On the commutation properties of finite convolution and differential operators I: commutation.

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Abstract

The commutation relation $KL = LK$ between finite convolution integral operator K and differential operator L has implications for spectral properties of K. We characterize all operators K admitting this commutation relation. Our analysis places no symmetry constraints on the kernel of K extending the well-known results of Morrison for real self-adjoint finite convolution integral operators.

1 Introduction

The need to understand spectral properties of finite convolution integral operators

$$(Ku)(x) = \int_{-1}^{1} k(x-y)u(y)\mathrm{d}y \tag{1.1}$$

acting on $L^2(-1,1)$ arises in a number of applications, including optics [6], radio astronomy [3], [4], electron microscopy [8], x-ray tomography [11], [23], noise theory [5] and medical imaging [2], [12], [13], [14]. In some cases it is possible to find a differential operator L which commutes with K (cf. [20, 18, 24, 12]),

$$KL = LK. \tag{C1}$$

If $k(z)$ is smooth the eigenfunctions of K also have to be smooth and hence can be chosen to be solutions of ordinary differential equations. More precisely, (C1) implies that eigenspaces E_λ of K are invariant under L, i.e. $L : E_\lambda \mapsto E_\lambda$. Now if L is diagonalizable, e.g. self-adjoint, or more generally, normal (for characterization of normality see Remark 7), then one can choose a basis for E_λ consisting of eigenfunctions of L. This permits to bring the vast literature on asymptotic properties of solutions of ordinary differential equations to bear on obtaining analytical information about the asymptotics of eigenvalues and eigenfunctions of integral operators. With this said, we will not be investigating spectral properties of differential operators that commute with integral operators. In our view questions about differential operators are much more tractable than questions about the integral operators, see e.g. [26], and our goal is to find all connections between the two questions.

The most famous example of this phenomenon is the band-and time limited operator of Landau, Pollak, and Slepian [16], [17], [20]–[22], corresponding to $k(z) = \frac{\sin(az)}{z}$ in (1.1) with $a > 0$. Sharp estimates for asymptotics of the eigenvalues of K were derived using its commutation with a second order symmetric differential operator, whose eigenfunctions are the well-known prolate spheroidal wave functions that first appeared in the context of quantum mechanics [19]. Another example is the result of Widom [24], where using comparison with special operators that commute with differential operators, the author obtained asymptotic behavior of the eigenvalues of a large class of integral operators with real-valued even kernels. A complete characterization of operators (1.1) with real even kernel commuting with symmetric second order differential operators was achieved by Morrison [18] (see also [25], [10]). We are interested in completing Morrison's characterization to include all complex-valued kernels $k(z)$. In this more general context the property of commutation must also be generalized, so as to permit the characterization of eigenfunctions as solutions of an eigenvalue problem for a second or fourth order differential operator.

A natural extension of commutation, as explained in the introductory section in [1] is

$$\begin{cases} KL_1 = L_2K \\ L_j^* = L_j, \qquad j = 1, 2 \end{cases}, \tag{C2}$$

where $L_j,\ j = 1, 2$ are differential operators with complex coefficients. This has implications for singular value decomposition of K. It is easy to check that (C2) reduces to a commutation relation for K^*K, indeed we have

$$L_1K^*K = K^*KL_1, \tag{1.2}$$

and therefore singular functions of K satisfy ODEs, in the sense explained above. In fact, commuting pairs (K, L), when K is non-compact can also provide instances where singular value decomposition of a related operator can be obtained via (C2), as was observed in [2], [12], [13], [14] in applications to truncated Hilbert transform operators, where $k(z) = 1/z$. In this setting the input function is considered on one interval while the output of K is defined on a different interval. Even though singularity of $k(z)$ destroys compactness of K, in was shown in the above cited papers that K^*K possesses a discrete spectrum and singular value decomposition for K can be obtained.

In this paper we give a complete list of pairs (K, L), satisfying commutation relation (C1), under the assumption that k is either analytic at the origin or has a simple[1] pole at 0, in which case the integral is understood in the principal value sense (cf. Theorem 1). Among the most important consequences of this characterization we mention two. The first is that any finite convolution operator K, with analytic kernel at the origin, admitting commutation must be similar to Morrison's operator (cf. Remark 4). The second is the complete description of singular kernels $k(z)$ for which K^*K in the sense described above has a discrete spectrum. As an example of one new result is the operator with kernel $k(z) = 1/\sin\left(\frac{\pi}{8}z\right)$ considered from $L^2(-1, 1) \to L^2(3, 5)$ and the operator $k(z) = \beta + \frac{1}{z}$, with $\beta \in i\mathbb{R}$, considered from $L^2(-1, 1) \to L^2(-b, b)$ (for any $b \neq 1$), both have discrete singular value decompositions (see Remark 10 for details and more examples).

The fact that aside from Morrison's class of compact self-adjoint finite convolution operators and their conjugates there are no essentially new examples is remedied in the second

[1] It is not hard to show that commutation is not possible for higher order poles.

part of this work [9], where we consider a new kind of commutation relation

$$\overline{K}L_1 = L_2K, \qquad K^* = K, \quad L_j^T = L_j,\ j = 1, 2, \tag{1.3}$$

which we call *sesquicommutation*. In this case we are able to prove that no nontrivial cases arise unless $L_1 = L_2 = L$. Moreover, the eigenspaces of the compact self-adjoint finite convolution operator K are invariant under the self-adjoint 4th order differential operator L^*L. To give one explicit example of the new such pair (K, L) obtained in [9] we define

$$k(z) = \frac{e^{-i\frac{\pi}{4}z}}{\cos\frac{\pi}{4}z} + \frac{ze^{i\frac{\pi}{4}z}}{\sin\frac{\pi}{2}z}, \qquad L = -\frac{\mathrm{d}}{\mathrm{d}y}\left[\cos\left(\tfrac{\pi y}{2}\right)\frac{\mathrm{d}}{\mathrm{d}y}\right] + \tfrac{\pi^2}{32}e^{i\frac{\pi y}{2}}. \tag{1.4}$$

2 Preliminaries

We assume that $zk(z) \in L^2((-2, 2), \mathbb{C})$ is analytic in a neighborhood of 0. This includes two cases: regular, when k is analytic at 0, and singular, when k has a simple pole at 0, in which case the integral is understood in the principal value sense. Further, assume that L, L_j are second order differential operators:

$$\begin{cases} Lu = \mathcal{a}u'' + \mathcal{b}u' + \mathcal{c}u, \\ \mathcal{a}(\pm 1) = 0,\ \mathcal{b}(\pm 1) = \mathcal{a}'(\pm 1), \end{cases} \tag{2.1}$$

where the indicated boundary conditions are necessary for the above commutation relations to hold. These are also necessary for the adjoint operator to be a differential operator as well. Thus various classes of operators, such as self-adjoint, symmetric or normal can be described by specifying additional constraints on the coefficients of L, always assuming that the boundary conditions in (2.1) hold.

When k is smooth in $[-2, 2]$, formulating commutation relations (C1) and (C2) in terms of the kernel $k(z)$ and the coefficients of L is a matter of integration by parts, which due to the imposed boundary conditions lead, respectively, to

$$\begin{aligned}[] [\mathcal{a}(y+z) - \mathcal{a}(y)]k''(z) + [2\mathcal{a}'(y) + \mathcal{b}(y+z) - \mathcal{b}(y)]k'(z) + \\ + [\mathcal{c}(y+z) - \mathcal{c}(y) + \mathcal{b}'(y) - \mathcal{a}''(y)]k(z) = 0, \end{aligned} \tag{R1}$$

$$\begin{aligned}[] [\mathcal{a}_2(y+z) - \mathcal{a}_1(y)]k''(z) + [2\mathcal{a}_1'(y) + \mathcal{b}_2(y+z) - \mathcal{b}_1(y)]k'(z) + \\ + [\mathcal{c}_2(y+z) - \mathcal{c}_1(y) + \mathcal{b}_1'(y) - \mathcal{a}_1''(y)]k(z) = 0, \end{aligned} \tag{R2}$$

where $\mathcal{a}_j, \mathcal{b}_j, \mathcal{c}_j$ denote the coefficients of L_j for $j = 1, 2$. Less obviously (see Remark 6), the same relation (R1) holds if k has a simple pole at 0.

The main idea of the proofs is to analyze these relations by taking sufficient number of derivatives in z and evaluating the result at $z = 0$. This allows one to find linear differential relations between the coefficients of the differential operators, narrowing down the set of possibilities to families of functions depending on finitely many parameters. Returning to the original relations (R1), (R2) we obtain necessary and sufficient conditions for commutation that can be completely analyzed, resulting in the explicit listing of all pairs (k, L) satisfying (R1).

Remark 1. The complete analysis of (C2) beyond the instances generated by (C1), can also be achieved by our approach, but will require substantially more work. We remark that in this case too it can be shown that either k is trivial or the coefficients of L_1 and L_2 are linear combinations of polynomials multiplied by exponentials.

3 Main Results

Definition 1. We will say that k (or operator K) is *trivial*, if it is a finite linear combination of exponentials $e^{\alpha z}$ or has the form $e^{\alpha z}p(z)$, where $p(z)$ is a polynomial. Note that in this case K is a finite-rank operator.

Remark 2. When K commutes with L, then MKM^{-1} commutes with MLM^{-1}. If M is the multiplication operator by $z \mapsto e^{\tau z}$, then MKM^{-1} is a finite convolution operator with kernel $k(z)e^{\tau z}$ (where k is the kernel of K) and MLM^{-1} is a second order differential operator with the same leading coefficient as L. Moreover, one can also add any complex constant to $\mathscr{c}(y)$ in (2.1), as well as multiply k, as well as L by arbitrary complex constants without affecting commutation. With this observation the results of Theorem 1 are stated up to such transformations in order to achieve the most concise form of the results.

In theorem below all parameters are complex, unless specified otherwise.

Theorem 1 (Commutation (C1))
Let K, L be given by (1.1) and (2.1) with $\mathscr{a}, \mathscr{b}, \mathscr{c}$ smooth in $[-2,2]$. Assume k is smooth in $[-2,2]\backslash\{0\}$ and either it

(i) is analytic at 0, not identically zero near 0 and is nontrivial in the sense of Definition 1.

(ii) has a simple pole at 0.

