have seen a lot of progress towards a more precise characterization of effective properties of composites. Yet, almost all of that progress is due to two approaches: use of variational principles and the "translation method". The former was pioneered by Hashin and Shtrikman [5], while the latter was discovered independently by Lurie and Cherkaev [9, 11] in the former Soviet Union and Murat and Tartar [15, 16] in France. The term "translation method" has been coined by Milton [13] and has become a universally accepted term since. The two methods mentioned above have brought about complete solutions to many G-closure problems [3, 4, 8, 9, 10, 11, 12, 16] as well as many optimal bounds on important functionals such as energy or complementary energy. The accumulated literature on the subject is too vast to allow a complete list (see, however, [13] for a review and comparison of the two methods). The results were almost always inequalities that the effective tensors had to satisfy. On rare occasions, however, researchers were discovering exact relations, or equalities for effective tensors [1, 4, 6, 7, 10, 12]. These results were in large respect byproducts of the inequalities derived using one of the two methods mentioned above.

This paper provides necessary conditions for an exact relation for effective tensors to hold by finding all exact relations for effective moduli of *laminates*. The already known exact relations from the literature above serve as a motivation to pose such a question. The key to the solution is a general lamination formula [13, formula (4.11)] of Milton that transforms the process of lamination into the process of taking convex combinations. The idea to use this formula in order to study the geometry of sets stable under lamination is due to Francfort and Milton [2].

Having outlined our sources we proceed to explain the idea. If there is an exact relation for effective tensors then the corresponding G-closure (the set of all possible effective tensors) must have empty interior. In all known such examples the G-closure lies on a smooth, even analytic manifold $\mathbb{M}$, possibly with boundary. The knowledge of the manifold itself is equivalent to establishing an exact relation for effective tensors without finding any actual G-closures. Obviously, if a piece of a manifold is stable under homogenization then the same piece must be stable under lamination. It means that if one applies the mapping W_n from [2, formula (3.21)] to this piece of the manifold the image must be a convex set. This convex set will have the same dimension as the original manifold as the mapping W_n is a diffeomorphism. But a "dimensionally challenged" convex set must lie in the hyperplane of the same dimension. This observation alone is hard to use as it must hold for a continuum many mappings W_n labeled by a vector n on the unit sphere in $\mathbb{R}^d$. The following observation comes to the rescue. If we restrict ourselves to polycrystals then if $C^* \in \mathbb{M}$ then so does every rotation of C^*. Therefore, by theorem 1 below, we need to require that only the W_{e_1} image of $\mathbb{M}$ lies in a hyperplane. Here e_1 denotes the first standard basis vector. Thus we have arrived at a remarkable conclusion that instead of searching for exact relations for effective moduli of laminates we may look for rotationally invariant manifolds whose W_{e_1} image lies in the hyperplane of the same dimension. These conditions on $\mathbb{M}$ are very restrictive. Every such manifold will contain a set stable under lamination and will be a serious candidate for an exact relation on effective tensors obtained by homogenization. The problem is that in general the lamination closure is different from G-closure [14]. However, in this special case we conjecture that stability under lamination implies stability under homogenization.

In the present paper we solve the above geometric problem and apply our solution to the cases of d-dimensional conductivity and 2-dimensional elasticity. For the conductivity we discover the well known 2-d result that the G-closure of the set of tensors with the same determinant belongs to the set of tensors with the same determinant. We also discover, and this is new, that there are no other exact relations for conductivity in any space dimension $d > 2$.

For the 2-dimensional elasticity we obtain 7 different exact relations. One of them was discovered by Hill [6, 7]. Two were found in [10, 12] and two more were described in [4]. One more is obtained as an intersection of the two manifolds from [4]. The last one appears to be new. It says that a mixture of isotropic materials with zero bulk modulus is again isotropic with zero bulk modulus.

2 A geometric problem.

We begin by a brief (and sloppy) review of the general Hilbert space setting of homogenization, as developed in [13], for readers familiar with that paper. Those who are unfamiliar with it we strongly encourage to read sections 2-4 of [13].

Let $\mathcal{H}_s = L^2(Q)$, where the unit cube Q represents a period cell in $\mathbb{R}^d$. This space has the orthogonal splitting

$$\mathcal{H}_s = \mathcal{U}_s \oplus \mathcal{F}_s \tag{2.1}$$

into the space of uniform (constant) and fluctuating functions. We then define a Hilbert space

$$\mathcal{H} = \mathcal{H}_s \otimes \mathcal{T}, \tag{2.2}$$

where $\mathcal{T}$ is an l-dimensional vector space of tensors over $\mathbb{R}^d$. Then

$$\mathcal{H} = \mathcal{U} \oplus \mathcal{F}, \quad \mathcal{U} = \mathcal{U}_s \otimes \mathcal{T}, \quad \mathcal{F} = \mathcal{F}_s \otimes \mathcal{T}. \tag{2.3}$$

Examples of $\mathcal{T}$ include (but are not limited to) $\mathbb{R}^d$ for conductivity and $\mathrm{Sym}(\mathbb{R}^d)$ for elasticity, where $\mathrm{Sym}(V)$ denotes the space of selfadjoint linear operators on the linear space V. In the former example $l = d$ in the latter $l = d(d+1)/2$. The inner product in $\mathcal{H}$ is determined by the inner product (u, v) in $\mathcal{T}$. It is important to assume that this inner product does not depend on the choice of an orthonormal basis in $\mathbb{R}^d$. Thus the rotation group $SO(d)$ has a representation into $SO(\mathcal{T})$, which we will denote $\Theta(\boldsymbol{R})$ for $\boldsymbol{R} \in SO(d)$. Now we assume that we have an orthogonal splitting of $\mathcal{F}$ into the scale-free, rotation-invariant, local in Fourier space subspaces (see [13] for definitions):

$$\mathcal{F} = \mathcal{E} \oplus \mathcal{J}. \tag{2.4}$$

Let $\Gamma_1(\hat{k})$ and $\Gamma_2(\hat{k})$ be projection operators onto $\mathcal{E}$ and $\mathcal{J}$ respectively in the Fourier space. These operators are functions of $\hat{k} = k/|k|$, which is equivalent to $\mathcal{E}$ and $\mathcal{J}$ being scale-free and local in Fourier space. The rotation invariance is equivalent to

$$\Gamma_i(\boldsymbol{R}\hat{k}) = \Theta(\boldsymbol{R})\Gamma_i(\hat{k})[\Theta(\boldsymbol{R})]^T, \quad i = 1, 2, \quad \boldsymbol{R} \in SO(d). \tag{2.5}$$

These conditions are satisfied for elasticity and conductivity settings [13]. Now we introduce the Milton's W_n function. For any $n \in \mathbb{R}^d$, $|n| = 1$, the function W_n is defined on an open dense subset $\mathcal{D} \subset \mathcal{Y} = \mathrm{Sym}(\mathcal{T})$ with values in $\mathcal{Y}$ by [2, formula (3.21)]

$$W_n(S) = [S - \Gamma_1(n)]^{-1}, \tag{2.6}$$

where S is related to the Hooke's law tensor C by [13, formula (4.6)]

$$S = (I - \frac{C}{c_0})^{-1}, \tag{2.7}$$

where I is the identity in algebra $\mathcal{Y}$ and c_0 is an arbitrary scalar. The formula for S is not important here except for the fact that if the coordinate system in $\mathbb{R}^d$ is rotated by $\boldsymbol{R}^T \in SO(d)$ then S changes to $\Theta(\boldsymbol{R})S[\Theta(\boldsymbol{R})]^T$. This formula will be used in section 5 to translate our results from S variables to C variables.

