On the commutation properties of finite convolution and differential operators.

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Abstract

Spectral properties of finite convolution operators can sometimes be understood if they can be related to differential operators that commute with them. We characterize all commuting pairs of such operators, placing no symmetry constraints on the kernel and extending prior commutation results for real self-adjoint finite convolution operators. We also introduce and fully analyze a new type of commutation, that we call sesquicommutation, that also has implications for spectral properties of integral operators. In particular, a new large class of compact self-adjoint finite convolution operators, whose eigenfunctions solve differential equations is explicitly described.

1 Introduction

The need to understand spectral properties of finite convolution integral operators

\[(Ku)(x) = \int_{-1}^{1} k(x-y)u(y)dy \tag{1.1}\]

acting on \(L^2(-1,1)\) arises in a number of applications, including optics [6], radio astronomy [3], [4], electron microscopy [8], x-ray tomography [11], [23], noise theory [5] and medical imaging [2], [12], [13], [14]. In some cases it is possible to find a differential operator \(L\) which commutes with \(K\) (cf. [20, 18, 24, 12]),

\[KL = LK.\]  \(\tag{C1}\)

If \(k(z)\) is smooth the eigenfunctions of \(K\) also have to be smooth and hence can be chosen to be solutions of ordinary differential equations. More precisely, (C1) implies that eigenspaces \(E_\lambda\) of \(K\) are invariant under \(L\), i.e. \(L : E_\lambda \mapsto E_\lambda\). Now if \(L\) is diagonalizable, e.g. self-adjoint, or more generally normal, then one can choose a basis for \(E_\lambda\) consisting of eigenfunctions of \(L\). This permits to bring the vast literature on asymptotic properties of solutions of ordinary differential equations to bear on obtaining analytical information about the asymptotics of eigenvalues and eigenfunctions of integral operators. With this said, we will not be investigating spectral properties of differential operators that commute with integral operators. In our view questions about differential operators are much more tractable than questions about the integral operators, see e.g. [26], and our goal is to find all connections between the two questions.

The most famous example of this phenomenon is the band-and time limited operator of Landau, Pollak, and Slepian [16], [17], [20]–[22], corresponding to \(k(z) = \frac{\sin(az)}{z}\) in (1.1) with \(a > 0\). Sharp estimates for asymptotics of the eigenvalues of \(K\) were derived using its commutation with a second order symmetric differential operator, whose eigenfunctions are the well-known prolate spheroidal wave functions that first appeared in the context of quantum mechanics [19]. Another example is the result of Widom [24], where using comparison with special operators that commute with differential operators, the author obtained asymptotic behavior of the eigenvalues of a large class of integral operators with real-valued even kernels. A complete characterization of operators (1.1) with real even kernel commuting with symmetric second order differential operators was achieved by Morrison [18] (see also [25], [10]). We are interested in completing Morrison’s characterization to include all complex-valued kernels \(k(z)\). In this more general context the property of commutation must also be generalized, so as to permit the characterization of eigenfunctions as solutions of an eigenvalue problem for a second or fourth order differential operator.
A natural extension of commutation, as explained in the introductory section in [1] is

\[
\begin{align*}
KL_1 &= L_2 K \\
L_j^* &= L_j, \quad j = 1, 2 
\end{align*}
\]  

(C2)

where \(L_j, j = 1, 2\) are differential operators with complex coefficients. This has implications for singular value decomposition of \(K\). It is easy to check that (C2) reduces to a commutation relation for \(K^*K\), indeed we have

\[L_1 K^* K = K^* K L_1,\]

(1.2)

and therefore singular functions of \(K\) satisfy ODEs, in the sense explained above. In fact, commuting pairs \((K, L)\), when \(K\) is non-compact can also provide instances where singular value decomposition of a related operator can be obtained via (C2), as was observed in [2], [12], [13], [14] in applications to truncated Hilbert transform operators, where \(k(z) = 1/z\). In this setting the input function is considered on one interval while the output of \(K\) is defined on a different interval. Even though singularity of \(k(z)\) destroys compactness of \(K\), in was shown in the above cited papers that \(K^*K\) possesses a discrete spectrum and singular value decomposition for \(K\) can be obtained.

In this paper we give a complete list of pairs \((K, L)\), satisfying commutation relation (C1), under the assumption that \(k\) is either analytic at the origin or has a simple\(^1\) pole at 0, in which case the integral is understood in the principal value sense (cf. Theorem 1). Among the most important consequences of this characterization we mention two. The first is that any finite convolution operator \(K\), with analytic kernel at the origin, admitting commutation must be similar to Morrison’s operator (cf. Remark 5). The second is the complete description of singular kernels \(k(z)\) for which \(K^*K\) in the sense described above has a discrete spectrum. As an example of one new result is the operator with kernel \(k(z) = 1/\sin(\pi z)\) considered from \(L^2(-1,1) \to L^2(3,5)\) and the operator \(k(z) = \beta + \frac{1}{z}\), with \(\beta \in \mathbb{R}\), considered from \(L^2(-1,1) \to L^2(-b, b)\) (for any \(b \neq 1\)), both have discrete singular value decompositions (see Remark 10 for details and more examples).

The fact that aside from Morrison’s class of compact self-adjoint finite convolution operators and their conjugates there are no essentially new examples is remedied in the second part of this work where we consider a new kind of commutation relation

\[
\mathcal{K}L_1 = L_2 K, \quad K^* = K, \quad L_j^T = L_j, \quad j = 1, 2,
\]

(1.3)

which we call \textit{sesquicommutation}.

We note that Morrison’s result lies in the intersection of commutation and sesquicommutation (with \(L_1 = L_2\)), when \(K\) is real and self-adjoint, since in this case sesquicommutation reduces to commutation.

The main case of interest is for self-adjoint operator \(K\). However, even if \(K\) is not self-adjoint (but compact) the sesquicommutation (1.3) permits us to relate singular values and functions of \(K\) to solutions of differential equations. It can be easily checked that (1.3) implies

\[L_1 K^* K = \mathcal{K}^* \mathcal{K} L_1.\]

(1.4)

Let now \(\lambda\) be a singular value of \(K\) corresponding to singular function \(u\), i.e. \(K^* Ku = \lambda u\), clearly \(\lambda \in \mathbb{R}\) and therefore we find \(\lambda L_1 u = K^* K L_1 u\). It follows that \(L_1 u\) is either zero, or an eigenfunction of \(K^*K\) with the same eigenvalue \(\lambda\). If the corresponding eigenspace of \(K^*K\) is one-dimensional, then there exists a complex number \(\sigma\) such that

\[L_1 u = \sigma u.\]

Otherwise, applying (1.4) to \(L_1 u\) we find that

\[K^* K (L_1^* L_1 u) = \lambda L_1^* L_1 u,
\]

hence eigenspaces of \(K^*K\) are invariant under the fourth order self-adjoint operator \(L_1^* L_1\). In particular, there exists an eigenbasis of \(K^*K\) consisting of eigenfunctions of \(L_1^* L_1\). Moreover, transposing the sesquicommutation relation and then taking adjoint we find \(KL_1^* = L_1^* \mathcal{K}\), which along with (1.3) implies

\(^1\)It is not hard to show that commutation is not possible for higher order poles.
Due to the imposed boundary conditions it is a matter of integration by parts to rewrite (1.3) as

\[ KL_1^* L_1 = L_2^* L_2 K. \]

In particular if \( L_1 = L_2 =: L \) we see that \( L^* L \) commutes with \( K \) (and also with \( K^* \)), hence eigenspaces of \( L^* L \) are invariant under \( K \) and \( K^* \).

Under the assumption that \( K \) is self-adjoint we prove in Theorem 3 that \( k \) is trivial (see Definition 1), unless \( L_1 = L_2 \) or \( L_1 = -L_2 \). We then show in Theorem 6 that the latter case yields only trivial kernels. The results in the former case are listed in Theorem 4, which presents a new class of finite convolution operators whose spectral properties will be amenable to analysis by means of differential equations.

As a particularly interesting example derived from sesquicommutation, we mention that the eigenfunctions of the compact self-adjoint integral operator \( K \) with kernel \( k(z) = \frac{e^{-i\frac{z}{2}}}{\cos \frac{z}{2}} + \frac{ze^{i\frac{z}{2}}}{\sin \frac{z}{2}} \) are eigenfunctions of the fourth order self-adjoint differential operator \( L^* L \), where

\[ L = -\frac{d}{dz} \left[ \cos \left( \frac{xz}{2} \right) \frac{d}{dy} \right] + \frac{x^2}{4z} e^{i\frac{xz}{2}}. \]

The corresponding integral operator \( K \) is self-adjoint and compact, since singularities at \( z = \pm 2 \) of \( k(z) \) are removable.