If (R1) holds, then (in case λ or $\mu = 0$ appropriate limits must be taken)

$$k(z) = \frac{\lambda}{\sinh\left(\frac{\lambda}{2}z\right)}\left(\alpha_1 \frac{\sinh(\mu z)}{\mu} + \alpha_2 \cosh(\mu z)\right) \tag{3.1}$$

$$\begin{cases} \mathscr{a}(y) = \frac{1}{\lambda^2}\left[\cosh(\lambda y) - \cosh\lambda\right] \\ \mathscr{b}(y) = \mathscr{a}'(y) \\ \mathscr{c}(y) = \left(\frac{\lambda^2}{4} - \mu^2\right)\mathscr{a}(y) \end{cases} \tag{3.2}$$

For some special choices of parameters, the differential operator commuting with K is more general than the one given by (3.2). Below we list all such cases:

1. $\alpha_1 = 0,\ \lambda = \pi i,\ \mu = \frac{2m+1}{4}\lambda$ with $m \in \mathbb{Z}$:

$$k(z) = \frac{\cos\left(\frac{\pi(2m+1)}{4}z\right)}{\sin\left(\frac{\pi}{2}z\right)} \qquad \text{and} \qquad \begin{cases} \mathscr{a}(y) = \alpha\left(e^{\pi i y} - e^{\pi i}\right) + \beta\left(e^{-\pi i y} - e^{-\pi i}\right) \\ \mathscr{b}(y) = \mathscr{a}'(y) \\ \mathscr{c}(y) = \frac{\pi^2}{4}\left[\frac{(2m+1)^2}{4} - 1\right]\mathscr{a}(y) \end{cases}$$

When $\alpha = \beta$ (3.2) is recovered.

2. $\alpha_1 = \mu = 0$, *then with* $\mathscr{a}_0(y) = \cosh(\lambda y) - \cosh\lambda$:

$$k(z) = \frac{1}{\sinh\left(\frac{\lambda}{2}z\right)} \qquad \textit{and} \qquad \begin{cases} \mathscr{a}(y) = \alpha\mathscr{a}_0(y) \\ \mathscr{b}(y) = \alpha\mathscr{a}_0'(y) + \beta\mathscr{a}_0(y) \\ \mathscr{c}(y) = \frac{\beta}{2}\mathscr{a}_0'(y) + \alpha\frac{\lambda^2}{4}\mathscr{a}_0(y) \end{cases}$$

When $\beta = 0$ (3.2) is recovered.

3. $\mu = \lambda = 0$, *then with $\mathscr{p}(y)$ an arbitrary polynomial of order at most two such that $\mathscr{p}'(0) = 0$:*

$$k(z) = \frac{1}{\beta} + \frac{1}{z} \qquad \textit{and} \qquad \begin{cases} \mathscr{a}(y) = (y^2 - 1)\mathscr{p}(y) \\ \mathscr{b}(y) = \mathscr{a}'(y) + \beta y\mathscr{p}'(y) - \beta\mathscr{p}''(y) \\ \mathscr{c}(y) = \beta\mathscr{p}'(y) \end{cases}$$

When $\mathscr{p}(y) \equiv 1$ (3.2) is recovered.

4. $\mu = \lambda = \alpha_1 = 0$, *then with $\mathscr{p}(y)$ an arbitrary polynomial of order at most two:*

$$k(z) = \frac{1}{z} \qquad \textit{and} \qquad \begin{cases} \mathscr{a}(y) = (y^2 - 1)\mathscr{p}(y) \\ \mathscr{b}(y) = a'(y) + \beta(y^2 - 1) \\ \mathscr{c}(y) = y\mathscr{p}'(y) + \beta y \end{cases}$$

When $\mathscr{p}(y) \equiv 1$ and $\beta = 0$ (3.2) is recovered.

Remark 3. If $\lambda \in i\mathbb{R}$, then $k(z)$ may become singular at $z \in [-2, 2]\backslash\{0\}$. In order to exclude these cases we need to require either

- $|\lambda| < \pi$, or
- $\pi \leq |\lambda| < 2\pi$ and $\alpha_1 = 0$, $\mu = \lambda\frac{2m+1}{4}$ for some $m \in \mathbb{Z}$

Remark 4. Morrison's result corresponds to the analytic case: $\alpha_2 = 0$ and when k is even and real-valued. According to Theorem 1 all integral operators with analytic $k(z)$ that commute with a differential operator are similar to Morrison's operator and therefore their spectrum can be determined using Morrison's results.

Remark 5. As we have already mentioned, the connections between the coefficient functions of the differential operators are obtained by differentiating the relation (R1) appropriate number of times and setting $z = 0$. Smoothness of coefficients, analyticity of k at zero (the fact that k is nontrivial and that it doesn't vanish near 0) are used at this stage, to argue that the differentiation procedure can be terminated at some point and the connections between the coefficient functions will follow. Thus, the original assumptions can be replaced by requiring appropriate degree of smoothness on k and the coefficient functions and that some expressions involving $k^{(j)}(0)$ are not zero. These expressions can be easily found from our analysis. For example the hypotheses of Theorem 1 (case (i)) can be replaced by $\mathscr{a}, \mathscr{b}, \mathscr{c}, k \in C^3$ and $k^2(0)k''(0) - k(0)k'(0) \neq 0$ (cf. Section 4). Analogous changes can be made in case (ii) of Theorem 1.

Remark 6. When k has a pole at zero, the commutation is understood in the principal value sense, namely

$$\lim_{\epsilon \to 0} \int_{[-1,1] \setminus B_\epsilon(x)} k(x-y) Lu(y) \mathrm{d}y - L \int_{[-1,1] \setminus B_\epsilon(x)} k(x-y) u(y) \mathrm{d}y = 0.$$

After integrating by parts, this can be rewritten as

$$\lim_{\epsilon \to 0} \int_{[-1,1] \setminus B_\epsilon(x)} F(x,y) u(y) \mathrm{d}y + \Phi(u,x,\epsilon) = 0,$$

where $F(x,y)$ is the left-hand side of (R1) with $z = x - y$ and

$$\begin{aligned} \Phi(u,x,\epsilon) = & k(\epsilon)\Big\{ \big[\mathscr{a}(x-\epsilon) - \mathscr{a}(x)\big] u'(x-\epsilon) + \big[\mathscr{b}(x-\epsilon) - \mathscr{b}(x) - \mathscr{a}'(x-\epsilon)\big] u(x-\epsilon) \Big\} - \\ & - k(-\epsilon)\Big\{ \big[\mathscr{a}(x+\epsilon) - \mathscr{a}(x)\big] u'(x+\epsilon) + \big[\mathscr{b}(x+\epsilon) - \mathscr{b}(x) - \mathscr{a}'(x+\epsilon)\big] u(x+\epsilon) \Big\} + \\ & + k'(\epsilon) u(x-\epsilon) \big[\mathscr{a}(x-\epsilon) - \mathscr{a}(x)\big] - k'(-\epsilon) u(x+\epsilon) \big[\mathscr{a}(x+\epsilon) - \mathscr{a}(x)\big]. \end{aligned}$$

Expanding $\Phi(u,x,\epsilon)$ in ϵ we observe that all terms up to $O(\epsilon)$ cancel out and hence, $\lim_{\epsilon \to 0} \Phi(u,x,\epsilon) = 0$. Therefore we conclude $F(x,y) = 0$ for $y \neq x$, resulting in the same relation (R1), as in smooth case.

Remark 7. As was discussed in the introduction one might want to check whether L (given by (2.1)) is normal: $LL^* = L^*L$. Recall that

$$L^* u = \overline{\mathscr{a}} u'' + (2\overline{\mathscr{a}}' - \overline{\mathscr{b}}) u' + (\overline{\mathscr{a}}'' - \overline{\mathscr{b}}' + \overline{\mathscr{c}}) u,$$

therefore we find

$$L = L^* \quad \Longleftrightarrow \quad \operatorname{Im} \mathscr{a} = 0, \quad \operatorname{Re} \mathscr{b} = \mathscr{a}' \quad \text{and} \quad \operatorname{Im} \mathscr{c} = \tfrac{1}{2} \operatorname{Im} \mathscr{b}'.$$

To analyze the normality relation, we first give the conditions for commutation of L with another differential operator $Du = \mathscr{A} u'' + \mathscr{B} u' + \mathscr{C} u$, assuming $\mathscr{a} \neq 0$. One can find that

$$\begin{aligned} LDu = & \mathscr{a}\mathscr{A} u^{(4)} + [\mathscr{a}(2\mathscr{A}' + \mathscr{B}) + \mathscr{b}\mathscr{A}]\, u^{(3)} + [\mathscr{a}(\mathscr{A}'' + 2\mathscr{B}' + \mathscr{C}) + \mathscr{b}(\mathscr{A}' + \mathscr{B}) + \mathscr{c}\mathscr{A}]\, u'' + \\ & + [\mathscr{a}(\mathscr{B}'' + 2\mathscr{C}') + \mathscr{b}(\mathscr{B}' + \mathscr{C}) + \mathscr{c}\mathscr{B}]\, u' + [\mathscr{a}\mathscr{C}'' + \mathscr{b}\mathscr{C}' + \mathscr{c}\mathscr{C}]\, u. \end{aligned}$$

Comparing this with an analogous expression for DLu and equating the coefficients of corresponding derivatives of u we obtain that $LD = DL$ is equivalent to

$$
\begin{cases}
\mathscr{a}\mathscr{A}' = \mathscr{A}\mathscr{a}' \\
2\mathscr{a}\mathscr{B}' + \mathscr{b}\mathscr{A}' = 2\mathscr{A}\mathscr{b}' + \mathscr{B}\mathscr{a}' \\
\mathscr{a}\mathscr{B}'' + 2\mathscr{a}\mathscr{C}' + \mathscr{b}\mathscr{B}' = \mathscr{A}\mathscr{b}'' + 2\mathscr{A}\mathscr{c}' + \mathscr{B}\mathscr{b}' \\
\mathscr{a}\mathscr{C}'' + \mathscr{b}\mathscr{C}' = \mathscr{A}\mathscr{c}'' + \mathscr{B}\mathscr{c}'
\end{cases}
\tag{3.3}
$$

The first equation of (3.3) implies $\mathscr{A} = \alpha\mathscr{a}$ for some $\alpha \in \mathbb{C}$. Using this in the second equation of (3.3) we get $\beta\mathscr{a} = (\mathscr{B} - \alpha\mathscr{b})^2$ for some $\beta \in \mathbb{C}$. The third relation reads

$$
\begin{aligned}
\mathscr{C}' = \alpha\mathscr{c}' - \tfrac{1}{2}(\mathscr{B}'' - \alpha\mathscr{b}'') + \frac{\mathscr{B}\mathscr{b}' - \mathscr{b}\mathscr{B}'}{2\mathscr{a}} = \\
= \alpha\mathscr{c}' + \frac{\beta}{2}\frac{\left(\mathscr{b}' - \frac{\mathscr{a}''}{2}\right)(\mathscr{B} - \alpha\mathscr{b}) - \left(\mathscr{b} - \frac{\mathscr{a}'}{2}\right)(\mathscr{B}' - \alpha\mathscr{b}')}{(\mathscr{B} - \alpha\mathscr{b})^2},
\end{aligned}
$$

where in the last step we used the identity $2\mathscr{a}(\mathscr{B}'' - \alpha\mathscr{b}'') = \mathscr{a}''(\mathscr{B} - \alpha\mathscr{b}) - \mathscr{a}'(\mathscr{B}' - \alpha\mathscr{b}')$. Integrating, we find $\mathscr{C} = \alpha\mathscr{c} + \frac{1}{2}f + \text{const}$, where

$$
f = \frac{\beta}{2}\,\frac{2\mathscr{b} - \mathscr{a}'}{\mathscr{B} - \alpha\mathscr{b}}.
$$