Lemma **1**

$$W_n(\Theta(\boldsymbol{R})S[\Theta(\boldsymbol{R})]^T) = \Theta(\boldsymbol{R})W_{R^T n}(S)[\Theta(\boldsymbol{R})]^T.$$

The proof follows from (2.5) and (2.6).

The key to our analysis is the theorem of Milton [13, formula (4.11)] or [2, formula (3.22)] that the set G of tensors is stable under lamination if and only if $W_n(G)$ is convex for any direction n. The main drawback of this theorem is that one has to "check" convexity for each n, i.e. infinitely many times. This small problem is is easily corrected by an additional assumption of rotational invariance of our set G. This assumption is satisfied by every polycrystalline G-closure.

Definition 1 *The set $G \subset \mathcal{Y} = \mathrm{Sym}(\mathcal{T})$ is called rotationally invariant if for any $S \in G$ and any $\boldsymbol{R} \in SO(d)$ we have $\Theta(\boldsymbol{R})S[\Theta(\boldsymbol{R})]^T \in G$.*

Theorem **1** *Suppose G is rotationally invariant and $W_{e_1}(G)$ is convex. Then G is stable under lamination.*

In view of this theorem it is convenient to denote the mapping W_{e_1} as W.

Proof. Due to the Milton's theorem all we need to show is that $W_n(G)$ is convex for any direction n. Let $\{S_1, S_2\} \subset G$ we will show that for any $\lambda \in (0, 1)$ and any $n \in \mathbb{R}^d$, $|n| = 1$ there exists $S^* \in G$ (depending on n and λ) such that $W_n(S^*) = \lambda W_n(S_1) + (1-\lambda)W_n(S_2)$. Let $\boldsymbol{R} \in SO(d)$ be such that $\boldsymbol{R}n = e_1$. Then by lemma 1 we have

$$\Theta(\boldsymbol{R})\Big(\lambda W_n(S_1) + (1-\lambda)W_n(S_2)\Big)[\Theta(\boldsymbol{R})]^T = \lambda W(\Theta(\boldsymbol{R})S_1[\Theta(\boldsymbol{R})]^T) + (1-\lambda)W(\Theta(\boldsymbol{R})S_2[\Theta(\boldsymbol{R})]^T). \tag{2.8}$$

By the rotational invariance of G we conclude that

$$\{\Theta(\boldsymbol{R})S_1[\Theta(\boldsymbol{R})]^T, \Theta(\boldsymbol{R})S_2[\Theta(\boldsymbol{R})]^T\} \subset G.$$

Thus, by convexity of $W(G)$ there exists $S_0 \in G$ such that the right hand side of (2.8) is equal to $W(S_0)$. Then by applying lemma 1 again we have

$$W(S_0) = \Theta(\boldsymbol{R})W_n([\Theta(\boldsymbol{R})]^T S_0 \Theta(\boldsymbol{R}))[\Theta(\boldsymbol{R})]^T. \tag{2.9}$$

Once again the rotational invariance of G implies that

$$S^* = [\Theta(\boldsymbol{R})]^T S_0 \Theta(\boldsymbol{R}) \in G.$$

Thus,

$$W_n(S^*) = \lambda W_n(S_1) + (1-\lambda)W_n(S_2) \tag{2.10}$$

and the theorem is proved. □

Now we are ready to consider the problem in the title of the paper and reduce it to a geometric one. Mathematically we are looking for smooth (real analytic) manifolds $\mathbb{M} \subset \mathcal{D}$ (possibly with boundary) that contain sets G stable under lamination. Suppose the manifold has dimension $k < N = l(l+1)/2$, then so does its diffeomorphic image $W(\mathbb{M})$. Assume that G has a nonempty interior in the induced topology of the manifold (otherwise there are more exact relations than the co-dimension of the manifold). Then so does $W(G)$. But $W(G)$ is convex. Therefore $W(G) \subset \Pi_k$, where Π_k is a k-hyperplane in the Euclidean space $\mathcal{Y}$ =Sym($\mathcal{T}$), dim$\mathcal{Y} = N$. Moreover $W(G)$ has a non-empty interior in the induced topology of Π_k. There is one more restriction that comes from physics.

Definition 2 *The operator $S \in \mathcal{Y}$ is called isotropic if for any $\boldsymbol{R} \in SO(d)$*

$$\Theta(\boldsymbol{R})S[\Theta(\boldsymbol{R})]^T = S.$$

We require the set G to have at least one isotropic point in its interior (in the induced topology) because regardless of the anisotropy of the original materials we can always arrange them to form an isotropic composite. Thus we will be looking for manifolds $\mathbb{M}$ that have rotation-invariant subsets G with nonempty interior in the induced topology of $\mathbb{M}$ containing an isotropic point, and such that $W(G) \subset \Pi_k$. In the next section we completely solve this geometric problem.

3 Solution of the geometric problem.

3.1 A necessary condition.

We are going to study the geometry of the manifold in a small (but finite) neighborhood of the isotropic tensor $S_0 \in \mathbb{M}$. Let us introduce a new notation that will simplify some of our formulas. The action of $\boldsymbol{R} \in SO(d)$ on $S \in \mathcal{Y}$ given by $\Theta(\boldsymbol{R})S[\Theta(\boldsymbol{R})]^T$ defines a representation of $SO(d)$ into $SO(\mathcal{Y})$. Let us denote this representation by $g(\boldsymbol{R})$, or simply g, when there is no confusion:

$$g(\boldsymbol{R})S = \Theta(\boldsymbol{R})S[\Theta(\boldsymbol{R})]^T. \tag{3.1}$$

Let $\mathcal{G}$ be the corresponding subgroup of $SO(\mathcal{Y})$:

$$\mathcal{G} = \{g(\boldsymbol{R}) \in SO(\mathcal{Y}) : \boldsymbol{R} \in SO(d)\}. \tag{3.2}$$

Let $\mathbb{T}_S\mathbb{M}$ denote the tangent space to $\mathbb{M}$ at S. Let $\mathcal{L}$ denote $\mathbb{T}_{S_0}\mathbb{M}$ the tangent space at the isotropic tensor $S_0 \in \mathbb{M}$. Since $\mathbb{M}$ is an embedded manifold in $\mathcal{Y}$, the tangent space $\mathcal{L}$ can be identified with a linear subspace of $\mathcal{Y}$.