## 2 Preliminaries

We assume that \( zk(z) \in L^2((-2, 2), \mathbb{C}) \) is analytic in a neighborhood of 0. This includes two cases: regular, when \( k \) is analytic at 0, and singular, when \( k \) has a simple pole at 0, in which case the integral is understood in the principal value sense. Further, assume that \( L, L_j \) are second order differential operators:

\[
\begin{aligned}
Lu &= \alpha u'' + \beta u' + \epsilon u, \\
\alpha(\pm 1) &= 0, \quad \beta(\pm 1) = \beta'(\pm 1),
\end{aligned}
\]

(2.1)

where the indicated boundary conditions are necessary for the above commutation relations to hold. These are also necessary for the adjoint operator to be a differential operator as well. Thus various classes of operators, such as self-adjoint, symmetric or normal can be described by specifying additional constraints on the coefficients of \( L \), always assuming that the boundary conditions in (2.1) hold.

When \( k \) is smooth in \([-2, 2]\), formulating commutation relations (C1) and (C2) in terms of the kernel \( k(z) \) and the coefficients of \( L \) is a matter of integration by parts, which due to the imposed boundary conditions lead, respectively, to

\[
[\alpha_2(y + z) - \alpha_1(y)]k''(z) + [2\alpha_1'(y) + \beta(y + z) - \beta(y)]k'(z) + [\epsilon_2(y + z) - \epsilon(y) + \beta'(y) - \epsilon''(y)]k(z) = 0,
\]

(R1)

\[
[\alpha_2(y + z) - \alpha_1(y)]k''(z) + [2\alpha_1'(y) + \beta_2(y + z) - \beta_1(y)]k'(z) + [\epsilon_2(y + z) - \epsilon_1(y) + \beta'_1(y) - \epsilon''_1(y)]k(z) = 0,
\]

(R2)

where \( \alpha_j, \beta_j, \epsilon_j \) denote the coefficients of \( L_j \) for \( j = 1, 2 \). Less obviously (see Remark 7), the same relation (R1) holds if \( k \) has a simple pole at 0.

For sesquicommutation (1.3) the operators \( L_j \) have to be of Sturm-Liouville type, since \( L = L^T \) implies that \( \beta = \alpha' \). Thus

\[
\begin{aligned}
L_j u &= (\beta_j u')' + e_j u, \\
\beta_j(\pm 1) &= 0,
\end{aligned}
\]

(2.2)

Due to the imposed boundary conditions it is a matter of integration by parts to rewrite (1.3) as

\[
\beta_1(y)k''(z) - \beta_2(y + z)k''(z) - \beta'_1(y)k'(z) - \beta'_2(y + z)k'(z) + \epsilon_1(y)k(z) - \epsilon_2(y + z)k(z) = 0.
\]

(R)
The main idea of the proof is to analyze (R) by differentiating it w.r.t. $z$ sufficient number of times and evaluating the result at $z = 0$. This allows one to find relations between the coefficient functions of the differential operators, and an ODE for the highest order coefficient. Once the form of the highest order coefficient is determined, we consequently find the forms of all the other coefficient functions. It turns out that the coefficient functions satisfy linear ODEs with constant coefficients, and therefore are equal to linear combinations of polynomials multiplied by exponentials. We then substitute these expressions into (R) and using the linear independence of functions $y^j e^{y \lambda_i}$, obtain equations for $k$. Then the task becomes to analyze how many of these equations can be satisfied by $k$ and how its form changes from one equation to another.

**Remark 1.** The reason that reduction of (1.3) to $L_1 = \pm L_2$ (see Section 7) works, is the self-adjointness assumption on $K$. This induces symmetry in (R). More precisely, (R) becomes a relation involving the even and odd parts (and their derivatives) of the function $k(z)e^{\frac{z}{2}}$. And as a result the relations for even and odd parts separate. We then prove that if $L_1 \neq \pm L_2$, then both even and odd parts of $k$ are determined in a way that $k$ becomes trivial.

The main idea of the proofs is to analyze these relations by taking sufficient number of derivatives in $z$ and evaluating the result at $z = 0$. This allows one to find linear differential relations between the coefficients of the differential operators, narrowing down the set of possibilities to families of functions depending on finitely many parameters. Returning to the original relations (R1), (R2) we obtain necessary and sufficient conditions for commutation that can be completely analyzed, resulting in the explicit listing of all pairs $(k, L)$ satisfying (R1).

**Remark 2.** The complete analysis of (C2) beyond the instances generated by (C1), can also be achieved by our approach, but will require substantially more work. We remark that in this case too it can be shown that either $k$ is trivial or the coefficients of $L_1$ and $L_2$ are linear combinations of polynomials multiplied by exponentials.

### 3 Main Results

#### 3.1 Commutation

**Definition 1.** We will say that $k$ (or operator $K$) is trivial, if it is a finite linear combination of exponentials $e^{\alpha z}$ or has the form $e^{\alpha z}p(z)$, where $p(z)$ is a polynomial. Note that in this case $K$ is a finite-rank operator.

**Remark 3.** When $K$ commutes with $L$, then $MKM^{-1}$ commutes with $MLM^{-1}$. If $M$ is the multiplication operator by $z \mapsto e^{\tau z}$, then $MKM^{-1}$ is a finite convolution operator with kernel $k(z)e^{\tau z}$ (where $k$ is the kernel of $K$) and $MLM^{-1}$ is a second order differential operator with the same leading coefficient as $L$. Moreover, one can also add any complex constant to $e(y)$ in (2.1), as well as multiply $k$, as well as $L$ by arbitrary complex constants without affecting commutation. With this observation the results of Theorem 1 are stated up to such transformations in order to achieve the most concise form of the results.

In theorem below all parameters are complex, unless specified otherwise.

**Theorem 1** (Commutation (C1))

Let $K, L$ be given by (1.1) and (2.1) with $a, \delta, e$ smooth in $[-2, 2]$. Assume $k$ is smooth in $[-2, 2] \setminus \{0\}$ and either it

(i) is analytic at 0, not identically zero near 0 and is nontrivial in the sense of Definition 1.

(ii) has a simple pole at 0.

If (R1) holds, then (in case $\lambda$ or $\mu = 0$ appropriate limits must be taken)

$$k(z) = \frac{\lambda}{\sinh \left(\frac{z}{2}\right)} \left( \frac{\sinh(\mu z)}{\mu} + \alpha_2 \cosh(\mu z) \right)$$

(3.1)

$$\begin{cases}
a(y) = \frac{1}{\lambda^2} \left[ \cosh(\lambda y) - \cosh \lambda \right] \\
\delta(y) = a'(y) \\
e(y) = \left( \frac{\lambda^2}{4} - \mu^2 \right) a(y)
\end{cases}$$

(3.2)
For some special choices of parameters, the differential operator commuting with $K$ is more general than the one given by (3.2). Below we list all such cases:

1. $\alpha_1 = 0$, $\lambda = \pi i$, $\mu = \frac{2m+1}{4} \lambda$ with $m \in \mathbb{Z}$:

$$k(z) = \frac{\cos \left( \frac{\pi(2m+1)}{4} z \right)}{\sin \left( \frac{\pi z}{2} \right)} \quad \text{and} \quad \begin{cases} a(y) = \alpha \left( e^{\pi iy} - e^{-\pi iy} \right) + \beta \left( e^{-\pi iy} - e^{-\pi iy} \right) \\ b(y) = e^\prime(y) \\ c(y) = \frac{\pi}{4} \left[ (2m+1)^2 - 1 \right] a(y) \end{cases}$$

When $\alpha = \beta$ (3.2) is recovered.

2. $\alpha_1 = \mu = 0$, then with $a_0(y) = \cosh(\lambda y) - \cosh \lambda$:

$$k(z) = \frac{1}{\sinh \left( \frac{\pi z}{2} \right)} \quad \text{and} \quad \begin{cases} a(y) = \alpha a_0(y) \\ b(y) = \alpha a_0(y) + \beta a_0(y) \\ c(y) = \frac{\beta}{4} a_0(y) + \alpha \frac{\lambda^2}{4} a_0(y) \end{cases}$$

When $\beta = 0$ (3.2) is recovered.