When $\mathscr{B} = \alpha\mathscr{b}$, then $\beta = 0$ and by convention we assume $f = 0$. Finally, substituting the expression for $\mathscr{C}$, the fourth equation of (3.3) can be simplified to

$$
2\beta\mathscr{c}' = (\mathscr{B} - \alpha\mathscr{b})f'' + \frac{\beta\mathscr{b}}{\mathscr{B} - \alpha\mathscr{b}}f' = [(\mathscr{B} - \alpha\mathscr{b})f']' + ff'.
$$

Now we integrate the last relation and putting everything together we conclude

$$
LD = DL \quad \Longleftrightarrow \quad
\begin{cases}
\mathscr{A} & = \alpha\mathscr{a}, \\
\beta\mathscr{a} & = (\mathscr{B} - \alpha\mathscr{b})^2, \\
\mathscr{C} & = \alpha\mathscr{c} + \frac{1}{2}f + \text{const}, \\
2\beta\mathscr{c} & = (\mathscr{B} - \alpha\mathscr{b})f' + \frac{1}{2}f^2 + \text{const}.
\end{cases}
$$

Write $L = L_0 + L_1$, where $2L_0 = L + L_*$ is self-adjoint and $2L_1 = L - L_*$ is skew-adjoint. Clearly L is normal, if and only if L_0 commutes with L_1. The coefficient of $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ in L_0 is $\operatorname{Re}\mathscr{a}$ and in L_1 is $i\operatorname{Im}\mathscr{a}$. The first equation for commutation of L_0, L_1 implies $\operatorname{Im}\mathscr{a} = \alpha\operatorname{Re}\mathscr{a}$ for some $\alpha \in \mathbb{R}$. W.l.o.g. we may take $\alpha = 0$. Indeed, L is normal if and only if $\tilde{L} = (1 - i\alpha)L$ is normal. Now the coefficient of $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ in $\tilde{L}_1$ is $\frac{1}{2}[(1 - i\alpha)\mathscr{a} - (1 + i\alpha)\overline{\mathscr{a}}] = 0$. Thus, w.l.o.g. $L = L_0 + L_1$ where L_0 is a second order self-adjoint operator and L_1 is of first order and skew-adjoint. Simplifying commutation relations for L_0, L_1 we find

$$LL^* = L^*L \quad \text{and} \quad L \neq L^*, \qquad \text{iff}$$

$$\begin{cases} L = L_0 + \gamma L_1, \quad \gamma \in \mathbb{R}\backslash\{0\}, \\ L_0 u = \mathscr{a} u'' + \mathscr{b}_0 u' + \mathscr{c}_0 u, \\ L_1 u = \mathscr{b}_1 u' + \mathscr{c}_1 u, \end{cases} \qquad \text{and} \qquad \begin{cases} \mathscr{a} \in \mathbb{R} \quad \text{and w.l.o.g. } \mathscr{a} > 0, \\ \mathscr{b}_1 = \sqrt{\mathscr{a}}, \\ \mathscr{c}_1 = \dfrac{2\mathscr{b}_0 - \mathscr{a}'}{\sqrt{\mathscr{a}}} + i\mathbb{R}, \\ \operatorname{Re} \mathscr{b}_0 = \mathscr{a}', \\ 4\mathscr{c}_0 = 2\mathscr{b}_0' - \mathscr{a}'' + \dfrac{(\mathscr{a}' - 2\mathscr{b}_0)(3\mathscr{a}' - 2\mathscr{b}_0)}{2\mathscr{a}} + \mathbb{R}. \end{cases}$$

The listed conditions in particular imply that L_0 is self adjoint and L_1 is skew-adjoint.

Theorem 1 characterizes solutions of the commutation relation $KLu = LKu$, where u is a smooth function on $[-1, 1]$. Up to this point we were assuming that $K : L^2(-1, 1) \mapsto L^2(-1, 1)$, but following [2], [12], [13], [14] we can consider K as an operator $K : L^2(-1, 1) \mapsto L^2(a, b)$ by restricting the variable x in $(Ku)(x)$ to (a, b), where (a, b) is the line segment connecting a to b in the complex plane. Now let $L_2 := L_{(a,b)}$ denote the operator L acting on (and returning) functions defined on the line segment (a, b) and similarly $L_1 := L_{(-1,1)}$. If both L_1 and L_2 are self-adjoint (in particular we need the coefficient of $\frac{\mathrm{d}^2}{\mathrm{d}y^2}$ in L to vanish at $\pm1, a$ and b) we get an example of commutation (C2): $KL_1 u = L_2 Ku$, where u is a smooth function on $[a, b]$. Below we present all such instances that can be deduced from the commutation relation $KL = LK$ (the results are given up to multiplication of $k(z)$ by $e^{\tau z}$, cf Remark 8 below).

Corollary 2. Let $K : L^2(-1, 1) \to L^2(a, b)$ be given by (1.1) and L be a differential operator given by (2.1), then the commutation relation

$$\begin{cases} KL_{(-1,1)} u = L_{(a,b)} K u & \qquad u \in C^\infty[-1, 1] \\ L^*_{(-1,1)} = L_{(-1,1)} \quad \text{and} \quad L^*_{(a,b)} = L_{(a,b)} \end{cases} \tag{3.4}$$

holds for the following choices of operators K, L and line segments (a, b):

1. k is given by (3.1), coefficients of L are given by (3.2) with

$$\begin{cases} \lambda, \mu \in \mathbb{R} \cup i\mathbb{R}, \quad \lambda \neq 0 \\ a = -1 + \frac{2\pi i n}{\lambda}, \quad b = 1 + \frac{2\pi i n}{\lambda} \quad , \ n \in \mathbb{Z} \end{cases}$$

   (When $\lambda \in i\mathbb{R}$ further restrictions of Remark 3 must be taken into account)

2. $k(z) = \dfrac{1}{\sinh\left(\frac{\lambda}{2} z\right)}$ and with $\mathscr{a}_0(y) = \cosh(\lambda y) - \cosh \lambda$:

$$\begin{cases} a(y) = \alpha a_0(y) \\ b(y) = \alpha a_0'(y) + \beta a_0(y) \\ c(y) = \frac{\beta}{2} a_0'(y) + \alpha \frac{\lambda^2}{4} a_0(y), \end{cases}$$

where $\beta \in i\mathbb{R}$, $\lambda \in \mathbb{R} \cup i\mathbb{R}$, $\alpha \in \mathbb{R}$ and $a = -1 + \frac{2\pi i n}{\lambda}$, $b = 1 + \frac{2\pi i n}{\lambda}$ with $n \in \mathbb{Z}$.

3. $k(z) = \dfrac{1}{\beta} + \dfrac{1}{z}$ and L has coefficients

$$\begin{cases} a(y) = (y^2 - 1)(y^2 - b^2) \\ b(y) = a'(y) + 2\beta(y^2 - 1) \\ c(y) = 2\beta y, \end{cases}$$

where $\beta \in i\mathbb{R}$, $a = -b$ and $b > 0$.

4. $k(z) = \dfrac{1}{z}$ and L has coefficients

$$\begin{cases} a(y) = (y^2 - 1)(y - a)(y - b) \\ b(y) = a'(y) + \beta(y^2 - 1) \\ c(y) = 2y^2 + (\beta - a - b)\, y, \end{cases}$$

where $\beta \in i\mathbb{R}$ and $a < b$ are real.

Proof. The proof immediately follows from Theorem 1 and discussion above, we just mention that in item 1 the restrictions $\lambda, \mu \in \mathbb{R} \cup i\mathbb{R}$ make L self-adjoint on $[-1, 1]$, the choice of a, b follows from the fact that coefficients of L are $\frac{2\pi i}{\lambda}$-periodic. Therefore, L is also self-adjoint on $[a, b]$. Similarly, in items 2, 3 and 4 the condition $\beta \in i\mathbb{R}$ guarantees self-adjointness of L. In item 3 we are forced to take $a = -b$, because in the corresponding commutation relation (item 3 of Theorem 1) $a(y) = (y^2 - 1)p(y)$ where $p'(0) = 0$, hence $p(y) = y^2 - b^2$. $\square$

Remark 8. Due to Remark 2 it is easy to check that in Corollary 2, in each of the four items K can be replaced by MKM^{-1} and L by MLM^{-1}, where M is multiplication operator by $e^{\tau z}$ and (in addition to given parameter restrictions) it must hold $\tau \in i\mathbb{R}$ in order for MLM^{-1} to be self-adjoint. Note that in this case M is a unitary operator, therefore MLM^{-1} is self-adjoint if and only if L is. However, for item 2 there is an additional case: $\tau \in \mathbb{C}$ and $\beta = 2i\alpha \operatorname{Im} \tau$.

Remark 9. Taking $\beta = 0$ in item 4 we obtain the commutation used in [2], [12], [13], [14] mentioned in the introduction. Indeed, since any real constant can be added to c we can rewrite $c(y) = 2\left(y - \frac{a+b}{4}\right)^2$, which is precisely the form of c used in those references.