Theorem **2** *The subspace $\mathcal{L} \subset \mathcal{Y}$ is $\mathcal{G}$-invariant.*

Proof. Let $\gamma(t) \in \mathbb{M}$ be a smooth curve with $\gamma(0) = S_0$. Let $X \in \mathcal{L}$ denote the tangent vector to γ at $t = 0$. By rotational invariance $\tilde{\gamma}(t) = g\gamma(t)$ is a smooth curve in $\mathbb{M}$[1]with $\tilde{\gamma}(0) = S_0$. Therefore the tangent vector to $\tilde{\gamma}$ at $t = 0$ must also belong to $\mathcal{L}$:

$$X \in \mathcal{L} \iff gX \in \mathcal{L}. \tag{3.3}$$

The theorem is proved. □

[1]Notice that the global group $\mathcal{G}$ maps a small neighborhood of S_0 in $\mathbb{M}$ into a small neighborhood of S_0 in $\mathbb{M}$.

Remark 1 *Notice that the same theorem applies to the tangent space at S_0 to the $N-1$ dimensional boundary of the polycrystalline G-closure with non-empty interior. Because of the fixed dimension, $N-1$, there are not many choices for $\mathcal{L}$. In the case of d-dimensional conductivity $\mathcal{L}$ is defined uniquely. For the d-dimensional elasticity there is a one-parameter family of choices. Unfortunately, there are cases where all such points S_0 are singular points of the boundary of the G-closure and do not have a tangent space.*

Theorem 2 immediately restricts $\mathcal{L}$ to just "a few" special choices. For example in the setting of d-dimensional conductivity there are only two choices for $\mathcal{L}$: the space of isotropic dxd matrices or the space of trace-free symmetric dxd matrices. In the setting of 2-D elasticity one can easily figure out the 10 choices for $\mathcal{L}$ from [1, formulas (2.24)–(2.26)] (see Appendix for the list).

Now we observe that since $SO(d)$ is a compact Lie group then the orthogonal complement of $\mathcal{L}$ is also $\mathcal{G}$-invariant. Moreover, we may split $\mathcal{Y}$ into the orthogonal sum of subspaces (of which $\mathcal{L}$ will be a partial sum) on each of which $\mathcal{G}$ acts irreducibly. Therefore we may choose an orthonormal basis for $\mathcal{Y}$ such that it may be grouped into bases for the subspaces entering the orthogonal decomposition of $\mathcal{Y}$ into the irreducible subspaces. Let us assume that the first k basis vectors span $\mathcal{L}$ and will be denoted $L_1, L_2, \ldots, L_k$. The rest of the basis spanning the orthogonal complement $\mathcal{L}^\perp$ will be denoted by $K_1, K_2, \ldots, K_{N-k}$.

Let U be a neighborhood of S_0 in $\mathbb{M}$. Then $W(U)$ must be a neighborhood of $W(S_0)$ in the hyperplane Π_k. The diffeomorphism W between U and $W(U)$ induces a linear isomorphism W_* between the corresponding tangent spaces. Therefore, the tangent space of Π_k at $W(S_0)$ will have a basis (not orthonormal in general)

$$P_i = W_* L_i, \quad i = 1, 2, \ldots, k. \tag{3.4}$$

But for a hyperplane we have

$$\Pi_k = \operatorname{Span}(P_1, \ldots, P_k) + W(S_0). \tag{3.5}$$

Since

$$W(S) = [S - \Gamma]^{-1}, \tag{3.6}$$

where $\Gamma = \Gamma_1(e_1)$, we can compute $W_* X$ explicitly. For every $X \in \mathcal{L}$

$$W_* X = -[S_0 - \Gamma]^{-1} X [S_0 - \Gamma]^{-1}. \tag{3.7}$$

We see that W_* is a self-adjoint operator on $\mathcal{Y}$. Therefore

$$M_j = {W_*}^{-1} K_j, \quad j = 1, \ldots, N-k \tag{3.8}$$

is a basis (not orthonormal in general) of the orthogonal complement of $\operatorname{Span}(P_1, \ldots, P_k)$:

$$(P_i, M_j) = (W_* L_i, {W_*}^{-1} K_j) = (L_i, K_j) = 0.$$

Thus we can give Π_k a dual definition:

$$\Pi_k = \{X \in \mathcal{Y} : (X, M_j) = (W(S_0), M_j), j = 1, \ldots, N-k\}. \tag{3.9}$$

This notation allows us to formulate the two conditions on $\mathbb{M}$ by a single formula. For any $v = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$

$$\sum_{i=1}^{k} \lambda_i P_i + W(S_0) \in \Pi_k$$

therefore for small v

$$W^{-1}\Big(\sum_{i=1}^{k} \lambda_i P_i + W(S_0)\Big) \in \mathbb{M}. \tag{3.10}$$

By rotational invariance

$$gW^{-1}\Big(\sum_{i=1}^{k}\lambda_i P_i + W(S_0)\Big) \in \mathbb{M}$$

for any $g \in \mathcal{G}$. Thus

$$W\left[gW^{-1}\Big(\sum_{i=1}^{k}\lambda_i P_i + W(S_0)\Big)\right] \in \Pi_k.$$

So, by (3.9) we get the desired formula

$$\left(M_r, W\left[gW^{-1}\Big(\sum_{i=1}^{k}\lambda_i P_i + W(S_0)\Big)\right]\right) = (M_r, W(S_0)) \tag{3.11}$$

for all $r = 1, \ldots, N-k$, for any v from a small neighborhood $\mathcal{N}$ of zero in $\mathbb{R}^k$ and for any $g \in \mathcal{G}$. We remark that (3.10) shows that $\mathcal{L}$ generates the sought after manifold $\mathbb{M}$ if it satisfies the invariance relation (3.11).