3. $\mu = \lambda = 0$, then with $p(y)$ an arbitrary polynomial of order at most two such that $p'(0) = 0$:

$$k(z) = \frac{1}{\beta} + \frac{1}{z} \quad \text{and} \quad \begin{cases} a(y) = (y^2 - 1) p(y) \\ b(y) = a'(y) + \beta y p'(y) - \beta p''(y) \\ c(y) = \beta p'(y) \end{cases}$$

When $p(y) \equiv 1$ (3.2) is recovered.

4. $\mu = \lambda = \alpha_1 = 0$, then with $p(y)$ an arbitrary polynomial of order at most two:

$$k(z) = \frac{1}{z} \quad \text{and} \quad \begin{cases} a(y) = (y^2 - 1) p(y) \\ b(y) = a'(y) + \beta(y^2 - 1) \\ c(y) = yp'(y) + \beta y \end{cases}$$

When $p(y) \equiv 1$ and $\beta = 0$ (3.2) is recovered.

Remark 4. If $\lambda \in i\mathbb{R}$, then $k(z)$ may become singular at $z \in [-2,2] \setminus \{0\}$. In order to exclude these cases we need to require either

- $|\lambda| < \pi$, or
- $\pi \leq |\lambda| < 2\pi$ and $\alpha_1 = 0$, $\mu = \lambda \frac{2m+1}{4}$ for some $m \in \mathbb{Z}$

Remark 5. Morrison’s result corresponds to the analytic case: $\alpha_2 = 0$ and when $k$ is even and real-valued. According to Theorem 1 all integral operators with analytic $k(z)$ that commute with a differential operator are similar to Morrison’s operator and therefore their spectrum can be determined using Morrison’s results.

Remark 6. As we have already mentioned, the connections between the coefficient functions of the differential operators are obtained by differentiating the relation (R1) appropriate number of times and setting $z = 0$. Smoothness of coefficients, analyticity of $k$ at zero (the fact that $k$ is nontrivial and that it doesn’t vanish near 0) are used at this stage, to argue that the differentiation procedure can be terminated at some point and the connections between the coefficient functions will follow. Thus, the original assumptions can be replaced by requiring appropriate degree of smoothness on $k$ and the coefficient functions and that some expressions involving $k^{(j)}(0)$ are not zero. These expressions can be easily found from our analysis. For example the hypotheses of Theorem 1 (case (i)) can be replaced by $a, b, c, k \in C^3$ and $k^2(0)k''(0) - k(0)k'(0) \neq 0$ (cf. Section 4). Analogous changes can be made in case (ii) of Theorem 1.
Remark 7. When \( k \) has a pole at zero, the commutation is understood in the principal value sense, namely

\[
\lim_{\epsilon \to 0} \int_{[-1,1]\setminus B_r(x)} k(x-y)Lu(y)dy - L \int_{[-1,1]\setminus B_r(x)} k(x-y)u(y)dy = 0.
\]

After integrating by parts, this can be rewritten as

\[
\lim_{\epsilon \to 0} \int_{[-1,1]\setminus B_r(x)} F(x,y)u(y)dy + \Phi(u, x, \epsilon) = 0,
\]

where \( F(x,y) \) is the left-hand side of (R1) with \( z = x - y \) and

\[
\Phi(u, x, \epsilon) = k(\epsilon) \left\{ \left[ a(x - \epsilon) - a(x) \right] u'(x) + \left[ b(x - \epsilon) - b(x) \right] u(x) \right\} -
\]

\[
-k(-\epsilon) \left\{ \left[ a(x + \epsilon) - a(x) \right] u'(x) + \left[ b(x + \epsilon) - b(x) \right] u(x) \right\} +
\]

\[
+k'(\epsilon) u(x) - \left[ a(x - \epsilon) - a(x) \right] - k'(-\epsilon) u(x) + \left[ a(x + \epsilon) - a(x) \right] u(x).
\]

Expanding \( \Phi(u, x, \epsilon) \) in \( \epsilon \) we observe that all terms up to \( O(\epsilon) \) cancel out and hence, \( \lim_{\epsilon \to 0} \Phi(u, x, \epsilon) = 0 \). Therefore we conclude \( F(x, y) = 0 \) for \( y \neq x \), resulting in the same relation (R1), as in smooth case.

Theorem 1 characterizes solutions of the commutation relation \( KLu = LKu \), where \( u \) is a smooth function on \([-1,1]\). Up to this point we were assuming that \( K : L^2(-1,1) \to L^2(-1,1) \), but following [2], [12], [13], [14] we can consider \( K \) as an operator \( K : L^2(-1,1) \to L^2(a, b) \) by restricting the variable \( x \) in \((Ku)(x)\) to \((a, b)\), where \((a, b)\) is the line segment connecting \( a \) to \( b \) in the complex plane. Now let \( L_2 := L_{(a,b)} \) denote the operator \( L \) acting on (and returning) functions defined on the line segment \((a, b)\) and similarly \( L_1 := L_{(-1,1)} \).

If both \( L_1 \) and \( L_2 \) are self-adjoint (in particular we need the coefficient of \( \frac{d^2}{dx^2} \) in \( L \) to vanish at \( \pm 1 \), \( a \) and \( b \)) we get an example of commutation (C2): \( KL_1u = L_2Ku \), where \( u \) is a smooth function on \([a,b]\). Below we present all such instances that can be deduced from the commutation relation \( KL = LK \) (the results are given up to multiplication of \( k(z) \) by \( e^{\pi z} \), cf Remark 8 below).

Corollary 2. Let \( K : L^2(-1,1) \to L^2(a, b) \) be given by (1.1) and \( L \) be a differential operator given by (2.1), then the commutation relation

\[
\begin{cases}
KL_{(-1,1)}u = L_{(a,b)}Ku & u \in C^\infty[-1,1] \\
L_{(-1,1)} = L_{(-1,1)} & L_{(a,b)} = L_{(a,b)}
\end{cases}
\]

holds for the following choices of operators \( K, L \) and line segments \((a, b)\):

1. \( k \) is given by (3.1), coefficients of \( L \) are given by (3.2) with

\[
\begin{cases}
\lambda, \mu \in \mathbb{R} \cup i\mathbb{R}, & \lambda \neq 0 \\
a = -1 + \frac{2\pi in}{\lambda}, & b = 1 + \frac{2\pi in}{\lambda}, & n \in \mathbb{Z}
\end{cases}
\]

(When \( \lambda \in i\mathbb{R} \) further restrictions of Remark 4 must be taken into account)

2. \( k(z) = \frac{1}{\sinh \left( \frac{z}{2} \right)} \) and with \( \alpha_0(y) = \cosh(\lambda y) - \cosh \lambda \):

\[
\begin{cases}
\alpha(y) = \alpha \alpha_0(y) \\
\beta(y) = \alpha \alpha_0(y) + \beta \alpha_0(y) \\
\gamma(y) = \frac{\beta}{2} \alpha_0(y) + \alpha^2 \gamma_0(y)
\end{cases}
\]

where \( \beta \in i\mathbb{R}, \lambda \in \mathbb{R} \cup i\mathbb{R}, \alpha \in \mathbb{R} \) and \( a = -1 + \frac{2\pi in}{\lambda}, \, b = 1 + \frac{2\pi in}{\lambda} \) with \( n \in \mathbb{Z} \).
3. $k(z) = \frac{1}{\beta} + \frac{1}{z}$ and $L$ has coefficients

$$
\begin{cases}
\phi(y) = (y^2 - 1)(y^2 - b^2) \\
\delta(y) = \phi'(y) + 2\beta(y^2 - 1) \\
\epsilon(y) = 2\beta y,
\end{cases}
$$

where $\beta \in i\mathbb{R}$, $a = -b$ and $b > 0$.

4. $k(z) = \frac{1}{z}$ and $L$ has coefficients

$$
\begin{cases}
\phi(y) = (y^2 - 1)(y - a)(y - b) \\
\delta(y) = \phi'(y) + \beta(y^2 - 1) \\
\epsilon(y) = 2y^2 + (\beta - a - b)y,
\end{cases}
$$

where $\beta \in i\mathbb{R}$ and $a < b$ are real.