Remark 10. Observe that in all of the cases $k(z)$ has a singularity and the corresponding operator K is not compact. The discreteness of the spectrum of K^*K is therefore non-obvious. Following [2, 12, 13, 14], the discreteness of the SVD decomposition comes from the discreteness of the spectrum of self-adjoint differential operators L_1 and L_2 in (C2), provided that *singularities of Ku are not at the end-points of the interval for the Sturm-Liouville eigenvalue problem for L_2*. In particular, the situation when $(-1,1)$ and (a,b) intersect does not in and of itself cause the appearance of the continuous spectrum. In the context of operators listed in Corollary 2 we can characterize all instances when the discreteness of SVD can be established. Let $\{z_j\}$ be the simple poles of k, and $\{y_j\}$—the zeros of $\mathscr{a}(y)$. If the set $\{y_j\} \setminus \{z_j \pm 1\}$ has at least two points, say a and b, then Ku is regular at points a, b and so (using (3.4)) K maps eigenfunctions of $L_{(-1,1)}$ to eigenfunctions of $L_{(a,b)}$, making the former the eigenfunctions of K^*K. We will call this case regular. In the list of operator pairs in Corollary 2 regular cases do occur. For example, operators in items 3 and 4 of Corollary 2 are regular, while operators in item 1 are regular if and only if for at least two different values of $n \in \mathbb{Z}$

$$\alpha_1 \sinh\left(\tfrac{2\pi i n\mu}{\lambda}\right) + \alpha_2 \cosh\left(\tfrac{2\pi i n\mu}{\lambda}\right) = 0.$$

In that case $\frac{\mu}{\lambda} \in \mathbb{Q}$ and there are countably many choices for a, b. For example taking $\alpha_1 = 0,\ \lambda = i\frac{\pi}{2},\ \mu = i\frac{\pi}{8}$ we can choose $a = 3,\ b = 5$ and obtain

$$k(z) = \frac{1}{\sin\left(\frac{\pi}{8}z\right)}, \qquad \begin{cases} \mathscr{a}(y) = \cos\left(\frac{\pi}{2}y\right), \\ \mathscr{b}(y) = \mathscr{a}'(y), \\ \mathscr{c}(y) = -\frac{3\pi^2}{64}\mathscr{a}(y). \end{cases}$$

4 Commutation, regular case

Lemma 3. Assume the setting of Theorem 1 case (i), then for some complex constants α, ν we have

$$\mathscr{a}'''(y) + \alpha\mathscr{a}(y) = 0, \qquad \mathscr{b}(y) = \mathscr{a}'(y), \qquad \mathscr{c}(y) = \nu\mathscr{a}(y). \tag{4.1}$$

Proof. Write $k(z) = \sum_{n=0}^{\infty} \frac{k_n}{n!} z^n$ near $z = 0$. The n-th derivative of (R1) w.r.t. z evaluated at $z = 0$ reads

$$\begin{aligned} 2\mathscr{a}'(y)k_{n+1} + [\mathscr{b}'(y) - \mathscr{a}''(y)]k_n + \sum_{j=0}^{n-1} C_j^n \mathscr{a}^{(n-j)}(y)k_{j+2} + \\ + \sum_{j=0}^{n-1} C_j^n \mathscr{b}^{(n-j)}(y)k_{j+1} + \sum_{j=0}^{n-1} C_j^n \mathscr{c}^{(n-j)}(y)k_j = 0, \end{aligned} \tag{4.2}$$

where $C_j^n = \binom{n}{j}$. The above relation for $n = 0$ gives

$$2k_1\mathscr{a}'(y) + [\mathscr{b}'(y) - \mathscr{a}''(y)]k_0 = 0. \tag{4.3}$$

Assume first $k_0 = 0$, then $k_1 = 0$ (otherwise the boundary conditions imply $\mathscr{a} = 0$). By induction one can conclude $k_j = 0$ for any j. Indeed, let $k_j = 0$ for $j = 0, ..., n$, then (4.2) reads

$$(n+2)\mathscr{a}'(y)k_{n+1} = 0.$$

Hence the boundary conditions imply $k_{n+1} = 0$. So if $k_0 = 0$, then $k(z)$ must be identically zero near $z = 0$, which we do not allow.

Thus $k_0 \neq 0$, and in view of Remark 2 we may assume $k_1 = k'(0) = 0$ (otherwise multiply $k(z)$ by $e^{-k_1/k_0 z}$). Taking into account the boundary conditions, from (4.3) we obtain $\mathscr{b}(y) = \mathscr{a}'(y)$. Now we substitute this in (4.2) with $n = 1$, integrate the result to find the expression for $\mathscr{c}$ in (4.1) with $\nu = -\frac{3k_2}{k_0}$. When $n = 2$ equation (4.2), after elimination of $\mathscr{b}$ and $\mathscr{c}$ becomes $k_3\mathscr{a}'(y) = 0$ and we conclude that $k_3 = 0$. When $n = 3$, we find

$$k_0k_2\mathscr{a}'''(y) + (5k_0k_4 - 9k_2^2)\mathscr{a}'(y) = 0.$$

If $k_2 = 0$, then $k_4 = 0$ and as can be immediately seen from (4.2), induction argument shows that $k_j = 0$ for all $j \geq 1$. Thus, we may assume $k_2 \neq 0$, in which case $\mathscr{a}$ satisfies the ODE in (4.1). $\square$

From (4.1) $\mathscr{a}$ has to have one of the following forms, with $a_j \in \mathbb{C}$

I. $\mathscr{a}(y) = a_1e^{\lambda y} + a_2e^{-\lambda y} + a_0$, with $0 \neq \lambda \in \mathbb{C}$

II. $\mathscr{a}(y) = a_2y^2 + a_1y + a_0$

• Assume case I holds, replacing the expressions for $\mathscr{a}, \mathscr{b}, \mathscr{c}$ from Lemma 3, (R1) becomes a linear combination of exponentials $e^{\pm\lambda y}$ with coefficients depending only on z, hence each coefficient must vanish. These can be simplified as $a_j\left\{k'' + \lambda\coth\left(\frac{\lambda}{2}z\right)k' + \nu k\right\} = 0$ for $j = 1, 2$. Of course, at least one of a_1, a_2 is different from zero and so we deduce

$$k'' + \lambda\coth\left(\tfrac{\lambda}{2}z\right)k' + \nu k = 0. \tag{4.4}$$

Setting $u(z) = k(z)\sinh\left(\frac{\lambda}{2}z\right)$, the above ODE becomes $u'' + \left(\nu - \frac{\lambda^2}{4}\right)u = 0$. So,

$$k(z) = \frac{\sinh(\mu z)}{\mu\sinh\left(\frac{\lambda}{2}z\right)} \qquad \mu^2 = \frac{\lambda^2}{4} - \nu.$$

When $\mu = 0$, the formula is understood in the limiting sense. Note that this is (3.1) with $\alpha_2 = 0$ (here α_2 refers to the parameter in formula (3.1), whose vanishing makes $k(z)$ analytic on $[-2, 2]$.) Because $\mathscr{a}(y)$ satisfies the boundary conditions we must have $a_1 = a_2$ or $\lambda \in \pi in$ for some $n \in \mathbb{Z}$. If $\lambda = \pi in$, then for k to be smooth in $[-2, 2]$ we must have $\mu \neq 0$, moreover $\sinh\left(\frac{2\mu m}{n}\right) = 0$ for any $m \in \mathbb{Z}$ with $\frac{m}{n} \in [-1, 1]$. In particular this should hold for $m = 1$, which implies $\mu = \frac{\lambda l}{2}$ for some $l \in \mathbb{Z}$, which in turn implies that k is a trigonometric polynomial, and hence is trivial. Thus we may assume $\lambda \notin \pi i\mathbb{Z}$, and so $a_1 = a_2$, showing that $\mathscr{a}(y) = \cosh(\lambda y) - \cosh\lambda$.

Now we show that if $\lambda \in i\mathbb{R}$, then it must hold $|\lambda| < \pi$. Otherwise, k is trivial. Indeed, assume $\lambda \in i\mathbb{R}$ and $|\lambda| \geq \pi$ we see that the denominator of $k(z)$ has additional zeros at $z = \pm\frac{2\pi i}{\lambda} \in [-2, 2]$. In order for k to be smooth, we require that its numerator also vanishes at these points. So $\sinh\left(\frac{2\pi i}{\lambda}\mu\right) = 0$ and hence $\mu = \frac{\lambda}{2}m$ for some $m \in \mathbb{Z}$. But then, again k is a trigonometric polynomial.

- Assume case II holds, then $\mathscr{a}(y) = a_2(y^2 - 1)$ and substituting into (R1) we find

$$zk'' + 2k' + \nu zk = 0. \tag{4.5}$$

Setting $u(z) = zk(z)$ the ODE turns into $u'' + \nu u = 0$, which corresponds to the limiting case $\lambda = 0$ in the formulas for k and $\mathscr{a}$ and concludes the proof of Theorem 1 case (i).

5 Commutation, singular case

Here we prove Theorems 1 case (ii). In the first subsection below we obtain the possible forms for the functions $\mathscr{a}, \mathscr{b}$ and $\mathscr{c}$. In the second one we do reduction of these forms, and finally in the third one we find k.

5.1 Forms of $\mathscr{a}, \mathscr{b}$ and $\mathscr{c}$

By the assumption $k(z) = z^{-1}(k_0 + k_1 z + ...)$, with $k_0 \neq 0$. So by rescaling we let $k_0 = 1$ and in view of Remark 2 we may assume $k_1 = 0$ (otherwise multiply $k(z)$ by $e^{-k_1/k_0 z}$). Multiply (R1) by z^3 and refer to the resulting relation by (E). Differentiate (E) three times w.r.t. z and let $z = 0$ to get

$$\mathscr{c}(y) = -\tfrac{1}{3}\mathscr{a}''(y) - 2k_2\mathscr{a}(y) + \tfrac{1}{2}\mathscr{b}'(y) + \text{const}. \tag{5.1}$$

Substitute this into (E), differentiate the result 4 times w.r.t. z and let $z = 0$, then

$$\mathscr{b}''' = \mathscr{a}^{(4)} + 24k_2\mathscr{a}'' - 72k_3\mathscr{a}' - 24k_2\mathscr{b}'. \tag{5.2}$$

In the fifth derivative of (E) we replace $b^{(4)}$ and b''' using the above relation, then the result reads

$$\alpha_1\mathscr{b}' = \mathscr{a}^{(5)} + 120k_2\mathscr{a}^{(3)} + \alpha_1\mathscr{a}'' + \alpha_2\mathscr{a}', \tag{5.3}$$

where $\alpha_1 = -1080k_3$ and the expression for α_2 is not important. Now if $\alpha_1 = 0$ we got a linear constant coefficient ODE for $\mathscr{a}$, otherwise we substitute the formula for $\mathscr{b}'$ from (5.3) into (5.2) and again obtain an ODE for $\mathscr{a}$, more precisely, for some constants $\beta_j \in \mathbb{C}$, either

(A) $\alpha_1 = 0$ and $\mathscr{a}^{(4)} + \beta_1\mathscr{a}'' + \beta_2\mathscr{a} = \beta_0$, or

(B) $\alpha_1 \neq 0$ and $\mathscr{a}^{(6)} + \beta_3\mathscr{a}^{(4)} + \beta_1\mathscr{a}'' + \beta_2\mathscr{a} = \beta_0$