3.2 Necessary and sufficient conditions.

Let $\mathcal{A}$ denote the Lie algebra of $\mathcal{G}$. Since we think of $\mathcal{G}$ as a subgroup of $SO(\mathcal{Y})$, we would like to think of $\mathcal{A}$ as a subalgebra of linear operators on $\mathcal{Y}$. Let $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ be a basis for $\mathcal{A}$, where $m = d(d-1)/2$ is the dimension of $SO(d)$. The action of $\mathcal{A}$ on $\mathcal{Y}$ is determined by (3.1):

$$\boldsymbol{a}_\alpha S = [A_\alpha, S] = A_\alpha S - SA_\alpha, \quad \alpha = 1, \ldots, m \tag{3.12}$$

for any $S \in \mathcal{Y}$, where $A_1, \ldots, A_m$ form a basis of a Lie algebra of a representation $\Theta(\boldsymbol{R})$. Since we think of $\{\Theta(\boldsymbol{R}) : \boldsymbol{R} \in SO(d)\}$ as a subgroup of $SO(\mathcal{T})$, we would like to think of operators A_α as a basis of a subalgebra of $\mathrm{Hom}(\mathcal{T}, \mathcal{T})$ the algebra of linear operators on $\mathcal{T}$. Notice that $\mathcal{Y}$ is another such subalgebra. Therefore the products in (3.12) make sense. Then we have the following necessary and sufficient conditions for $\mathcal{L}$ to generate the manifold $\mathbb{M}$ with desired properties.

Theorem **3** *The manifold $\mathbb{M}$ generated by a $\mathcal{G}$-invariant subspace $\mathcal{L}$ by (3.10) satisfies (3.11) if and only if the following finite collection of identities are satisfied:*

$$(A_\alpha, [W_0, L_i K_r L_j]) = 0, \quad \alpha = 1, \ldots, m, \quad r = 1, \ldots, N-k, \quad i, j = 1, \ldots, k \tag{3.13}$$

Proof. Let us prove first that the conditions above are necessary. Initially we will obtain three sets of conditions and then prove that two of them are always satisfied. The third one will then be reduced to (3.13).

Observe that if we let $g \in \mathcal{G}$ to be the identity then the equation (3.11) is identically satisfied for every $v = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$. Then it is a natural idea to differentiate (3.11) with respect to g at the identity using the explicit formula (3.6) for W. The equation (3.11) becomes

$$\left(\left\{(\sum_{i=1}^{k}\lambda_i gP_i + gW_0)^{-1} + g\Gamma - \Gamma\right\}^{-1}, M_r\right) = (M_r, W_0), \quad v \in \mathcal{N}, \quad r = 1, \ldots, N-k. \tag{3.14}$$

Differentiating in $g \in \mathcal{G}$ at the identity, we obtain

$$\left(\boldsymbol{a}_\alpha W_0 - W_0(\boldsymbol{a}_\alpha\Gamma)W_0 + \sum_{i=1}^{k}\lambda_i(\boldsymbol{a}_\alpha P_i - P_i(\boldsymbol{a}_\alpha\Gamma)W_0 - W_0(\boldsymbol{a}_\alpha\Gamma)P_i) - \sum_{i,j=1}^{k}\lambda_i\lambda_j P_i(\boldsymbol{a}_\alpha\Gamma)P_j, M_r\right) = 0 \tag{3.15}$$

for all $\alpha = 1, \ldots, m$. Since (3.15) must be valid for every $v \in \mathcal{N}$ we get the promised three sets of conditions:

$$(\boldsymbol{a}_\alpha W_0 - W_0(\boldsymbol{a}_\alpha\Gamma)W_0, M_r) = 0, \tag{3.16}$$

$$(\boldsymbol{a}_\alpha P_i - P_i(\boldsymbol{a}_\alpha\Gamma)W_0 - W_0(\boldsymbol{a}_\alpha\Gamma)P_i, M_r) = 0, \tag{3.17}$$

$$(P_i(\boldsymbol{a}_\alpha\Gamma)P_j + P_j(\boldsymbol{a}_\alpha\Gamma)P_i, M_r) = 0. \tag{3.18}$$

In order to simplify these conditions we would like to express them in terms of basis operators L_i and K_r and A_α instead of P_i and M_r and $\boldsymbol{a}_\alpha$. This is easily accomplished by the formulas (3.4), (3.6), (3.7), (3.8) and (3.12). We obtain

$$([A_\alpha, S_0], K_r) = 0, \tag{3.19}$$

$$(A_\alpha, W_0L_iK_rS_0 - S_0K_rL_iW_0) = (A_\alpha, \Gamma W_0L_iK_r - K_rL_iW_0\Gamma), \tag{3.20}$$

$$(A_\alpha, [\Gamma, W_0L_iK_rL_jW_0 + W_0L_jK_rL_iW_0]) = 0, \tag{3.21}$$

Now we use (3.6) to get

$$\Gamma W_0 = -I + S_0W_0, \tag{3.22}$$

where I denotes the identity in the algebra $\mathcal{Y}$. Applying (3.22) to (3.20) and (3.21) we get:

$$([A_\alpha, S_0], W_0L_iK_r + K_rL_iW_0) = ([A_\alpha, L_i], K_r), \tag{3.23}$$

$$(A_\alpha, [W_0, L_i, K_rL_j]) = ([A_\alpha, S_0], W_0L_iK_rL_jW_0). \tag{3.24}$$

First we observe that $[A_\alpha, S_0] = 0$ because A_α commutes with the isotropic tensor S_0 for every $\alpha = 1, \ldots, m$. Then we observe that $[A_\alpha, L_i]$ is a linear combination of basis operators L_i, because $\mathcal{L}$ is a $\mathcal{G}$-invariant subspace. Thus

$$([A_\alpha, L_i], K_r) = 0$$

and (3.19) and (3.23) are identically satisfied, while (3.24) is reduced to (3.13).

Now let us show sufficiency, which is a corollary of the connectedness of $SO(d)$. The derivation of our equations is equivalent to saying that if $S \in \mathbb{M}$ and $\boldsymbol{a} \in \mathcal{A}$ then $\boldsymbol{a}S \in \mathbb{T}_S\mathbb{M}$, where $\mathbb{M}$ is given by (3.10). Then for a given $\boldsymbol{a} \in \mathcal{A}$ consider a smooth vector field $X(S)$ in the neighborhood of S_0 on $\mathbb{M}$:

$$X(S) = \boldsymbol{a}S. \tag{3.25}$$

Now fix S and consider a curve $\xi(t) = \exp(\boldsymbol{a}t)S$. Obviously, this curve solves the ODE:

$$\begin{cases} \dot{\xi}(t) &= X(\xi(t)), \\ \xi(0) &= S. \end{cases} \tag{3.26}$$

Therefore $\xi(t) \in \mathbb{M}$ for small t. Since the *exp* map is surjective on the small neighborhood of identity in a Lie group $\mathcal{G}$, we conclude that there exists a neighborhood U of the identity in $\mathcal{G}$ such that for every $g \in U$ we have $gS \in \mathbb{M}$. It remains to note two facts. One is that the whole group G maps a small neighborhood of $S_0 \in \mathbb{M}$ into a small neighborhood of S_0, and the other one is that for a connected Lie group any neighborhood of the identity generates the whole group. Thus $gS \in \mathbb{M}$ for any $g \in \mathcal{G}$ and for any S sufficiently close to S_0.□

In conclusion we will describe a subclass of solutions of (3.13).