**Proof.** The proof immediately follows from Theorem 1 and discussion above, we just mention that in item 1 the restrictions $\lambda, \mu \in \mathbb{R} \cup i\mathbb{R}$ make $L$ self-adjoint on $[-1, 1]$, the choice of $a, b$ follows from the fact that coefficients of $L$ are $\frac{2\pi}{a_k}$-periodic. Therefore, $L$ is also self-adjoint on $[a, b]$. Similarly, in items 2, 3 and 4 the condition $\beta \in i\mathbb{R}$ guarantees self-adjointness of $L$. In item 3 we are forced to take $a = -b$, because in the corresponding commutation relation (item 3 of Theorem 1) $\phi(y) = (y^2 - 1)\rho(y)$ where $\rho'(0) = 0$, hence $\rho(y) = y^2 - b^2$.

**Remark 8.** Due to Remark 3 it is easy to check that in Corollary 2, in each of the four items $K$ can be replaced by $MKM^{-1}$ and $L$ by $MLM^{-1}$, where $M$ is multiplication operator by $e^{\tau z}$ and (in addition to given parameter restrictions) it must hold $\tau \in i\mathbb{R}$ in order for $MLM^{-1}$ to be self-adjoint. Note that in this case $M$ is a unitary operator, therefore $MLM^{-1}$ is self-adjoint if and only if $L$ is. However, for item 2 there is an additional case: $\tau \in \mathbb{C}$ and $\beta = 2i\alpha \text{Im } \tau$.

**Remark 9.** Taking $\beta = 0$ in item 4 we obtain the commutation used in [2], [12], [13], [14] mentioned in the introduction. Indeed, since any real constant can be added to $\phi$ we can rewrite $\phi(y) = 2(y - \frac{a+b}{4})^2$, which is precisely the form of $\phi$ used in those references.

**Remark 10.** Observe that in all of the cases $k(z)$ has a singularity and the corresponding operator $K$ is not compact. The discreteness of the spectrum of $K^*K$ is therefore non-obvious. Following [2, 12, 13, 14], the discreteness of the SVD decomposition comes from the discreteness of the spectrum of self-adjoint differential operators $L_1$ and $L_2$ in (C2), provided that *singularities of $Ku$ are not at the end-points of the interval for the Sturm-Liouville eigenvalue problem for $L_2$*. In particular, the situation when $(-1, 1)$ and $(a, b)$ intersect does not in and of itself cause the appearance of the continuous spectrum. In the context of operators listed in Corollary 2 we can characterize all instances when the discreteness of SVD can be established. Let $\{z_j\}$ be the simple poles of $k$, and $\{y_j\}$ — the zeros of $\phi(y)$. If the set $\{y_j\} \setminus \{z_j \pm 1\}$ has at least two points, say $a$ and $b$, then $Ku$ is regular at points $a, b$ and so (using (3.3)) $K$ maps eigenfunctions of $L_{(-1, 1)}$ to eigenfunctions of $L_{(a,b)}$, making the former the eigenfunctions of $K^*K$. We will call this case regular. In the list of operator pairs in Corollary 2 regular cases do occur. For example, operators in items 3 and 4 of Corollary 2 are regular, while operators in item 1 are regular if and only if for at least two different values of $n \in \mathbb{Z}$

$$
\alpha_1 \sinh \left(\frac{2\pi n^2}{X}\right) + \alpha_2 \cosh \left(\frac{2\pi n^2}{X}\right) = 0.
$$

In that case $\frac{X}{\pi} \in \mathbb{Q}$ and there are countably many choices for $a, b$. For example taking $\alpha_1 = 0$, $\lambda = i\frac{\pi}{2}$, $\mu = i\frac{\pi}{8}$ we can choose $a = 3$, $b = 5$ and obtain

$$
k(z) = \frac{1}{\sin \left(\frac{\pi}{8}z\right)},
\begin{cases}
\phi(y) = \cos \left(\frac{\pi}{8}y\right), \\
\delta(y) = \phi'(y), \\
\epsilon(y) = -\frac{3\pi^2}{64} \phi(y).
\end{cases}
$$
3.2 Sesquicommutation

Let us assume that

(A) $K$ is self-adjoint, so $k(-z) = \overline{k(z)}$, \quad $z \in [-2, 2]$.

**Theorem 3** (Reduction of sesquicommutation)

Let $K, L_1, L_2$ be given by (1.1) and (2.2) with $\theta_j, c_j, k$ smooth in $[-2, 2]$. Assume $k$ is nontrivial, (A) holds, and $k$ is analytic at 0, but not identically zero near 0. Then (1.3) implies either $L_1 = L_2$ or $L_1 = -L_2$.

**Remark 11.** Let $M$ be the multiplication operator by $z \mapsto e^{\tau z}$ with $\tau \in i\mathbb{R}$, then $MKM^{-1}$ is a finite convolution operator with kernel $k(z)e^{\tau z}$ (where $k$ is the kernel of $K$), which is also self-adjoint since so is $K$. If $K$ sesquicommutes with $L$, i.e. $KL = LK$, then $MKM^{-1}$ sesquicommutes with $M^{-1}LM^{-1}$. With this observation the results of Theorem 4 are stated up to multiplication of $k$ by $e^{\tau z}$, i.e. we chose a convenient constant $\tau$ in order to more concisely state the results.

**Theorem 4** ($L_1 = L_2$)

Let $K, L_1, L_2$ be given by (1.1) and (2.2), with $L_1 = L_2$ and let their coefficient functions be $\theta$ and $c$. Let $\theta, c$ be smooth in $[-2, 2]$. Further, assume $k$ is nontrivial, (A) holds, $k$ is analytic at 0, but not identically zero near 0. Then (1.3) implies (all the used parameters are real, unless stated otherwise)

1. $k(z) = \frac{\gamma \sinh \frac{\mu z}{\mu \sinh \gamma z}}{\mu \sinh \gamma z}$,

$$
\begin{cases}
\theta(y) = \frac{1}{2\mu^2} [\cosh(2\gamma y) - \cosh(2\gamma)], \\
c(y) = (\gamma^2 - \mu^2)\theta(y) + c_0,
\end{cases}
$$

where $\mu \in \mathbb{R} \cup i\mathbb{R}$ and $c_0 \in \mathbb{C}$.

2. $k(z) = a e^{-i\mu z} + \frac{\sin \mu z}{z}$, \quad $\alpha \neq 0$ and

$$
\begin{cases}
\theta(y) = y^2 - 1, \\
c(y) = i\mu\theta'(y) + \mu^2\theta(y) + \frac{\mu}{\alpha},
\end{cases}
$$

3. $k(z) = \frac{\sinh(2\mu_2)\sinh(\mu_1 z)e^{-i\frac{\pi z}{2}} + \sinh(2\mu_1)\sinh(\mu_2 z)e^{i\frac{\pi z}{2}}}{\mu_1 \mu_2 \sin \frac{\pi z}{2}}$ and

$$
\begin{cases}
\theta(y) = -\cos \frac{\pi y}{2}, \\
c(y) = i\mu_2^2 - \mu_1^2 \theta'(y) - \left(\frac{\mu_1^2 + \mu_2^2}{4}\right)\theta(y),
\end{cases}
$$

where $\mu_1, \mu_2 \in \mathbb{R} \cup i\mathbb{R}$. In the special case $\mu_1 = i\mu; \mu_2 = i(\mu \pm \frac{\pi}{2})$ with $\mu \in \mathbb{R}$, to $c(y)$ a complex multiple of $e^{-2i(\frac{\pi}{4} \pm \mu)y}$ can be added.

**Remark 12.**

(i) In items 1 and 3, if $\mu, \mu_1$ or $\gamma = 0$, one takes appropriate limits. Note that $k$ can be multiplied by arbitrary real constant and $L_1 = L_2$ by a complex one.

(ii) Using the same proof techniques one can easily check that under the given assumptions of the theorem, no kernel would satisfy the sesquicommutation relation, when $L_1 = L_2$ is a first order operator.

(iii) In item 1, $K$ is real valued and self-adjoint, in particular sesquicommutation reduces to commutation and we recover Morrison’s result.
(iv) Widom’s theory of asymptotics of eigenvalues applies only if \( k(z) \) has an even extension to \( \mathbb{R} \) such that \( \hat{k}(\xi) \) is nonnegative and monotone decreasing, at least when \( \xi \to \infty \). Item 2 corresponds to \( \hat{k}(\xi) \) being a characteristic function of an interval plus a delta-function, centered anywhere one likes. Item 3 is the most puzzling, it is unknown if there is an extension whose Fourier transform is nonnegative and monotone decreasing. Item 1 are all even kernels.