Therefore, using the fact that ODEs in (A) and (B) contain only even derivatives of $\mathscr{a}$, we can conclude that in either case $\mathscr{a}$ has one of the following forms, with $p_j, a_j, \tilde{a}_j \in \mathbb{C}$; $\lambda_j, \lambda, \mu \in \mathbb{C}\backslash\{0\}$ and $\lambda \neq \pm\mu$ and $\lambda_j \neq \pm\lambda_l$ for $j \neq l$,

I. 1) $\mathscr{a}(y) = \sum_{j=1}^{3}(a_j e^{\lambda_j y} + \tilde{a}_j e^{-\lambda_j y}) + a_0$

2) $\mathscr{a}(y) = \sum_{j=1}^{2}(a_j e^{\lambda_j y} + \tilde{a}_j e^{-\lambda_j y}) + \sum_{j=0}^{2} p_j y^j$

3) $\mathscr{a}(y) = a_1 e^{\lambda y} + \tilde{a}_1 e^{-\lambda y} + \sum_{j=0}^{4} p_j y^j$

II. 1) $\mathscr{a}(y) = (a_1 y + \tilde{a}_1)e^{\lambda y} + (a_2 y + \tilde{a}_2)e^{-\lambda y} + a_3 e^{\mu y} + \tilde{a}_3 e^{-\mu y} + a_0$

2) $\mathscr{a}(y) = (a_1 y + \tilde{a}_1)e^{\lambda y} + (a_2 y + \tilde{a}_2)e^{-\lambda y} + p_2 y^2 + p_1 y + p_0$

III. $\mathscr{a}(y) = (a_2 y^2 + a_1 y + a_0)e^{\lambda y} + (\tilde{a}_2 y^2 + \tilde{a}_1 y + \tilde{a}_0)e^{-\lambda y} + a_3$

IV. $\mathscr{a}(y) = \sum_{j=0}^{6} a_j y^j$

If $\alpha_1 \neq 0$, then from (5.3) we see that $\mathscr{b}$ has exactly the same form as $\mathscr{a}$. Assume now $\alpha_1 = 0$, if $k_2 = 0$ we find from (5.2) that $\mathscr{b}(y) = \mathscr{a}'(y) + p_2(y^2 - 1)$, if $k_2 \neq 0$, then $\mathscr{b}$ is of the same form as $\mathscr{a}$ only it might contain two extra exponentials $e^{\pm\sqrt{-24k_2}y}$, if those differ from all the exponentials appearing in $\mathscr{a}$, otherwise if one of them coincides, say with $e^{\lambda y}$, then the polynomial multiplying the latter gets one degree higher. Finally, $\mathscr{c}$ is of the same form as $\mathscr{b}$.

5.2 Reduction

Our goal is to reduce the cases I–IV and conclude that $\mathscr{a}(y)$ can have one of the two forms $a_1 e^{\lambda y} + a_2 e^{-\lambda y} + a_0$ or $\sum_{j=0}^{6} a_j y^j$. Moreover, $\mathscr{b}$ and $\mathscr{c}$ must have exactly the same form as $\mathscr{a}$, but possibly with different constants b_j, c_j instead of a_j. This reduction will be achieved by the three lemmas below.

Lemma 4. If the functions $\mathscr{a}, \mathscr{b}, \mathscr{c}$ contain an exponential term, the polynomial multiplying it must be constant.

Proof. See the appendix. □

Lemma 5. The functions $\mathscr{a}, \mathscr{b}, \mathscr{c}$ cannot contain two exponentials $e^{\lambda y}, e^{\mu y}$ with $\mu \neq \pm\lambda$.

Proof. Consider a typical exponential term in $\mathscr{a}, \mathscr{b}$ and $\mathscr{c}$ (due to Lemma 4 the polynomial multiplying it must be a constant), namely

$$\mathscr{a} \leftrightarrow a_0 e^{\lambda y}, \qquad \mathscr{b} \leftrightarrow b_0 e^{\lambda y}, \qquad \mathscr{c} \leftrightarrow c_0 e^{\lambda y},$$

where $a_0 \neq 0$. The equation coming from $e^{\lambda y}$ after substituting these forms into (R1) is (obtained analogously to the first equation of (6.2) in the appendix)

$$a_0(e^{\lambda z} - 1)k'' + \left[2a_0\lambda + b_0(e^{\lambda z} - 1)\right] k' + \left[b_0\lambda - a_0\lambda^2 + c_0(e^{\lambda z} - 1)\right] k = 0.$$

After changing the variables $u(z) = k(z)(e^{\lambda z} - 1)$ it becomes

$$a_0 u'' + (b_0 - 2a_0\lambda)u' + (a_0\lambda^2 - b_0\lambda + c_0)u = 0. \tag{5.4}$$

Then, with $\nu = -\frac{b_0}{2a_0}$ and $\alpha_1, \alpha_2 \in \mathbb{C}$ we have

$$k(z) = \frac{e^{(\nu+\lambda)z}}{e^{\lambda z} - 1} \cdot \begin{cases} \alpha_1 z + \alpha_2, & \mu := \sqrt{\frac{b_0^2}{4a_0^2} - \frac{c_0}{a_0}} = 0 \\ \alpha_1 \sinh(\mu z) + \alpha_2 \cosh(\mu z), & \mu \neq 0 \end{cases} \tag{5.5}$$

We claim that the set $\{\lambda, -\lambda\}$ is determined by the functions given above. In other words, up to the sign, λ is determined by k. This will prove that in $\mathcal{a}(y)$, there cannot be another exponential $e^{\mu y}$ with $\mu \neq \pm\lambda$, because the equation coming from $e^{\mu y}$ will lead to a formula for k incompatible with (5.5). Computing the residue of k at the pole $z = 0$ we find $k_0 = \frac{\alpha_2}{\lambda}$, hence it is enough to show that α_2 is determined up to the sign. Let k be given by the second formula of (5.5) (in the other case the same argument will apply), write $\mu = \mu_1 + i\mu_2$ and $\lambda = \lambda_1 + i\lambda_2$.

Let $\lambda_1 \neq 0$ and $\mu_1 \neq 0$, then w.l.o.g. we may assume $\mu_1 > 0$, otherwise negate (α_1, μ). If $\lambda_1 > 0$ we find

$$k(z) \sim \begin{cases} \frac{1}{2}(\alpha_1 + \alpha_2)e^{(\nu+\mu)z}, & z \to +\infty, \\ \frac{1}{2}(\alpha_1 - \alpha_2)e^{(\nu+\lambda-\mu)z}, & z \to -\infty. \end{cases}$$

Therefore, α_2 is equal to the difference of coefficients in the asymptotics of k at plus and minus infinities. But when $\lambda_1 < 0$, by writing down the asymptotics, one can see that the same difference gives $-\alpha_2$.

Let now $\lambda_1 \neq 0$ and $\mu_1 = 0$, we find $k(z) \sim e^{\nu z}(i\alpha_1 \sin(\mu_2 z) + \alpha_2 \cos(\mu_2 z))$ as $z \to +\infty$ if $\lambda_1 > 0$, and when $\lambda_1 < 0$ the same formula holds, but the RHS multiplied by $-e^{\lambda z}$. Again we see that α_2 is determined up to the sign.

Let $\lambda_1 = 0$ and $\mu_2 \neq 0$, we may assume $\mu_2 > 0$, otherwise negate (α_1, μ), then

$$k(iz) \sim \begin{cases} \frac{1}{2}(\alpha_1 - \alpha_2)e^{i(\nu+\lambda-\mu)z}, & z \to +\infty, \\ \frac{1}{2}(\alpha_1 + \alpha_2)e^{i(\nu+\mu)z}, & z \to -\infty. \end{cases}$$

Finally, the case $\lambda_1 = \mu_2 = 0$ can be treated similarly.

Remains to note that $\mathcal{b}, \mathcal{c}$ cannot have an exponential $e^{\mu y}$ with $\mu \neq \pm\lambda$ either (we assume $a_0 e^{\lambda y}$ appears in $\mathcal{a}$). Indeed, if $\tilde{b}_0 e^{\mu y}$ and $\tilde{c}_0 e^{\mu y}$ appear in $\mathcal{b}$ and $\mathcal{c}$ respectively, then for k we obtain an equation like (5.4), but with $a_0 = 0$ and b_0, c_0 replaced with $\tilde{b}_0, \tilde{c}_0$, hence $k(z) = e^{(\mu+\tilde{\nu})z}/(e^{\mu z} - 1)$ with $\tilde{\nu} = -\tilde{c}_0/\tilde{b}_0$. But this is of the same form as (5.5), hence as we showed μ is determined up to its sign. In other words the two formulas for k are compatible only if $\mu = \pm\lambda$.

□

Lemma 6. The functions $\mathcal{a}, \mathcal{b}, \mathcal{c}$ cannot contain an exponential and a polynomial at the same time.

Proof. Let $a_5 e^{\lambda y} + \sum_{j=0}^{4} a_j y^j$, with $a_5 \neq 0$ be part of $\mathcal{a}$. The functions $\mathcal{b}, \mathcal{c}$ also have such parts, but with possibly different constants b_j, c_j. From the above lemma we know that k is given by (5.5) (with a_0 replaced by a_5). One can check that once these expressions for $\mathscr{a}, \mathscr{b}$ and $\mathscr{c}$ are substituted into (R1), the factors y^4 get canceled and the equation corresponding to y^3 reads

$$a_4 z k'' + (b_4 z + 2a_4)k' + (c_4 z + b_4)k = 0. \tag{5.6}$$

Let us first show that $a_4 = 0$. For the sake of contradiction assume $a_4 \neq 0$, then the solution, with $\omega = -\frac{b_4}{2a_4}$, is given by

$$k(z) = \frac{e^{\omega z}}{z} \cdot \begin{cases} \beta_1 z + \beta_2, & \eta := \sqrt{\frac{b_4^2}{4a_4^2} - \frac{c_4}{a_4}} = 0, \\ \beta_1 \sinh(\eta z) + \beta_2 \cosh(\eta z), & \eta \neq 0. \end{cases} \tag{5.7}$$

We note that this is not compatible with (5.5), because cross multiplying the two formulas we get (with f, g being the second multiplying factors from (5.5) and (5.7), respectively)

$$z e^{(\nu+\lambda)z} f(z) = e^{\omega z}(e^{\lambda z} - 1) g(z).$$

If $g(z) = \beta_1 \sinh(\eta z) + \beta_2 \cosh(\eta z)$, we use the linear independence of $z e^{\gamma z}$ and $e^{\tilde{\gamma} z}$ to conclude that $k = 0$. Let $g(z) = \beta_1 z + \beta_2$, if f is given by the first formula the above relation reads

$$\alpha_1 z^2 e^{(\nu+\lambda)z} + \alpha_2 z e^{(\nu+\lambda)z} + \beta_1 z e^{\omega z} - \beta_1 z e^{(\omega+\lambda)z} = \beta_2 e^{(\omega+\lambda)z} - \beta_2 e^{\omega z}.$$

Because $\lambda \neq 0$, the exponentials on RHS are linearly independent, hence we conclude that $\beta_2 = 0$, which contradicts to k having a pole at zero. When f is given by the second formula the same argument applies.