THEOREM **4** *A left (and right) annihilator of a $\mathcal{G}$-invariant subspace is a $\mathcal{G}$-invariant subspace satisfying (3.13).*

Proof. Let Λ be a $\mathcal{G}$-invariant subspace. Define

$$\mathcal{L} = \{L : \ L\Lambda = 0\}. \tag{3.27}$$

Observe that $\mathcal{L}$ is a $\mathcal{G}$-invariant subspace. Indeed,

$$0 = g(L\Lambda) = (gL)(g\Lambda) = (gL)\Lambda.$$

Now let us write down the definition of the orthogonal complement to $\mathcal{L}$:

$$\mathcal{K} = \{K : \quad (K, L) = 0, \ \ \forall L : \quad L\Lambda_i = 0, \ \ i = 1, \ldots, \dim \Lambda\}, \tag{3.28}$$

where Λ_i's form a basis for Λ. The definition of $\mathcal{K}$ suggests a reformulation based on the Fredholm alternative. Let

$$\lambda : \mathcal{Y} \to \mathrm{Hom}(\mathcal{T}, \mathcal{T})^{\dim \Lambda}, \quad (\lambda Y)_i = Y\Lambda_i, \tag{3.29}$$

where $\mathrm{Hom}(\mathcal{T}, \mathcal{T})$ denotes the space of all linear operators on $\mathcal{T}$. Then by Fredholm alternative (3.28) is equivalent to

$$\mathcal{K} = \{K : \quad K = \lambda^* Z, \ \ Z \in \mathrm{Hom}(\mathcal{T}, \mathcal{T})^{\dim \Lambda}\}, \tag{3.30}$$

where $\lambda^* : \mathrm{Hom}(\mathcal{T}, \mathcal{T})^{\dim \Lambda} \to \mathcal{Y}$ is the adjoint map of λ. It is easy to calculate that

$$\lambda^* Z = \sum_{i=1}^{\dim \Lambda} \left(Z_i \Lambda_i + \Lambda_i Z_i^T \right). \tag{3.31}$$

Finally, (3.27), (3.31) imply that for any $\{L_1, L_2\} \subset \mathcal{L}$ and for any $K \in \mathcal{K}$

$$L_1 K L_2 = 0. \tag{3.32}$$

Thus an invariant subspace $\mathcal{L}$ given by (3.27) satisfies (3.13). □

If the subspace $\mathcal{L}$ in the above theorem is non-trivial then so is the annihilator of $\mathcal{L}$ that contains at least Λ. Thus the theorem produces pairs of mutually annihilating subspaces satisfying (3.13). In section 5 summary items 1(a) and 3(b) form such a pair.

4 An example: conductivity.

In the case of conductivity we have

$$\mathcal{T} = \mathbb{R}^d, \quad \Theta(\boldsymbol{R}) = \boldsymbol{R}, \quad \Gamma = e_1 \otimes e_1, \quad \mathcal{Y} = \mathrm{Sym}(\mathbb{R}^d), \quad S_0 = s_0 I. \tag{4.1}$$

Therefore,

$$W_0 = \frac{1}{s_0} I + \frac{e_1 \otimes e_1}{s_0(s_0 - 1)}. \tag{4.2}$$

Applying the conditions (3.13) for this particular setting we obtain that e_1 must be an eigenvalue of all matrices $L_i K_r L_j + L_j K_r L_i$.

It will follow from general results of part II of this paper that

$$\mathcal{Y} = \{tI : t \in \mathbb{R}\} \oplus \{A : \mathbf{Tr} A = 0\} \tag{4.3}$$

is the required decomposition of $\mathcal{Y}$ into a direct sum of $\mathcal{G}$-invariant irreducible subspaces. In dimensions 2 and 3 this fact follows from the well-known form of isotropic Hooke's law: κ times the projection onto the scalar matrices plus μ times the projection onto the trace-free matrices. Shur's lemma then implies the decomposition (4.3). Thus we have just two choices for $\mathcal{L}$.

Let

$$\mathcal{L} = \{tI : t \in \mathbb{R}\}. \tag{4.4}$$

Then for an exact relation to hold we need to require that e_1 be an eigenvector for K_r, i.e. for any trace-free matrix. This is clearly false in any space dimension.

Let

$$\mathcal{L} = \{A : \mathbf{Tr} A = 0\}. \tag{4.5}$$

Then for an exact relation to hold we need to require that

$$e_1 \text{ is an eigenvector for } L_i L_j + L_j L_i. \tag{4.6}$$

If $d>2$ then we can easily choose two different trace free matrices L_1 and L_2 such that $\mathbf{Tr}(L_1L_2)=0$ that violate (4.6). In particular, if $d=3$ we can take

$$L_1=\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\quad L_2=\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}. \tag{4.7}$$

Then

$$L_1L_2+L_2L_1=\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}. \tag{4.8}$$

For $d>3$ we can take the two matrices L_1' and L_2' that contain L_1 and L_2 as their upper left 3x3 blocks with the rest of the elements zero. Then $L_1'L_2'$ will contain (4.8) as its upper left 3x3 block with the rest of the elements zero.

If $d=2$ then it is easy to check that $\mathcal{L}$ given by (4.5) satisfies (4.6). This means that for $d=2$ the subspace $\mathcal{L}$ given by (4.5) generates a manifold corresponding to the exact relation for effective moduli of laminates. The equation of this manifold is

$$\mathbf{Tr}(W_0^{-1}W(S)W_0^{-1})=\mathbf{Tr}(W_0^{-1}) \tag{4.9}$$

A simple Maple calculation then gives the well-known result:

$$\det\sigma^*=\text{constant}. \tag{4.10}$$

5 An example: two-dimensional elasticity.

In the case of 2-d elasticity we have

$$\mathcal{T}=\mathrm{Sym}(\mathbb{R}^2),\quad \mathcal{Y}=\mathrm{Sym}(\mathcal{T}). \tag{5.1}$$

For any $\boldsymbol{A}\in\mathcal{T}$ we let

$$\Theta(\boldsymbol{R})\boldsymbol{A}=\boldsymbol{RAR}^T. \tag{5.2}$$

It will be convenient to represent elements of $\mathcal{Y}$ by symmetric 3x3 matrices in the following orthonormal basis of $\mathcal{T}$ [1, 10, 12]:

$$\mathbf{B_1}=\frac{1}{\sqrt{2}}\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \mathbf{B_2}=\frac{1}{\sqrt{2}}\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad \mathbf{B_3}=\frac{1}{\sqrt{2}}\begin{bmatrix}0&1\\1&0\end{bmatrix}. \tag{5.3}$$