From the discussion in the introduction we immediately obtain:

**Corollary 5.** Let \( K \) be one of the operators of Theorem 4 and let \( L \) be corresponding operator that sesquicommutates with it (i.e. \( \bar{K}L = LK \)), then \( L^*L \) commutes with \( K \). In particular, the eigenfunctions of \( K \) are eigenfunctions of the fourth order self-adjoint differential operator \( L^*L \). Moreover, if eigenspaces of \( K \) are one-dimensional, then eigenfunction \( u \) of \( K \) satisfies second order differential equation \( Lu = \sigma \pi \) for some \( \sigma \in \mathbb{C} \).

**Remark 13.** The example mentioned in the introduction is obtained from item 3 of Theorem 4 by choosing \( m_2 = 0, \ m_1 = \frac{4}{27} \).

**Theorem 6 \((L_1 = -L_2)\)**

Let \( K, L_1, L_2 \) be given by (1.1) and (2.2), with \( L_1 = -L_2 \) and let the coefficients of \( L_1 \) be \( \theta \) and \( c \). Let \( \theta, c, k \) be smooth in \([-2, 2]\). Further, assume \((A)\) holds, \( k \) is analytic at 0, but not identically zero near 0. If \((1.3)\) holds true, then \( k \) is trivial.

## 4 Commutation, regular case

**Lemma 7.** Assume the setting of Theorem 1 case (i), then for some complex constants \( \alpha, \nu \) we have

\[
\varphi'''(y) + \alpha \varphi'(y) = 0, \quad \vartheta(y) = \varphi'(y), \quad \epsilon(y) = \nu \varphi(y).
\]

**Proof.** Write \( k(z) = \sum_{n=0}^{\infty} \frac{k_n}{n!} z^n \) near \( z = 0 \). The \( n \)-th derivative of (R1) w.r.t. \( z \) evaluated at \( z = 0 \) reads

\[
2 \varphi'(y)k_{n+1} + [\vartheta'(y) - \varphi''(y)]k_n + \sum_{j=0}^{n-1} C_j^n \varphi(n-j)(y)k_{j+2} + \sum_{j=0}^{n-1} C_j^n \vartheta(n-j)(y)k_{j+1} + \sum_{j=0}^{n-1} C_j^n \epsilon(n-j)(y)k_j = 0 \quad (4.2)
\]

where \( C_j^n = \binom{n}{j} \). The above relation for \( n = 0 \) gives

\[
2k_1 \varphi'(y) + [\vartheta'(y) - \varphi''(y)]k_0 = 0. \quad (4.3)
\]

Assume first \( k_0 = 0 \), then \( k_1 = 0 \) (otherwise the boundary conditions imply \( \varphi = 0 \)). By induction one can conclude \( k_j = 0 \) for any \( j \). Indeed, let \( k_j = 0 \) for \( j = 0, ..., n \), then (4.2) reads

\[
(n + 2) \varphi'(y)k_{n+1} = 0.
\]

Hence the boundary conditions imply \( k_{n+1} = 0 \). So if \( k_0 = 0 \), then \( k(z) \) must be identically zero near \( z = 0 \), which we do not allow.

Thus \( k_0 \neq 0 \), and in view of Remark 3 we may assume \( k_1 = k'(0) = 0 \) (otherwise multiply \( k(z) \) by \( e^{-k_1/k_0 z} \)). Taking into account the boundary conditions, from (4.3) we obtain \( \vartheta(y) = \varphi'(y) \). Now we substitute this in (4.2) with \( n = 1 \), integrate the result to find the expression for \( \varphi \) in (4.1) with \( \nu = -\frac{3k_2}{k_0} \). When \( n = 2 \) equation (4.2), after elimination of \( \vartheta \) and \( \epsilon \) becomes \( k_3 \varphi'(y) = 0 \) and we conclude that \( k_3 = 0 \). When \( n = 3 \), we find

\[
k_0k_2 \varphi'''(y) + (5k_0k_4 - 9k_2^2) \varphi'(y) = 0.
\]

If \( k_2 = 0 \), then \( k_4 = 0 \) and as can be immediately seen from (4.2), induction argument shows that \( k_j = 0 \) for all \( j \geq 1 \). Thus, we may assume \( k_2 \neq 0 \), in which case \( \varphi \) satisfies the ODE in (4.1).

\[\square\]

From (4.1) \( \varphi \) has to have one of the following forms, with \( a_j \in \mathbb{C} \)
I. \( a(y) = a_1 e^{\lambda y} + a_2 e^{-\lambda y} + a_0 \), with \( 0 \neq \lambda \in \mathbb{C} \)

II. \( a(y) = a_2 y^2 + a_1 y + a_0 \)

- Assume case I holds, replacing the expressions for \( a, \beta, c \) from Lemma 7, (R1) becomes a linear combination of exponentials \( e^{\pm \lambda y} \) with coefficients depending only on \( z \), hence each coefficient must vanish. These can be simplified as \( a_j \{ k'' + \lambda \coth \left( \frac{\lambda}{2} z \right) k' + \nu k \} = 0 \) for \( j = 1, 2 \). Of course, at least one of \( a_1, a_2 \) is different from zero and so we deduce

\[
k'' + \lambda \coth \left( \frac{\lambda}{2} z \right) k' + \nu k = 0. \tag{4.4}
\]

Setting \( u(z) = k(z) \sinh \left( \frac{\lambda}{2} z \right) \), the above ODE becomes \( u'' + \left( \nu - \frac{\lambda^2}{4} \right) u = 0 \). So,

\[
k(z) = \frac{\sinh(\mu z)}{\mu \sinh \left( \frac{\lambda}{2} z \right)}, \quad \mu^2 = \frac{\lambda^2}{4} - \nu.
\]

When \( \mu = 0 \), the formula is understood in the limiting sense. Note that this is (3.1) with \( \alpha_2 = 0 \) (here \( \alpha_2 \) refers to the parameter in formula (3.1), whose vanishing makes \( k(z) \) analytic on \([-2, 2] \).) Because \( a(y) \) satisfies the boundary conditions we must have \( a_1 = a_2 \) or \( \lambda \in \pi i n \) for some \( n \in \mathbb{Z} \). If \( \lambda = \pi n \), then for \( k \) to be smooth in \([-2, 2] \) we must have \( \mu \neq 0 \), moreover \( \sinh \left( \frac{2\pi m}{n} \right) = 0 \) for any \( m \in \mathbb{Z} \) with \( \frac{m}{n} \in [-1, 1] \). In particular this should hold for \( m = 1 \), which implies \( \mu = \frac{n}{4} \) for some \( l \in \mathbb{Z} \), which in turn implies that \( k \) is a trigonometric polynomial, and hence is trivial. Thus we may assume \( \lambda \notin \pi i n \), and so \( a_1 = a_2 \), showing that \( a(y) = \cosh(\lambda y) - \cosh \lambda \).

Now we show that if \( \lambda \in i \mathbb{R} \), then it must hold \( |\lambda| < \pi \). Otherwise, \( k \) is trivial. Indeed, assume \( \lambda \in i \mathbb{R} \) and \( |\lambda| \geq \pi \) we see that the denominator of \( k(z) \) has additional zeros at \( z = \pm \frac{2\pi}{\lambda} \) in \([-2, 2] \). In order for \( k \) to be smooth, we require that its numerator also vanishes at these points. So \( \sinh \left( \frac{2\pi}{\lambda} \right) = 0 \) and hence \( \mu = \frac{n}{\pi} \) for some \( m \in \mathbb{Z} \). But then, again \( k \) is a trigonometric polynomial.

- Assume case II holds, then \( a(y) = a_2 (y^2 - 1) \) and substituting into (R1) we find

\[
z k'' + 2 k' + \nu z k = 0. \tag{4.5}
\]

Setting \( u(z) = z k(z) \) the ODE turns into \( u'' + \nu u = 0 \), which corresponds to the limiting case \( \lambda = 0 \) in the formulas for \( k \) and \( a \) and concludes the proof of Theorem 1 case (i).

5 Commutation, singular case

Here we prove Theorems 1 case (ii). In the first subsection below we obtain the possible forms for the functions \( a, \beta \) and \( c \). In the second one we do reduction of these forms, and finally in the third one we find \( k \).