Thus, $a_4 = 0$, if $b_4 \neq 0$ we find $k(z) = e^{\omega z}/z$, but now $\omega = -c_4/b_4$. This has the same form as (5.7), hence again it is incompatible with (5.5). Therefore, $b_4 = 0$ and obviously $c_4 = 0$. With this information, the equation corresponding to y^2 is as (5.6) with all subscripts changed from 4 to 3. Hence, the same procedure works and eventually we conclude $a_j = b_j = c_j = 0$ for $j = 1, ..., 4$.

□

5.3 Finding k

The analysis of the previous subsection shows that we have two possible forms ($\lambda \neq 0$)

$$\text{I. } \mathscr{a}(y) = a_1 e^{\lambda y} + a_2 e^{-\lambda y} + a_0, \qquad \text{II. } \mathscr{a}(y) = \sum_{j=0}^{6} a_j y^j.$$

Moreover we also showed that in each case $\mathscr{b}, \mathscr{c}$ are exactly of the same form as $\mathscr{a}$, only with possibly different constants b_j, c_j instead of a_j.

5.3.1 Case I

Assume case I holds, substituting the expressions for $\mathscr{a}, \mathscr{b}, \mathscr{c}$ into (R1) we find that a linear combination of $e^{pm\lambda y}$ is zero, hence the coefficient of each exponential must vanish. Like this we obtain two ODEs for k. More precisely,

$$a_1(e^{\lambda z}-1)k'' + \left[2a_1\lambda + b_1(e^{\lambda z}-1)\right]k' + \left[b_1\lambda - a_1\lambda^2 + c_1(e^{\lambda z}-1)\right]k = 0,$$
$$a_2(e^{-\lambda z}-1)k'' + \left[-2a_2\lambda + b_2(e^{-\lambda z}-1)\right]k' + \left[-b_2\lambda - a_2\lambda^2 + c_2(e^{-\lambda z}-1)\right]k = 0.$$

Note that the second equation is obtained from the first one if we negate λ and change the subscripts of a_1, b_1, c_1 from 1 to 2. Consider the following cases:

Case I.1. $a_1 = a_2 = 0$, then $\mathscr{a} \equiv 0$ and from the boundary conditions $\mathscr{b}(\pm 1) = 0$. W.l.o.g. let $b_1 \neq 0$ solving the first ODE for k we get, with $\nu = -\frac{c_1}{b_1}$

$$k(z) = \frac{e^{(\nu+\lambda)z}}{e^{\lambda z}-1} = \frac{e^{\left(\nu+\frac{\lambda}{2}\right)z}}{2\sinh\left(\frac{\lambda}{2}z\right)}.$$

For this to satisfy also the second ODE we need $c_2 = -(\nu+\lambda)b_2$. One can check that for k to be smooth in $[-2,2]\backslash\{0\}$, we cannot have $\lambda = \pi i n$, therefore the boundary conditions on $\mathscr{b}$ imply $b_1 = b_2$ and so $\mathscr{b}(y) = \cosh(\lambda y) - \cosh\lambda$. Now if $\lambda \in i\mathbb{R}$, for the same reason we require $|\lambda| < \pi$. From the relation (5.1) we see that $\mathscr{c}(y) = \frac{1}{2}\mathscr{b}'(y)$. After ignoring the exponential in the numerator of the formula for k (see Remark 2) we obtain

$$k(z) = \frac{1}{\sinh\left(\frac{\lambda}{2}z\right)}, \qquad \begin{cases} \mathscr{a}(y) = 0, \\ \mathscr{b}(y) = \cosh(\lambda y) - \cosh\lambda, \\ \mathscr{c}(y) = \frac{1}{2}\mathscr{b}'(y). \end{cases} \tag{5.8}$$

Case I.2. If $a_1 \neq 0$ (the case $a_2 \neq 0$ can be treated analogously) by rescaling let us take $a_1 = \frac{1}{2}$, then as the formula (5.5) was obtained we get, by w.l.o.g. choosing $\nu = -\lambda/2$, or equivalently $b_1 = \lambda a_1$ (see Remark 2) that

$$k(z) = \frac{1}{\sinh\left(\frac{\lambda}{2}z\right)} \cdot \begin{cases} \alpha_1 z + \alpha_2, & \mu := \sqrt{b_1^2 - 2c_1} = 0, \\ \alpha_1\sinh(\mu z) + \alpha_2\cosh(\mu z), & \mu \neq 0. \end{cases}$$

• Let k be given by the first formula. It is easy to check that $\lambda = \pi i n$, with $n \in \mathbb{Z}$ contradicts to the smoothness assumption on k, so the boundary conditions imply that $a_1 = a_2$ and therefore $\mathscr{a}(y) = \cosh(\lambda y) - \cosh\lambda$. Because of the same reason, when $\lambda \in i\mathbb{R}$ we need a further restriction $|\lambda| < \pi$. The boundary conditions $\mathscr{b}(\pm 1) = \mathscr{a}'(\pm 1)$ then imply

$$b_2 = -\tfrac{\lambda}{2}, \quad b_0 = 0 \quad \Rightarrow \quad \mathscr{b}(y) = \tfrac{\lambda}{2}e^{\lambda y} - \tfrac{\lambda}{2}e^{-\lambda y} = \mathscr{a}'(y).$$

Now, k has to satisfy also the second ODE, so we substitute the expression for k there and simplify the result to find

$$e^{-\frac{\lambda}{2}z}(\alpha_1 z + \alpha_2)\left(c_2 - \tfrac{\lambda^2}{8}\right) = 0,$$

which clearly implies $c_2 = \frac{\lambda^2}{8}$. But because this was the case $\mu = 0$ we have $c_1 = \frac{b_1^2}{2} = \frac{\lambda^2}{8}$ and therefore we conclude that $\mathscr{c}(y) = \frac{\lambda^2}{2}\mathscr{a}(y)$. Thus, we proved (3.1) and (3.2) of Theorem 1 in the limiting case $\mu = 0$. Moreover, when $\alpha_1 = 0$ we obtain the same kernel as in (5.8), hence we can take a linear combination of the differential operator of this case and the one in (5.8) and K will still commute with it. This proves item 2 of Theorem 1.

• Let k be given by the second formula. When $\lambda \in i\mathbb{R}$ there are further restrictions for parameters. Let us analyze them. Firstly, if $\lambda \in i\mathbb{R}$ with $|\lambda| \geq 2\pi$, then the denominator of k has zeros at $\pm\frac{2\pi i}{\lambda}, \pm\frac{4\pi i}{\lambda} \in [-2,2]$, which cannot be canceled out by the numerator, therefore $|\lambda| < 2\pi$. So there are two cases: when $|\lambda| < \pi$, k is smooth in $[-2,2]\backslash\{0\}$ and when $\pi \leq |\lambda| < 2\pi$ the denominator of k has zeros at $\pm\frac{2\pi i}{\lambda} \in [-2,2]$, which can be canceled out by the numerator if and only if $\alpha_1 = 0$ and $\cosh\left(\frac{2\pi i\mu}{\lambda}\right) = 0$, i.e. $\mu = \lambda\frac{2m+1}{4}$ for some $m \in \mathbb{Z}$. This is summarized in Remark 3.

Let us substitute the expression for k into the second ODE, multiply the result by $e^{\frac{\lambda}{2}z}$. After simplification we obtain

$$\left[(\mu^2 a_2 + \tfrac{\lambda^2 a_2}{4} + \tfrac{b_2\lambda}{2} + c_2)\alpha_1 + \mu\alpha_2(a_2\lambda + b_2)\right]\sinh(\mu z) + \\ + \left[(\mu^2 a_2 + \tfrac{\lambda^2 a_2}{4} + \tfrac{b_2\lambda}{2} + c_2)\alpha_2 + \mu\alpha_1(a_2\lambda + b_2)\right]\cosh(\mu z) = 0.$$

By linear independence we conclude that the coefficients of $\sinh(\mu z), \cosh(\mu z)$ must be zero. Or equivalently their sum and difference must be zero, but these equations can be written as

$$\begin{cases} (\alpha_1 + \alpha_2)\left((\mu + \frac{\lambda}{2})[(\mu + \frac{\lambda}{2})a_2 + b_2] + c_2\right) = 0, \\ (\alpha_1 - \alpha_2)\left((\mu - \frac{\lambda}{2})[(\mu - \frac{\lambda}{2})a_2 - b_2] + c_2\right) = 0. \end{cases} \tag{5.9}$$

The boundary conditions $\mathscr{a}(\pm 1) = 0$ imply that $a_0 = -a_1 e^{\lambda} - a_2 e^{-\lambda}$ and

$$(a_1 - a_2)(e^{\lambda} - e^{-\lambda}) = 0.$$

a) Let $a_2 = a_1$, then $\mathscr{a}(y) = \cosh(\lambda y) - \cosh(\lambda)$ and from the boundary conditions $\mathscr{b}(\pm 1) = \mathscr{a}'(\pm 1)$ we find $\mathscr{b}(y) = \mathscr{a}'(y)$ as was discussed above. Now in this case (5.9) simplifies to

$$\begin{cases} (\alpha_1 + \alpha_2)\left(\frac{\lambda^2}{4} - \mu^2 - 2c_2\right) = 0, \\ (\alpha_1 - \alpha_2)\left(\frac{\lambda^2}{4} - \mu^2 - 2c_2\right) = 0. \end{cases}$$

But because both α_1, α_2 are not zero at the same time, we get $c_2 = \frac{1}{2}(\frac{\lambda^2}{4} - \mu^2)$. From the definition of μ we see that also $c_1 = \frac{1}{2}(\frac{\lambda^2}{4} - \mu^2)$. And using the freedom of choosing c_0 we conclude that we may write $\mathscr{c}(y) = (\frac{\lambda^2}{4} - \mu^2)\mathscr{a}(y)$. This proves (3.1) and (3.2) of Theorem 1 in the case $\mu \neq 0$.

b) Let $e^{\lambda} = e^{-\lambda}$, i.e. $\lambda = \pi i n$ for some $n \in \mathbb{Z}$. But the above discussion implies that $\alpha_1 = 0$, $\lambda = \pi i$ (or $-\pi i$, but this would lead to the same results) and $\mu = \lambda\frac{2m+1}{4}$ with $m \in \mathbb{Z}$. In this case (5.9) implies

$$b_2 = -\lambda a_2, \qquad c_2 = a_2\left(\frac{\lambda^2}{4} - \mu^2\right).$$

Recalling that $b_1 = \lambda a_1$, the boundary conditions $\mathscr{b}(\pm 1) = \mathscr{a}'(\pm 1)$ imply $b_0 = 0$ and so far we have $\mathscr{a}(y) = a_1(e^{\lambda y} - e^{\lambda}) + a_2(e^{-\lambda y} - e^{-\lambda})$ and $\mathscr{b}(y) = \mathscr{a}'(y)$. Finally, again from the definition of μ we have $c_1 = a_1(\frac{\lambda^2}{4} - \mu^2)$. This and the above formula for c_2 (and the freedom of choosing c_0) allow one to write $\mathscr{c}(y) = (\frac{\lambda^2}{4} - \mu^2)\mathscr{a}(y)$. This proves item 1 of Theorem 1. Of course to start with we assumed $a_1 \neq 0$ and we normalized $a_1 = \frac{1}{2}$, but when considering the case $a_2 \neq 0$ we can allow a_1 to vanish. This explains why there are no restrictions on α, β in item 1 of Theorem 1.