Then

$$S_0=\begin{bmatrix}s_1&0&0\\0&s_2&0\\0&0&s_2\end{bmatrix},\quad \Gamma=\frac{1}{2}\begin{bmatrix}1&1&0\\1&1&0\\0&0&2\end{bmatrix}. \tag{5.4}$$

The decomposition of $\mathcal{Y}$ into the orthogonal sum of irreducible subspaces is easily inferred from [1, formulas (2.24)–(2.26)]:

$$\mathcal{Y}=\mathcal{L}_1\oplus\mathcal{L}_2\oplus\mathcal{L}_3\oplus\mathcal{L}_4, \tag{5.5}$$

where

$$\mathcal{L}_1=\mathrm{Span}(X_1),\quad \mathcal{L}_2=\mathrm{Span}(X_2),\quad \mathcal{L}_3=\mathrm{Span}(X_3,X_4),\quad \mathcal{L}_4=\mathrm{Span}(X_5,X_6), \tag{5.6}$$

and

$$X_1=\begin{bmatrix}\cos t&0&0\\0&\frac{1}{\sqrt{2}}\sin t&0\\0&0&\frac{1}{\sqrt{2}}\sin t\end{bmatrix},\quad X_2=\begin{bmatrix}-\sin t&0&0\\0&\frac{1}{\sqrt{2}}\cos t&0\\0&0&\frac{1}{\sqrt{2}}\cos t\end{bmatrix},$$

$$X_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad X_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad X_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{5.7}$$

Finally the one-dimensional Lie algebra of the representation $\Theta(\boldsymbol{R})$ is spanned by

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{5.8}$$

in the basis (5.3).

Now we are ready to check conditions (3.13). There are 10 choices for $\mathcal{L}$ (since $\mathcal{L}_1$ becomes $\mathcal{L}_2$ when parameter t is replaced by $t + \pi/2$). For each of the ten choices we compute the scalars $p_{\alpha rij} = (A_\alpha, [W_0, L_i K_r L_j])$ using a Maple program. These scalars depend on components s_i of S_0 and a parameter t. We get an exact relation whenever we can choose the values of the parameters to make the array $p_{\alpha rij}$ identically zero. We list the ten arrays computed by Maple in the Appendix. Examining each of the ten tables we easily pick out all those cases (7 in total) for which $p_{\alpha rij} = 0$ for all α, r, i and j. Below we give a summary of what we have discovered. We have translated our results from the S variables to the actual Hooke's laws C by means of (2.7). We will use $\mathbb{H}$ to denote the corresponding manifold in C variables. We still represent elasticity tensors as 3x3 matrices in the basis (5.3).

Summary.

1. dim$\mathbb{M}$= 1
   (a) $\mathcal{L}$ =Span(X_1) for $\sin t = 0$.
   This case corresponds to the result of Hill [6, 7] that an effective Hooke's law of a composite made of components with the same shear modulus has that same shear modulus.
   (b) $\mathcal{L}$ =Span(X_2) with $\sin t = 0$ and $(S_0)_{11} = 1$.
   This relation says that a mixture of isotropic materials with zero bulk modulus is again isotropic with zero bulk modulus.
2. dim$\mathbb{M}$= 2
   (a) $\mathcal{L}$ =Span(X_3, X_4) with $(S_0)_{11} = 1$
   This case describes a family of manifolds

$$\mathbb{H}_t = \{C = c \otimes c + t^2 T | \ c_1^2 = t^2 + c_2^2 + c_3^2\}, \tag{5.9}$$

   where

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.10}$$

   and c_1, c_2 and c_3 are coordinates of the 2 by 2 symmetric matrix c in the basis (5.3). This manifold contains only degenerate Hooke's laws. It is the intersection of items 3(a) and 4 below.
   (b) $\mathcal{L}$ =Span(X_5, X_6)
   Here we have a two-parameter family of manifolds $\mathbb{H}_{\kappa,t}$. The manifolds contain Hooke's laws of square symmetry with a given bulk modulus κ and variable shear moduli μ_1 and μ_2 that satisfy

$$(\frac{1}{\kappa} + \frac{1}{\mu_1})(\frac{1}{\kappa} + \frac{1}{\mu_2}) = t. \tag{5.11}$$

   This case was treated by Lurie and Cherkaev in [10].

3. dim$\mathbb{M}$= 3
   (a) $\mathcal{L}$ =Span(X_1, X_3, X_4) with $\sin t = 0$
   This case describes a one-parameter family of manifolds

$$\mathbb{H}_t = \{C = c \otimes c + tT| \ \ c \in \mathrm{Sym}(\mathbb{R}^2)\}. \tag{5.12}$$

   This exact relation was discussed in [4]. Item 1(a) is contained here as a submanifold.
   (b) $\mathcal{L}$ =Span(X_2, X_5, X_6) with $\sin t = 0$
   Here we have a one-parameter family of manifolds $\mathbb{H}_\kappa$. The manifolds contain Hooke's laws of square symmetry with a given bulk modulus κ. This exact relation was discovered in [12]. Item 2(b) is contained here as a submanifold.
4. dim$\mathbb{M}$= 5, $\mathcal{L}$ =Span$(X_2, X_3, X_4, X_5, X_6)$ with $\sin t = 0$ and $(S_0)_{11} = 1$
   Here we have a single manifold

$$\mathbb{H} = \{C : \ \ \det C = 0\} \tag{5.13}$$

   passing through isotropic tensors with *bulk modulus* equal to zero. This exact relation was described in [4, section 4.3]. Item 1(b) is contained here as a submanifold.

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A Appendix.

Here we give the results of the Maple program computing arrays $p_{\alpha rij} = (A_\alpha, [W_0, L_i K_r L_j])$ for all possible choices of rotationally invariant subspaces $\mathcal{L}$ in the context of Section 5. We display only elements with $i \le j$ since $p_{\alpha rij} = p_{\alpha rji}$. The array elements depend on 3 parameters $s_1 \ge 0$, $s_2 > 0$ and $t \in [0, 2\pi)$. Below each array we list all values of the parameters, if any, that make all array entries vanish.

1. $\mathcal{L} = \mathcal{L}_1$.

$$\begin{aligned}
p_{1,1,1,1} &= 0\\
p_{1,2,1,1} &= 0\\
p_{1,3,1,1} &= -\frac{\cos(t)\sqrt{2}\sin(t)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,4,1,1} &= 0\\
p_{1,5,1,1} &= -\frac{1-\cos(t)^2 - s_1 + s_1\cos(t)^2}{2\,s_1\,s_2{}^2 - 3\,s_1\,s_2 + s_1 - s_2{}^2 + s_2}
\end{aligned}$$

This array becomes zero if $\sin t = 0$, or if $\cos t = 0$ and $s_1 = 1$. The first possibility corresponds to the result of Hill [6, 7], the second—to a new exact relation.