5.1 Forms of \( a, \beta \) and \( c \)

By the assumption \( k(z) = z^{-1}(k_0 + k_1 z + ...) \), with \( k_0 \neq 0 \). So by rescaling we let \( k_0 = 1 \) and in view of Remark 3 we may assume \( k_1 = 0 \) (otherwise multiply \( k(z) \) by \( e^{-k_1/k_0 z} \)). Multiply (R1) by \( z^3 \) and refer to the resulting relation by (E). Differentiate (E) three times w.r.t. \( z \) and let \( z = 0 \) to get

\[
c(y) = -\frac{1}{2} a''(y) - 2 k_2 a(y) + \frac{1}{2} \beta'(y) + \text{const.} \tag{5.1}
\]

Substitute this into (E), differentiate the result 4 times w.r.t. \( z \) and let \( z = 0 \), then

\[
\beta'''' = a^{(4)} + 24 k_2 a'' - 72 k_3 a' - 24 k_2 \beta'. \tag{5.2}
\]

In the fifth derivative of (E) we replace \( \beta^{(4)} \) and \( \beta'''' \) using the above relation, then the result reads

\[
\alpha_1 \beta' = a^{(5)} + 120 k_3 a^{(3)} + \alpha_1 a'' + \alpha_2 a', \tag{5.3}
\]

where \( \alpha_1 = -1080 k_3 \) and the expression for \( \alpha_2 \) is not important. Now if \( \alpha_1 = 0 \) we got a linear constant coefficient ODE for \( a \), otherwise we substitute the formula for \( \beta' \) from (5.3) into (5.2) and again obtain an ODE for \( a \), more precisely, for some constants \( \beta_j \in \mathbb{C} \), either
(A) \( \alpha_1 = 0 \) and \( \phi^{(4)} + \beta_1 \phi'' + \beta_2 \phi = \beta_0 \), or
(B) \( \alpha_1 \neq 0 \) and \( \phi^{(6)} + \beta_3 \phi^{(4)} + \beta_1 \phi'' + \beta_2 \phi = \beta_0 \)

Therefore, using the fact that ODEs in (A) and (B) contain only even derivatives of \( \phi \), we can conclude that in either case \( \phi \) has one of the following forms, with \( p_j, a_j, \tilde{a}_j \in \mathbb{C}; \lambda_j, \lambda, \mu \in \mathbb{C} \setminus \{0\} \) and \( \lambda \neq \pm \mu \) and \( \lambda_j \neq \pm \lambda_l \) for \( j \neq l \),

I. \( \phi(y) = \sum_{j=1}^{3} (a_j e^{\lambda_j y} + \tilde{a}_j e^{-\lambda_j y}) + a_0 \)

II. \( \phi(y) = (a_1 y + \tilde{a}_1) e^{\lambda y} + (a_2 y + \tilde{a}_2) e^{-\lambda y} + a_3 e^{\mu y} + \tilde{a}_3 e^{-\mu y} + a_0 \)

III. \( \phi(y) = (a_2 y^2 + a_1 y + a_0) e^{\lambda y} + \tilde{a}_2 y^2 + \tilde{a}_1 y + \tilde{a}_0) e^{-\lambda y} + a_3 \)

IV. \( \phi(y) = \sum_{j=0}^{6} a_j y^j \)

If \( \alpha_1 \neq 0 \), then from (5.3) we see that \( \phi \) has exactly the same form as \( \phi \). Assume now \( \alpha_1 = 0 \), if \( k_2 = 0 \) we find from (5.2) that \( \phi(y) = \phi'(y) + p_2(y^2 - 1) \), if \( k_2 \neq 0 \), then \( \phi \) is of the same form as \( \phi \). If those differ from all the exponentials appearing in \( \phi \), otherwise if one of them coincides, say with \( e^{\lambda y} \), then the polynomial multiplying the latter gets one degree higher. Finally, \( \phi \) is of the same form as \( \phi \).

5.2 Reduction

Our goal is to reduce the cases I–IV and conclude that \( \phi(y) \) can have one of the two forms \( a_1 e^{\lambda y} + a_2 e^{-\lambda y} + a_0 \) or \( \sum_{j=0}^{6} a_j y^j \). Moreover, \( \phi \) and \( \phi \) must have exactly the same form as \( \phi \), but possibly with different constants \( b_j, c_j \) instead of \( a_j \). This reduction will be achieved by the three lemmas below.

**Lemma 8.** If the functions \( \phi, \psi, \chi \) contain an exponential term, the polynomial multiplying it must be constant.

*Proof.* See the appendix.

**Lemma 9.** The functions \( \phi, \psi, \chi \) cannot contain two exponentials \( e^{\lambda y}, e^{\mu y} \) with \( \mu \neq \pm \lambda \).

*Proof.* Consider a typical exponential term in \( \phi, \psi \) and \( \chi \) (due to Lemma 8 the polynomial multiplying it must be a constant), namely

\[
\phi \leftrightarrow a_0 e^{\lambda y}, \quad \psi \leftrightarrow b_0 e^{\lambda y}, \quad \chi \leftrightarrow c_0 e^{\lambda y},
\]

where \( a_0 \neq 0 \). The equation coming from \( e^{\lambda y} \) after substituting these forms into (R1) is (obtained analogously to the first equation of (10.2) in the appendix)

\[
a_0(e^{\lambda z} - 1)k'' + [2a_0 \lambda + b_0(e^{\lambda z} - 1)] k' + [b_0 \lambda - a_0 \lambda^2 + c_0(e^{\lambda z} - 1)] k = 0.
\]

After changing the variables \( u(z) = k(z)(e^{\lambda z} - 1) \) it becomes

\[
a_0u'' + (b_0 - 2a_0 \lambda)u' + (a_0 \lambda^2 - b_0 \lambda + c_0)u = 0.
\]

(5.4)
We note that this is not compatible with (5.5), because cross multiplying the two formulas we get (with $\alpha_2$ given by the second formula of (5.5) (in the other case the same argument will apply), write $a_0 \sinh(\mu z) + \alpha_2 \cosh(\mu z)$, hence it is enough to show that $\alpha_2$ is determined up to the sign. Let $k$ be given by the second formula of (5.5) (in the other case the same argument will apply), write $\mu = \mu_1 + i\mu_2$ and $\lambda = \lambda_1 + i\lambda_2$.

Let $\lambda_1 \neq 0$ and $\mu_1 \neq 0$, then w.l.o.g. we may assume $\mu_1 > 0$, otherwise negate $(\alpha_1, \mu)$. If $\lambda_1 > 0$ we find

$$k(z) \sim \begin{cases} \frac{1}{2}(\alpha_1 + \alpha_2)e^{(\nu + \mu)z}, & z \to +\infty, \\ \frac{1}{2}(\alpha_1 - \alpha_2)e^{(\nu + \mu - \mu)z}, & z \to -\infty. \end{cases}$$

Therefore, $\alpha_2$ is equal to the difference of coefficients in the asymptotics of $k$ at plus and minus infinities. But when $\lambda_1 < 0$, by writing down the asymptotics, one can see that the same difference gives $-\alpha_2$.

Let now $\lambda_1 \neq 0$ and $\mu_1 = 0$, we find $k(z) \sim e^{\nu z}(a_1 \sinh(\mu z) + \alpha_2 \cosh(\mu z))$ as $z \to +\infty$ if $\lambda_1 > 0$, and when $\lambda_1 < 0$ the same formula holds, but the RHS multiplied by $-e^{\lambda z}$. Again we see that $\alpha_2$ is determined up to the sign.

Let $\lambda_1 = 0$ and $\mu_2 \neq 0$, we may assume $\mu_2 > 0$, otherwise negate $(\alpha_1, \mu)$, then

$$k(iz) \sim \begin{cases} \frac{1}{2}(\alpha_1 - \alpha_2)e^{(\nu + \mu - \mu)z}, & z \to +\infty, \\ \frac{1}{2}(\alpha_1 + \alpha_2)e^{(\nu + \mu)z}, & z \to -\infty. \end{cases}$$

Finally, the case $\lambda_1 = \mu_2 = 0$ can be treated similarly.

Remains to note that $\phi, \psi$ cannot have an exponential $e^{\mu z}$ with $\mu \neq \pm \lambda$ either (we assume $a_0 e^{\lambda z}$ appears in $\phi$). Indeed, if $b_0 e^{\mu z}$ and $c_0 e^{\mu z}$ appear in $\phi$ and $\psi$ respectively, then for $k$ we obtain an equation like (5.4), but with $a_0 = 0$ and $b_0, c_0$ replaced with $b_0, c_0$, hence $k(z) = e^{(\mu + \bar{\nu})z}/(e^{\mu z} - 1)$ with $\nu = -c_0/b_0$. But this is of the same form as (5.5), hence as we showed $\mu$ is determined up to its sign. In other words the two formulas for $k$ are compatible only if $\mu = \pm \lambda$.

$$\square$$

Lemma 10. The functions $\phi, \psi, \epsilon$ cannot contain an exponential and a polynomial at the same time.