5.3.2 Case II

Assume case II holds, substituting the expressions for $\mathscr{a}, \mathscr{b}, \mathscr{c}$ into (R1) we find that a linear combination of monomials y^j is zero, hence the coefficient of each y^j must vanish (one can check that y^6 cancels out). These relations can be conveniently written as

$$\begin{aligned}&\left[\frac{\mathscr{a}^{(j)}(z)}{j!} - a_j\right] k'' + \left[\frac{\mathscr{b}^{(j)}(z)}{j!} - b_j + 2(j+1)a_{j+1}\right] k' + \\ &+ \left[\frac{\mathscr{c}^{(j)}(z)}{j!} - c_j + (j+1)b_{j+1} - (j+1)(j+2)a_{j+2}\right] k = 0, \qquad j = 0, ..., 5,\end{aligned} \tag{5.10}$$

with the convention that $a_7 = 0$. Let $\deg(\mathscr{a}) = m$, $\deg(\mathscr{b}) = n$ and $\deg(\mathscr{c}) = s$.

Case II.1. Let $\mathscr{a} \equiv 0$, then $\mathscr{b}(\pm 1) = 0$ and hence $n \geq 2$. By scaling we let $b_n = 1$. We are going to show that n cannot be strictly larger than 2 and so $n = 2$. Note that $s \leq n$, otherwise the above relation with $j = s - 1$ reads $c_s z k = 0$, which implies $k = 0$ since $c_s \neq 0$ by the definition of s. Now (5.10) with $j = n - 1$ reads

$$zk' + [1 + c_n z]k = 0, \tag{5.11}$$

whose solution is given by $k(z) = \alpha \frac{e^{-c_n z}}{z}$, where $\alpha \in \mathbb{C}$. Invoking Remark 2 we may w.l.o.g. assume $c_n = 0$. The relation with $j = n - 2$ becomes

$$\left[\tfrac{n}{2}z^2 + b_{n-1}z\right] k' + \left[c_{n-1}z + b_{n-1}\right] k = 0.$$

Substituting $k(z) = \frac{1}{z}$ into this equation we obtain $c_{n-1} = \frac{n}{2}$. Now, if $n > 2$ we consider the relation for $j = n - 3$, which reads

$$\left[\tfrac{n(n-1)}{6}z^3 + \tfrac{n-1}{2}b_{n-1}z^2 + b_{n-2}z\right] k' + \left[\tfrac{n-1}{2}c_{n-1}z^2 + c_{n-2}z + b_{n-2}\right] k = 0.$$

Again substituting the expression for k and using the expression for c_{n-1} we obtain

$$\tfrac{n(n-1)}{12}z + c_{n-2} + \tfrac{n-1}{2}b_{n-1} = 0, \tag{5.12}$$

which is a contradiction. Thus our conclusion is that $n = 2$, in which case $\mathscr{b}(y) = y^2 - 1$, $c_2 = 0$, $c_1 = 1$ and hence $\mathscr{c}(y) = y$, and we obtain the operator in item 4 of Theorem 1 when $\mathscr{p} = 0$.

Case II.2. Let $\mathscr{a} \neq 0$, then $m \geq 2$. By scaling we let $a_m = 1$. Let us first show that $n \leq m$. For the sake of contradiction assume $n > m$. If also $s > n$, then (5.10) with $j = s - 1$ reads $c_s z k = 0$, which is a contradiction and therefore $s \leq n$. Now (5.10) with $j = n - 1$ reads

$$zk' + [1 + c_n z]\, k = 0,$$

with the convention that $c_n = 0$ if $s < n$. As in the previous case w.l.o.g. we assume $c_n = 0$ so that $k(z) = \frac{1}{z}$. Using these and looking at (5.10) for $j = n - 2$ and $j = n - 3$ we obtain exactly the same contradiction (5.12) as in the previous case (only with a different free constant).

Thus $n \leq m$, and it is easy to see that also $s \leq m$. The relation (5.10) for $j = m - 1$ reads

$$zk'' + (2 + b_m z)k' + (b_m + c_m z)k = 0, \tag{5.13}$$

whose solution is, with $\alpha_1, \alpha_2 \in \mathbb{C}$

$$k(z) = \frac{e^{-\frac{b_m}{2} z}}{z} \cdot \begin{cases} \alpha_1 \sinh(\mu z) + \alpha_2 \cosh(\mu z), & \mu^2 := \frac{b_m^2}{4} - c_m \neq 0, \\ \alpha_1 z + \alpha_2, & \mu = 0. \end{cases} \tag{5.14}$$

Invoking Remark 2 let us w.l.o.g. assume $b_m = 0$. Then from (5.13)

$$k''(z) = -\frac{2k'(z) + c_m z k(z)}{z}. \tag{5.15}$$

The relation (5.10) for $j = m - 2$ (after dividing it by $m - 1$) is

$$\left[a_{m-1} z + \tfrac{m}{2} z^2\right] k'' + (b_{m-1} z + 2a_{m-1})\, k' + \left[c_{m-1} z + \tfrac{m}{2} c_m z^2 + b_{m-1} - m\right] k = 0.$$

Substituting k'' from (5.15) into this equation we obtain

$$(b_{m-1} - m) z k' + [(c_{m-1} - c_m a_{m-1}) z + b_{m-1} - m]\, k = 0. \tag{5.16}$$

Let us now consider the cases for different values of m:

a) let $m = 2$, then $\mathscr{a}(y) = y^2 - 1$, $b_2 = 0$ further the boundary conditions imply $b_1 = 2$, $b_0 = 0$ and hence $\mathscr{b}(y) = 2y$. Then (5.16) reads $c_1 k = 0$, hence $c_1 = 0$ and so $\mathscr{c}(y) = c_2 y^2$. $k(z)$ is determined from (5.14), where $\mu^2 = -c_2$. This proves formulas (3.1) and (3.2) of Theorem 1 in the limiting case $\lambda = 0$.

b) let $m = 3$, then $\mathscr{a}(y) = (y^2 - 1)(y - \sigma)$ and $b_3 = 0$. In particular we see that $a_2 = -\sigma$ and $a_1 = -1$. From the boundary conditions $b_0 = 2 - b_2$; $b_1 = -2\sigma = 2a_2$. The relation (5.10) with $j = m - 3 = 0$ reads

$$(z^3 + a_2 z^2 + a_1 z)k'' + (b_2 z^2 + b_1 z + 2a_1)k' + (c_3 z^3 + c_2 z^2 + c_1 z)k = 0.$$

Substituting k'' from (5.15) this simplifies to

$$(b_2-2)z^2k' + [(c_2 - c_3a_2)z^2 + (c_1+c_3)z]k = 0,$$

and combining this with (5.16) we obtain

$$zk' + (c_1 + c_3 - b_2 + 3)\, k = 0.$$

But because k has a simple pole at 0, we must have $c_1 + c_3 - b_2 + 3 = 1$, hence $c_3 = b_2 - c_1 - 2$. Then $k(z) = 1/z$, substituting this expression into (5.13) we conclude $c_1 = b_2 - 2$ and hence $c_3 = 0$. Next we substitute it into (5.16) to find $c_2 = 0$. Thus

$$\begin{cases} \mathscr{a}(y) = (y^2-1)(y-\sigma) \\ \mathscr{b}(y) = b_2y^2 - 2\sigma y + 2 - b_2 \\ \mathscr{c}(y) = (b_2-2)y \end{cases}$$

This proves item 4 of Theorem 1, when $\beta = b_2 - 3$ and $\mathscr{p}$ is a first order polynomial.

c) let $m = 4$, then $\mathscr{a}(y) = (y^2-1)(y-\sigma_1)(y-\sigma_2)$, $b_4 = 0$. Note that $a_3 = -\sigma_1 - \sigma_2; a_2 = \sigma_1\sigma_2 - 1$. Further, from the boundary conditions on $\mathscr{b}$ we get $b_1 = 2(a_2+2) - b_3$ and $b_0 = -b_2 + 2a_3$. From (5.14) k has two possible forms, assume first $k(z) = \frac{1}{z}(\alpha_1 z + \alpha_2)$ in which case $c_4 = \frac{b_4^2}{4} = 0$. Since k has a simple pole at the origin $\alpha_2 \neq 0$ and let us normalize $\alpha_2 = 1$. (5.16) in this case reads $(b_3-4)zk' + (c_3z + b_3 - 4)k = 0$. Substituting the expression for k into this equation we obtain

$$c_3\alpha_1 z + c_3 + (b_3 - 4)\alpha_1 = 0,$$

which implies that $c_3 = 0$ and

$$\alpha_1(b_3 - 4) = 0. \tag{5.17}$$

The relations (5.10) with $j = m-3$ and $j = m-4$ read respectively as

$$(4z^3 + 3a_3z^2 + 2a_2z)k'' + (3b_3z^2 + 2b_2z + 4a_2)k' + 2(c_2z - 3a_3 + b_2)k = 0, \tag{5.18}$$

$$(z^4+a_3z^3+a_2z^2+a_1z)k''+(b_3z^3+b_2z^2+b_1z+2a_1)k'+(c_2z^2+c_1z-2a_2+b_1)k = 0. \tag{5.19}$$

Now, (5.17) implies that we should consider two cases:

• If $\alpha_1 = 0$, we substitute $k(z) = \frac{1}{z}$ into (5.18) and find $c_2 = \frac{3}{2}b_3 - 4$. Finally substitution into (5.19) gives

$$\frac{b_3 - 4}{2} z + 2a_3 - b_2 + c_1 = 0,$$

therefore $b_3 = 4$ and $c_1 = -2a_3 + b_2$. Putting everything together we obtain

$$\begin{cases} a(y) = (y^2 - 1)(y - \sigma_1)(y - \sigma_2) \\ b(y) = 4y^3 + b_2 y^2 + 2(\sigma_1\sigma_2 - 1)y - b_2 - 2(\sigma_1 + \sigma_2) \\ c(y) = 2y^2 + (b_2 + 2\sigma_1 + 2\sigma_2)y \end{cases}$$

This proves item 4 of Theorem 1, when $\beta = b_2 + 3(\sigma_1 + \sigma_2)$ and p is a second order polynomial.