2. $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$.

$$\begin{aligned}
p_{1,1,1,1} &= 0\\
p_{1,1,1,2} &= 0\\
p_{1,1,2,2} &= 0\\
p_{1,2,1,1} &= -\frac{\cos(t)\sqrt{2}\sin(t)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,2,1,2} &= -\frac{1}{2}\,\frac{\sqrt{2}\left(-1+2\cos(t)^2\right)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,2,2,2} &= \frac{\cos(t)\sqrt{2}\sin(t)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,3,1,1} &= 0\\
p_{1,3,1,2} &= 0\\
p_{1,3,2,2} &= 0\\
p_{1,4,1,1} &= -\frac{-s_1 + s_1\cos(t)^2 + 1 - \cos(t)^2}{2\,s_1\,s_2{}^2 - 3\,s_1\,s_2 + s_1 - s_2{}^2 + s_2}\\
p_{1,4,1,2} &= \frac{\sin(t)\cos(t)\,(s_1 - 1)}{2\,s_1\,s_2{}^2 - 3\,s_1\,s_2 + s_1 - s_2{}^2 + s_2}\\
p_{1,4,2,2} &= \frac{\cos(t)^2\,(s_1 - 1)}{2\,s_1\,s_2{}^2 - 3\,s_1\,s_2 + s_1 - s_2{}^2 + s_2}
\end{aligned}$$

This array can never become zero because of the entries (1,2,1,1) and (1,2,1,2).

3. $\mathcal{L} = \mathcal{L}_3$.

$$
\begin{aligned}
p_{1,1,1,1} &= 0 \\
p_{1,1,1,2} &= \frac{\cos(t)\,(s_1 - 1)}{2\,s_1\,s_2^2 - 3\,s_1\,s_2 + s_1 - s_2^2 + s_2} \\
p_{1,1,2,2} &= 0 \\
p_{1,2,1,1} &= 0 \\
p_{1,2,1,2} &= -\frac{\sin(t)\,(s_1 - 1)}{2\,s_1\,s_2^2 - 3\,s_1\,s_2 + s_1 - s_2^2 + s_2} \\
p_{1,2,2,2} &= 0 \\
p_{1,3,1,1} &= 0 \\
p_{1,3,1,2} &= 0 \\
p_{1,3,2,2} &= 0 \\
p_{1,4,1,1} &= 0 \\
p_{1,4,1,2} &= 0 \\
p_{1,4,2,2} &= 0
\end{aligned}
$$

This array becomes zero if $s_1 = 1$. This is the intersection of the two exact relations from [4].

4. $\mathcal{L} = \mathcal{L}_4$.

$$
\begin{aligned}
p_{1,1,1,1} &= 0 \\
p_{1,1,1,2} &= 0 \\
p_{1,1,2,2} &= 0 \\
p_{1,2,1,1} &= 0 \\
p_{1,2,1,2} &= 0 \\
p_{1,2,2,2} &= 0 \\
p_{1,3,1,1} &= 0 \\
p_{1,3,1,2} &= 0 \\
p_{1,3,2,2} &= 0 \\
p_{1,4,1,1} &= 0 \\
p_{1,4,1,2} &= 0 \\
p_{1,4,2,2} &= 0
\end{aligned}
$$

This case corresponds to the exact relation discovered by Lurie and Cherkaev [10].

5. $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_3$.

$$
\begin{aligned}
p_{1,1,1,1} &= 0\\
p_{1,1,1,2} &= 0\\
p_{1,1,1,3} &= \frac{1}{2}\,\frac{\cos(t)\sin(t)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,1,2,2} &= 0\\
p_{1,1,2,3} &= -\frac{\sin(t)\,(s_1 - 1)}{2\,s_1\,s_2^2 - 3\,s_1\,s_2 + s_1 - s_2^2 + s_2}\\
p_{1,1,3,3} &= 0\\
p_{1,2,1,1} &= 0\\
p_{1,2,1,2} &= 0\\
p_{1,2,1,3} &= \frac{1}{2}\,\frac{\sqrt{2}\sin(t)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,2,2,2} &= 0\\
p_{1,2,2,3} &= 0\\
p_{1,2,3,3} &= 0\\
p_{1,3,1,1} &= -\frac{-s_1 + s_1\cos(t)^2 + 1 - \cos(t)^2}{2\,s_1\,s_2^2 - 3\,s_1\,s_2 + s_1 - s_2^2 + s_2}\\
p_{1,3,1,2} &= -\frac{1}{2}\,\frac{\sqrt{2}\sin(t)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,3,1,3} &= 0\\
p_{1,3,2,2} &= 0\\
p_{1,3,2,3} &= 0\\
p_{1,3,3,3} &= 0
\end{aligned}
$$

This array becomes zero if $\sin t = 0$. It corresponds to the exact relation discovered by Grabovsky and Milton [4].

6. $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_4$.

$$
\begin{aligned}
p_{1,1,1,1} &= 0\\
p_{1,1,1,2} &= 0\\
p_{1,1,1,3} &= \frac{\cos(t)\sin(t)\,(s_1 - 1)}{2\,s_1\,s_2^2 - 3\,s_1\,s_2 + s_1 - s_2^2 + s_2}\\
p_{1,1,2,2} &= 0\\
p_{1,1,2,3} &= 0\\
p_{1,1,3,3} &= 0\\
p_{1,2,1,1} &= 0\\
p_{1,2,1,2} &= 0\\
p_{1,2,1,3} &= -\frac{\cos(t)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,2,2,2} &= 0\\
p_{1,2,2,3} &= 0\\
p_{1,2,3,3} &= 0\\
p_{1,3,1,1} &= -\frac{\cos(t)\sqrt{2}\sin(t)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,3,1,2} &= \frac{\cos(t)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,3,1,3} &= 0\\
p_{1,3,2,2} &= 0\\
p_{1,3,2,3} &= 0\\
p_{1,3,3,3} &= 0
\end{aligned}
$$

This array becomes zero if $\cos t = 0$. It corresponds to the exact relation of Lurie, Cherkaev and Fedorov [12].