Proof. Let $a_5 e^{\lambda y} + \sum_{j=0}^4 a_j y^j$, with $a_5 \neq 0$ be part of $\phi$. The functions $\theta, \epsilon$ also have such parts, but with possibly different constants $b_j, c_j$. From the above lemma we know that $k$ is given by (5.5) (with $a_0$ replaced by $a_5$). One can check that once these expressions for $\phi, \psi$ and $\epsilon$ are substituted into (R1), the factors $y^4$ get canceled and the equation corresponding to $y^3$ reads

$$a_4 z k'' + (b_4 z + 2a_4)k' + (c_4 z + b_4)k = 0. \tag{5.6}$$

Let us first show that $a_4 = 0$. For the sake of contradiction assume $a_4 \neq 0$, then the solution, with $\omega = -b_4/a_4$, is given by

$$k(z) = \frac{e^{\omega z}}{z} \begin{cases} \beta_1 z + \beta_2, & \eta := \sqrt{\frac{b_4^2}{a_4^2} - \frac{c_4}{a_4}} = 0, \\ \beta_1 \sinh(\eta z) + \beta_2 \cosh(\eta z), & \eta \neq 0. \end{cases} \tag{5.7}$$

We note that this is not compatible with (5.5), because cross multiplying the two formulas we get (with $f, g$ being the second multiplying factors from (5.5) and (5.7), respectively)

$$ze^{(\nu + \lambda)z} f(z) = e^{\omega z}(e^{\lambda z} - 1)g(z).$$

If $g(z) = \beta_1 \sinh(\eta z) + \beta_2 \cosh(\eta z)$, we use the linear independence of $ze^\gamma z$ and $e^{\gamma z}$ to conclude that $k = 0$. Let $\nu = \beta_1 z + \beta_2$, if $f$ is given by the first formula the above relation reads
\[ a_1 z^2 e^{(\nu+\lambda)z} + a_2 z e^{(\nu+\lambda)z} + \beta_1 z e^{\omega z} - \beta_1 z e^{(\omega+\lambda)z} = \beta_2 e^{(\omega+\lambda)z} - \beta_2 e^{\omega z}. \]

Because \( \lambda \neq 0 \), the exponentials on RHS are linearly independent, hence we conclude that \( \beta_2 = 0 \), which contradicts to \( k \) having a pole at zero. When \( f \) is given by the second formula the same argument applies.

Thus, \( a_4 = 0 \), if \( b_1 \neq 0 \) we find \( k(z) = e^{\nu z}/z \), but now \( \omega = -c_1/b_4 \). This has the same form as (5.7), hence again it is incompatible with (5.5). Therefore, \( b_1 = 0 \) and obviously \( c_4 = 0 \). With this information, the equation corresponding to \( y^2 \) is as (5.6) with all subscripts changed from 4 to 3. Hence, the same procedure works and eventually we conclude \( a_j = b_j = c_j = 0 \) for \( j = 1, \ldots, 4 \).

\[ \square \]

5.3 Finding \( k \)

The analysis of the previous subsection shows that we have two possible forms (\( \lambda \neq 0 \))

\[
\begin{align*}
\text{I. } \varrho(y) &= a_1 e^{\lambda y} + a_2 e^{-\lambda y} + a_0, & \text{II. } \varrho(y) &= \sum_{j=0}^{6} a_j y^j.
\end{align*}
\]

Moreover we also showed that in each case \( \vartheta, \varrho \) are exactly of the same form as \( \varrho \), only with possibly different constants \( b_j, c_j \) instead of \( a_j \).

5.3.1 Case I

Assume case I holds, substituting the expressions for \( \varrho, \vartheta, \varrho \) into (R1) we find that a linear combination of \( e^{pm\lambda y} \) is zero, hence the coefficient of each exponential must vanish. Like this we obtain two ODEs for \( k \). More precisely,

\[
\begin{align*}
a_1(e^{\lambda z} - 1)k'' + [2a_1 \lambda + b_1(e^{\lambda z} - 1)]k' + [b_1 \lambda - a_1 \lambda^2 + c_1(e^{\lambda z} - 1)]k &= 0, \\
a_2(e^{-\lambda z} - 1)k'' + [-2a_2 \lambda + b_2(e^{-\lambda z} - 1)]k' + [-b_2 \lambda - a_2 \lambda^2 + c_2(e^{-\lambda z} - 1)]k &= 0.
\end{align*}
\]

Note that the second equation is obtained from the first one if we negate \( \lambda \) and change the subscripts of \( a_1, b_1, c_1 \) from 1 to 2. Consider the following cases:

**Case I.1.** \( a_1 = a_2 = 0 \), then \( \varrho \equiv 0 \) and from the boundary conditions \( \vartheta(\pm 1) = 0 \). W.l.o.g. let \( b_1 \neq 0 \) solving the first ODE for \( k \) we get, with \( \nu = -\frac{\mu}{b_1} \)

\[
k(z) = e^{(\nu + \lambda)z} = e^{(\nu + \frac{\mu}{2})z} / 2 \sinh \left( \frac{\mu}{2} z \right).
\]

For this to satisfy also the second ODE we need \( c_2 = -(\nu + \lambda)b_2 \). One can check that for \( k \) to be smooth in \([ -2, 2 \) \( \setminus \{ 0 \} \), we cannot have \( \lambda = \pi i n \), therefore the boundary conditions on \( \vartheta \) imply \( b_1 = b_2 \) and so \( \vartheta(y) = \cosh(\lambda y) - \cosh \lambda \). Now if \( \lambda \in i\mathbb{R} \), for the same reason we require \( |\lambda| < \pi \). From the relation (5.1) we see that \( \varrho(y) = \frac{1}{2} \vartheta'(y) \). After ignoring the exponential in the numerator of the formula for \( k \) (see Remark 3) we obtain

\[ k(z) = \frac{1}{\sinh \left( \frac{\mu}{2} z \right)}, \quad \begin{cases} \varrho(y) = 0, \\ \vartheta(y) = \cosh(\lambda y) - \cosh \lambda, \\ \varrho(y) = \frac{1}{2} \vartheta'(y). \end{cases} \quad (5.8) \]

**Case I.2.** If \( a_1 \neq 0 \) (the case \( a_2 \neq 0 \) can be treated analogously) by rescaling let us take \( a_1 = \frac{1}{2} \), then as the formula (5.5) was obtained we get, by w.l.o.g. choosing \( \nu = -\lambda/2 \), or equivalently \( b_1 = \lambda a_1 \) (see Remark 3) that

\[
k(z) = \frac{1}{\sinh \left( \frac{\mu}{2} z \right)} \cdot \begin{cases} a_1 z + a_2, \\ a_1 \sinh(\mu z) + a_2 \cosh(\mu z), \quad \mu := \sqrt{b_1^2 - 2c_1} = 0, \\ \mu \neq 0. \end{cases}
\]
Let $k$ be given by the first formula. It is easy to check that $\lambda = \pi i n$, with $n \in \mathbb{Z}$ contradicts to the smoothness assumption on $k$, so the boundary conditions imply that $a_1 = a_2$ and therefore $e(y) = \cosh(\lambda y) - \cosh(\lambda)$. Because of the same reason, when $\lambda \in i \mathbb{R}$ we need a further restriction $|\lambda| < \pi$. The boundary conditions $\theta(\pm 1) = e'(\pm 1)$ then imply

$$b_2 = -\frac{\lambda}{2}, \quad b_0 = 0 \implies \theta(y) = \frac{\lambda}{2} e^{\lambda y} - \frac{\lambda}{2} e^{-\lambda y} = e'(y).$$

Now, $k$ has to satisfy also the second ODE, so we substitute the expression for $k$ there and simplify the result to find

$$e^{-\frac{\lambda}{2}i}(\alpha_1 z + \alpha_2)\left(c_2 - \frac{\lambda^2}{8}\right) = 0,$$

which clearly implies $c_2 = \frac{\lambda^2}{8}$. But because this was the case $\mu = 0$ we have $c_1 = \frac{\lambda^2}{4} = \frac{\lambda^2}{8}$ and therefore we conclude that $e(y) = \frac{\lambda^2}{2} a(y)$. Thus, we proved (3.1) and (3.2) of Theorem 1 in the limiting case $\mu = 0$. Moreover, when $a_1 = 0$ we obtain the same kernel as in (5.8), hence we can take a linear combination of the differential operator of this case and the one in (5.8) and $K$ will still commute with it. This proves item 2 of Theorem 1.