- If $\alpha_1 \neq 0$, we get $b_3 = 4$, substituting $k(z) = \alpha_1 + \frac{1}{z}$ into (5.18) we obtain

$$c_2 \alpha_1 z + (b_2 - 3a_3)\alpha_1 + c_2 - 2 = 0,$$

hence we deduce $c_2 = 0$ and $\alpha_1(b_2 - 3a_3) = 2$. Finally, we substitute k into (5.19) and obtain $c_1 = -3a_3 + b_2$ and $a_3(b_2 - 3a_3) = 0$, but because $b_2 - 3a_3 \neq 0$ we get $a_3 = 0$, i.e. $\sigma_1 = -\sigma_2$. Then also $\alpha_1 = \frac{2}{b_2}$, $k(z) = \dfrac{2}{b_2} + \dfrac{1}{z}$ and

$$\begin{cases} a(y) = (y^2 - 1)(y^2 - \sigma_1^2), \\ b(y) = 4y^3 + b_2 y^2 - 2(\sigma_1^2 + 1)y - b_2, \\ c(y) = b_2 y. \end{cases}$$

This establishes item 3 of Theorem 1 with $\beta = b_2/2$.

Let now $k(z) = \frac{1}{z}(\alpha_1 \sinh(\mu z) + \alpha_2 \cosh(\mu z))$, with $\mu^2 = -c_4 \neq 0$. One can check by subsequent substitutions into (5.16), (5.18) and (5.19) that this case is impossible.

d) Subsequent substitutions show also that $m \geq 5$ is impossible.

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6 Appendix

Here we prove Lemma 4, stating that if the functions a, b, c contain an exponential term, the polynomial multiplying it must be a constant. So let us concentrate on a typical exponential term in a, b and c, namely

$$a \leftrightarrow e^{\lambda y} \sum_{j=0}^{2} a_j y^j, \qquad b \leftrightarrow e^{\lambda y} \sum_{j=0}^{3} b_j y^j, \qquad c \leftrightarrow e^{\lambda y} \sum_{j=0}^{3} c_j y^j.$$

The goal is to show that all the coefficients vanish, except possibly for a_0, b_0, c_0. We are going to substitute these expressions into (R1). The result becomes a linear combination of terms $y^j e^{\lambda y}$, hence the coefficient of each such terms must vanish. Below we analyze these coefficients, which are in fact ODEs for k.

1. First let us show that the polynomials in $\mathcal{b}$ and $\mathcal{c}$ cannot be of higher order, than the polynomial in $\mathcal{a}$, i.e. $b_3 = c_3 = 0$. The equations corresponding to $y^3 e^{\lambda y}$ and $y^2 e^{\lambda y}$ are

$$\begin{aligned} b_3(e^{\lambda z}-1)k' + \left[b_3\lambda + c_3(e^{\lambda z}-1)\right]k = 0,\\ 3(b_3k' + c_3k)e^{\lambda z}z + (a_2k'' + b_2k' + c_2k)e^{\lambda z} + (2\lambda a_2 - b_2)k' - a_2k'' -\\ -[\lambda^2 a_2 - b_2\lambda + c_2 - 3b_3]k = 0. \end{aligned} \tag{6.1}$$

Assume $b_3 \neq 0$, from the first equation $k(z) = e^{\left(\lambda - \frac{c_3}{b_3}\right)z}/(e^{\lambda z}-1)$. Invoking Remark 2 w.l.o.g. we assume $c_3 = \lambda b_3$ in which case $k(z) = 1/(e^{\lambda z}-1)$. Substitute this into the second equation and multiplying the result by $(e^{\lambda z}-1)^2$ we obtain

$$(a_2\lambda^2 - b_2\lambda + c_2)e^{2\lambda z} + (2b_2\lambda - 2a_2\lambda^2 + 3b_3 - 2c_2)e^{\lambda z} - 3b_3\lambda z e^{\lambda z} + a_2\lambda^2 - b_2\lambda + c_2 - 3b_3 = 0.$$

The functions $e^{2\lambda z}, e^{\lambda z}, ze^{\lambda z}$ and 1 are linearly independent, hence the coefficient of each one must vanish. But we see that the coefficient of $ze^{\lambda z}$ is $3b_3\lambda \neq 0$, which is a contradiction. Thus, $b_3 = 0$ and therefore also $c_3 = 0$.

2. We now show that $a_2 = 0$. The equations corresponding to $y^2 e^{\lambda y}$ and $ye^{\lambda y}$ are

$$\begin{aligned} a_2(e^{\lambda z}-1)k'' + \left[2a_2\lambda + b_2(e^{\lambda z}-1)\right]k' + \left[b_2\lambda - a_2\lambda^2 + c_2(e^{\lambda z}-1)\right]k = 0,\\ 2(a_2k'' + b_2k' + c_2k)e^{\lambda z}z + (a_1k'' + b_1k' + c_1k)e^{\lambda z} + (2\lambda a_1 + 4a_2 - b_1)k' -\\ -a_1k'' - [\lambda^2 a_1 + (4a_2 - b_1)\lambda + c_1 - 2b_2]k = 0. \end{aligned} \tag{6.2}$$

Assume $a_2 \neq 0$, and by normalization let us assume $a_2 = 1$. Solving the first equation we get (as was done in (5.5))

$$k(z) = \frac{e^{\left(\lambda - \frac{b_2}{2}\right)z}}{e^{\lambda z}-1} \cdot \begin{cases} \alpha_1 z + \alpha_2, & \mu := \sqrt{\frac{b_2^2}{4} - c_2} = 0\\ \alpha_1 e^{\mu z} + \alpha_2 e^{-\mu z}, & \mu \neq 0 \end{cases}$$

Using Remark 2 let us w.l.o.g. assume $b_2 = 2\lambda$.

Let k be given by the first formula. Since $\alpha_2 \neq 0$ we may normalize it to be one, so $k(z) = \frac{\alpha_1 z + \alpha_2}{e^{\lambda z}-1}$ and $c_2 = \frac{b_2^2}{4}$. Substituting this expression into the second equation of (6.2) and multiplying the result by $(e^{\lambda z}-1)^3$ we obtain

$$\begin{aligned} (p_1z + p_2)e^{3\lambda z} + \left[2\lambda^2\alpha_1 z^2 + \left((2 - 3\alpha_1 a_1)\lambda^2 + (3b_1 - 8)\alpha_1\lambda - 3c_1\alpha_1\right)z + p_3\right]e^{2\lambda z} +\\ +(p_4z^2 + p_5z + p_6)e^{\lambda z} + p_7z + p_8 = 0, \end{aligned}$$

where p_j are constants depending on $a_1, b_1, c_1, \alpha_1, \lambda$ and their particular expressions are not important. From linear independence the coefficient of $z^2 e^{2\lambda z}$ must vanish, which implies $\alpha_1 = 0$, but then the coefficient of $ze^{2\lambda z}$ becomes $2\lambda^2 \neq 0$, which leads to a contradiction.

Let k be given by the second formula, then $c_2 = \frac{b_2^2}{4} - \mu^2$ and $\mu \neq 0$. Substituting k into the second equation of (6.2) and multiplying the result by $e^{\mu z}(e^{\lambda z} - 1)^3$ we obtain

$$\begin{aligned} &\alpha_1(\mu + \tfrac{\lambda}{2})ze^{(2\mu+\lambda)z} - \alpha_1(\mu - \tfrac{\lambda}{2})ze^{(2\mu+2\lambda)z} + \alpha_2(\mu + \tfrac{\lambda}{2})ze^{2\lambda z} - \alpha_2(\mu - \tfrac{\lambda}{2})ze^{\lambda z} = \\ &= q_0 + q_1 e^{\lambda z} + q_2 e^{2\lambda z} + q_3 e^{3\lambda z} + q_4 e^{2\mu z} + q_5 e^{(2\mu+\lambda)z} + q_6 e^{(2\mu+2\lambda)z} + q_7 e^{(2\mu+3\lambda)z}, \end{aligned} \tag{6.3}$$

where q_j are constants whose particular expressions are not important. Note that the functions on LHS of (6.3) are linearly independent from the ones on RHS. If all the exponents on LHS are distinct then the coefficients multiplying them must be zero. In particular $\alpha_1(\mu + \frac{\lambda}{2}) = 0$ and $\alpha_1(\mu - \frac{\lambda}{2}) = 0$, which imply $\alpha_1 = 0$. Analogously, $\alpha_2 = 0$ leading to $k = 0$. Now assume the exponents on LHS of (6.3) are not distinct, then there are two possibilities:

a) $2\mu + \lambda = 2\lambda$, hence $\lambda = 2\mu$ and LHS of (6.3) becomes $2\mu(\alpha_1 + \alpha_2)ze^{4\mu z}$. Hence $\alpha_1 = -\alpha_2$, which then implies

$$k(z) = \frac{2\alpha_1 \sinh(\mu z)}{e^{\lambda z} - 1}.$$

This contradicts to the assumption that k has a simple pole at the origin.

b) $2\mu + 2\lambda = \lambda$, hence $\lambda = -2\mu$. Similarly, this case also leads to a contradiction.

3. To show $b_2 = c_2 = 0$, we can apply the same argument of 1, because once we established $a_2 = 0$ the equations in (6.2) are exactly the ones in (6.1), the only difference is that in the latter we need to replace b_3, c_3 by $\frac{2}{3}b_2, \frac{2}{3}c_2$ and a_2, b_2, c_2 by a_1, b_1, c_1 respectively. After this, in an analogous way to 2, we show that $a_1 = 0$, again the equations corresponding to $ye^{\lambda y}$ and $e^{\lambda y}$ are exactly the ones in (6.2) only a_2, b_2, c_2 need to be replaced by $\frac{a_1}{2}, \frac{b_1}{2}, \frac{c_1}{2}$ and a_1, b_1, c_1 by a_0, b_0, c_0 respectively. Finally, again as in 1, we establish that also $b_1 = c_1 = 0$.

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