7. $\mathcal{L} = \mathcal{L}_3 \oplus \mathcal{L}_4$.

$$
\begin{aligned}
p_{1,1,1,1} &= 0\\
p_{1,1,1,2} &= \frac{\cos(t)\,(s_1-1)}{2\,s_1\,s_2^2-3\,s_1\,s_2+s_1-s_2^2+s_2}\\
p_{1,1,1,3} &= 0\\
p_{1,1,1,4} &= -\frac{1}{2}\,\frac{\sqrt{2}\sin(t)}{2\,s_1\,s_2-s_1-s_2}\\
p_{1,1,2,2} &= 0\\
p_{1,1,2,3} &= \frac{1}{2}\,\frac{\sqrt{2}\sin(t)}{2\,s_1\,s_2-s_1-s_2}\\
p_{1,1,2,4} &= 0\\
p_{1,1,3,3} &= 0\\
p_{1,1,3,4} &= 0\\
p_{1,1,4,4} &= 0\\
p_{1,2,1,1} &= 0\\
p_{1,2,1,2} &= -\frac{\sin(t)\,(s_1-1)}{2\,s_1\,s_2^2-3\,s_1\,s_2+s_1-s_2^2+s_2}\\
p_{1,2,1,3} &= 0\\
p_{1,2,1,4} &= -\frac{1}{2}\,\frac{\sqrt{2}\cos(t)}{2\,s_1\,s_2-s_1-s_2}\\
p_{1,2,2,2} &= 0\\
p_{1,2,2,3} &= \frac{1}{2}\,\frac{\sqrt{2}\cos(t)}{2\,s_1\,s_2-s_1-s_2}\\
p_{1,2,2,4} &= 0\\
p_{1,2,3,3} &= 0\\
p_{1,2,3,4} &= 0\\
p_{1,2,4,4} &= 0
\end{aligned}
$$

This array can never be zero because of the entries (1,1,2,3) and (1,2,1,4).

8. $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4$.

$$
\begin{aligned}
p_{1,1,1,1} &= 0\\
p_{1,1,1,2} &= 0\\
p_{1,1,1,3} &= \frac{1}{2}\,\frac{\cos(t)\sin(t)}{2\,s_1\,s_2-s_1-s_2}\\
p_{1,1,1,4} &= 0\\
p_{1,1,1,5} &= \frac{\cos(t)\sin(t)\,(s_1-1)}{2\,s_1\,s_2^2-3\,s_1\,s_2+s_1-s_2^2+s_2}\\
p_{1,1,2,2} &= 0\\
p_{1,1,2,3} &= -\frac{\sin(t)\,(s_1-1)}{2\,s_1\,s_2^2-3\,s_1\,s_2+s_1-s_2^2+s_2}\\
p_{1,1,2,4} &= 0\\
p_{1,1,2,5} &= -\frac{1}{2}\,\frac{\sqrt{2}\cos(t)}{2\,s_1\,s_2-s_1-s_2}\\
p_{1,1,3,3} &= 0\\
p_{1,1,3,4} &= \frac{1}{2}\,\frac{\sqrt{2}\cos(t)}{2\,s_1\,s_2-s_1-s_2}\\
p_{1,1,3,5} &= 0\\
p_{1,1,4,4} &= 0\\
p_{1,1,4,5} &= 0\\
p_{1,1,5,5} &= 0
\end{aligned}
$$

This array is zero if $\cos t = 0$ and $s_1 = 1$. This case corresponds to the exact relation described in [4, section 4.3].

9. $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$.

$$
\begin{aligned}
p_{1,1,1,1} &= 0\\
p_{1,1,1,2} &= 0\\
p_{1,1,1,3} &= 0\\
p_{1,1,1,4} &= \frac{1}{2}\,\frac{\sqrt{2}\sin(t)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,1,2,2} &= 0\\
p_{1,1,2,3} &= 0\\
p_{1,1,2,4} &= \frac{1}{2}\,\frac{\sqrt{2}\cos(t)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,1,3,3} &= 0\\
p_{1,1,3,4} &= 0\\
p_{1,1,4,4} &= 0\\
p_{1,2,1,1} &= -\frac{-s_1 + s_1\cos(t)^2 + 1 - \cos(t)^2}{2\,s_1\,{s_2}^2 - 3\,s_1\,s_2 + s_1 - {s_2}^2 + s_2}\\
p_{1,2,1,2} &= \frac{\cos(t)\sin(t)\,(s_1 - 1)}{2\,s_1\,{s_2}^2 - 3\,s_1\,s_2 + s_1 - {s_2}^2 + s_2}\\
p_{1,2,1,3} &= -\frac{1}{2}\,\frac{\sqrt{2}\sin(t)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,2,1,4} &= 0\\
p_{1,2,2,2} &= \frac{\cos(t)^2\,(s_1 - 1)}{2\,s_1\,{s_2}^2 - 3\,s_1\,s_2 + s_1 - {s_2}^2 + s_2}\\
p_{1,2,2,3} &= -\frac{1}{2}\,\frac{\sqrt{2}\cos(t)}{2\,s_1\,s_2 - s_1 - s_2}\\
p_{1,2,2,4} &= 0\\
p_{1,2,3,3} &= 0\\
p_{1,2,3,4} &= 0\\
p_{1,2,4,4} &= 0
\end{aligned}
$$

This array can never be zero because of the entries (1,1,1,4) and (1,1,2,4).

10. $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_4$.

$$
\begin{aligned}
p_{1,1,1,1} &= 0 \\
p_{1,1,1,2} &= 0 \\
p_{1,1,1,3} &= 0 \\
p_{1,1,1,4} &= -\frac{\cos(t)}{2\,s_1\,s_2 - s_1 - s_2} \\
p_{1,1,2,2} &= 0 \\
p_{1,1,2,3} &= 0 \\
p_{1,1,2,4} &= \frac{\sin(t)}{2\,s_1\,s_2 - s_1 - s_2} \\
p_{1,1,3,3} &= 0 \\
p_{1,1,3,4} &= 0 \\
p_{1,1,4,4} &= 0 \\
p_{1,2,1,1} &= -\frac{\cos(t)\sqrt{2}\sin(t)}{2\,s_1\,s_2 - s_1 - s_2} \\
p_{1,2,1,2} &= -\frac{1}{2}\,\frac{\sqrt{2}\left(-1 + 2\cos(t)^2\right)}{2\,s_1\,s_2 - s_1 - s_2} \\
p_{1,2,1,3} &= \frac{\cos(t)}{2\,s_1\,s_2 - s_1 - s_2} \\
p_{1,2,1,4} &= 0 \\
p_{1,2,2,2} &= \frac{\cos(t)\sqrt{2}\sin(t)}{2\,s_1\,s_2 - s_1 - s_2} \\
p_{1,2,2,3} &= -\frac{\sin(t)}{2\,s_1\,s_2 - s_1 - s_2} \\
p_{1,2,2,4} &= 0 \\
p_{1,2,3,3} &= 0 \\
p_{1,2,3,4} &= 0 \\
p_{1,2,4,4} &= 0
\end{aligned}
$$

This array can never be zero because of the entries (1,1,1,4) and (1,1,2,4).