Let $k$ be given by the second formula. When $\lambda \in i \mathbb{R}$ there are further restrictions for parameters. Let us analyze them. Firstly, if $\lambda \in i \mathbb{R}$ with $|\lambda| \geq 2\pi$, then the denominator of $k$ has zeros at $\pm 2\pi i \frac{m}{2}, \pm 2\pi i = [-2, 2]$, which cannot be canceled out by the numerator, therefore $|\lambda| < 2\pi$. So there are two cases: when $|\lambda| < \pi$, $k$ is smooth in $[-2, 2] \setminus \{0\}$ and when $\pi \leq |\lambda| < 2\pi$ the denominator of $k$ has zeros at $\pm 2\pi i \frac{m}{2} \in [-2, 2]$, which can be canceled out by the numerator if and only if $a_1 = 0$ and $\cosh(\frac{2\pi i m}{\lambda}) = 0$, i.e. $\mu = \lambda^{2m+1}$ for some $m \in \mathbb{Z}$. This is summarized in Remark 4.

Let us substitute the expression for $k$ into the second ODE, multiply the result by $e^{\frac{\lambda}{2}z}$. After simplification we obtain

$$\left[\left(\mu^2 a_2 + \frac{\lambda^2 a_2}{4} + \frac{b_0}{2} + c_2\right) \alpha_1 + \mu a_2 (a_2 \lambda + b_2)\right] \sinh(\mu z) +$$

$$+ \left[\left(\mu^2 a_2 + \frac{\lambda^2 a_2}{4} + \frac{b_0}{2} + c_2\right) \alpha_2 + \mu a_2 (a_2 \lambda + b_2)\right] \cosh(\mu z) = 0.$$

By linear independence we conclude that the coefficients of $\sinh(\mu z), \cosh(\mu z)$ must be zero. Or equivalently their sum and difference must be zero, but these equations can be written as

$$\begin{cases}
(\alpha_1 + \alpha_2) \left(\mu + \frac{\lambda}{2}\right) [(\mu + \frac{\lambda}{2}) a_2 + b_2] + c_2 = 0, \\
(\alpha_1 - \alpha_2) \left(\mu - \frac{\lambda}{2}\right) [(\mu - \frac{\lambda}{2}) a_2 - b_2] + c_2 = 0.
\end{cases}$$

The boundary conditions $e.(\pm 1) = 0$ imply that $a_0 = -a_1 e^\lambda - a_2 e^{-\lambda}$ and

$$(a_1 - a_2) (e^\lambda - e^{-\lambda}) = 0.$$

a) Let $a_2 = a_1$, then $e(y) = \cosh(\lambda y) - \cosh(\lambda)$ and from the boundary conditions $\theta(\pm 1) = e'(\pm 1)$ we find $\theta(y) = e'(y)$ as was discussed above. Now in this case (5.9) simplifies to

$$\begin{cases}
(\alpha_1 + \alpha_2) \left(\frac{\lambda^2}{4} - \mu^2 - 2c_2\right) = 0, \\
(\alpha_1 - \alpha_2) \left(\frac{\lambda^2}{4} - \mu^2 - 2c_2\right) = 0.
\end{cases}$$

But because both $\alpha_1, \alpha_2$ are not zero at the same time, we get $c_2 = \frac{1}{2} \left(\frac{\lambda^2}{4} - \mu^2\right)$. From the definition of $\mu$ we see that also $c_1 = \frac{1}{2} \left(\frac{\lambda^2}{4} - \mu^2\right)$. And using the freedom of choosing $c_0$ we conclude that we may write $e(y) = (\frac{\lambda^2}{4} - \mu^2) a(y)$. This proves (3.1) and (3.2) of Theorem 1 in the case $\mu \neq 0$.

b) Let $e^\lambda = e^{-\lambda}$, i.e. $\lambda = \pi i n$ for some $n \in \mathbb{Z}$. But the above discussion implies that $\alpha_1 = 0, \lambda = \pi i (or \ -\pi i, but this would lead to the same results) and $\mu = \lambda^{2m+1}$ with $m \in \mathbb{Z}$. In this case (5.9) implies

$$b_2 = -\lambda a_2, \quad c_2 = a_2 \left(\frac{\lambda^2}{4} - \mu^2\right).$$
Recalling that $b_1 = \lambda a_1$, the boundary conditions $\theta(\pm 1) = \alpha'(\pm 1)$ imply $b_0 = 0$ and so far we have $\alpha(y) = a_1(e^{\lambda y} - e^{-\lambda}) + a_2(e^{-\lambda y} - e^{\lambda y})$ and $\theta(y) = \alpha'(y)$. Finally, again from the definition of $\mu$ we have $c_1 = a_1(\frac{1}{4} - \mu^2)$. This and the above formula for $c_2$ (and the freedom of choosing $c_0$) allow one to write $\epsilon(y) = (\frac{1}{4} - \mu^2)\alpha(y)$. This proves item 1 of Theorem 1. Of course to start with we assumed $a_2 \neq 0$, but when considering the case $a_2 = 0$ we can allow $a_1$ to vanish. This explains why there are no restrictions on $\alpha, \beta$ in item 1 of Theorem 1.

5.3.2 Case II

Assume case II holds, substituting the expressions for $\alpha, \beta, \epsilon$ into (R1) we find that a linear combination of monomials $y^j$ is zero, hence the coefficient of each $y^j$ must vanish (one can check that $y^0$ cancels out). These relations can be conveniently written as

$$
\left[ \frac{\alpha^{(j)}(z)}{j!} - a_j \right] k'' + \left[ \frac{\theta^{(j)}(z)}{j!} - b_j + 2(j+1)a_{j+1} \right] k' + \left[ \frac{\epsilon^{(j)}(z)}{j!} - c_j + (j+1)b_{j+1} - (j+1)(j+2)a_{j+2} \right] k = 0, \quad j = 0, \ldots, 5,
$$

(5.10)

with the convention that $a_7 = 0$. Let $\deg(\alpha) = m$, $\deg(\theta) = n$ and $\deg(\epsilon) = s$.

**Case II.1.** Let $\alpha \equiv 0$, then $\theta(\pm 1) = 0$ and hence $n \geq 2$. By scaling we let $b_n = 1$. We are going to show that $n$ cannot be strictly larger than 2 and so $n = 2$. Note that $s \leq n$, otherwise the above relation with $j = s - 1$ reads $c_s z k = 0$, which implies $k = 0$ since $c_s \neq 0$ by the definition of $s$. Now (5.10) with $j = n - 1$ reads

$$
z k' + [1 + c_n z] k = 0,
$$

(5.11)

whose solution is given by $k(z) = \alpha e^{-\frac{c_n}{z}}$, where $\alpha \in \mathbb{C}$. Invoking Remark 3 we may w.l.o.g. assume $c_n = 0$. The relation with $j = n - 2$ becomes

$$
\left[ \frac{n}{2} z^2 + b_{n-1} z \right] k' + [c_{n-1} z + b_{n-1}] k = 0.
$$

Substituting $k(z) = \frac{1}{z}$ into this equation we obtain $c_{n-1} = \frac{n}{2}$. Now, if $n > 2$ we consider the relation for $j = n - 3$, which reads

$$
\left[ \frac{n(n-1)}{6} z^3 + \frac{n-1}{2} b_{n-1} z^2 + b_{n-2} z \right] k' + \left[ \frac{n-1}{2} c_{n-1} z^2 + c_{n-2} z + b_{n-2} \right] k = 0.
$$

Again substituting the expression for $k$ and using the expression for $c_{n-1}$ we obtain

$$
\frac{n(n-1)}{12} z + c_{n-2} + \frac{n-1}{2} b_{n-1} = 0,
$$

(5.12)

which is a contradiction. Thus our conclusion is that $n = 2$, in which case $\theta(y) = y^2 - 1$, $c_2 = 0$, $c_1 = 1$ and hence $\epsilon(y) = y$, and we obtain the operator in item 4 of Theorem 1 when $\rho = 0$.

**Case II.2.** Let $\alpha \neq 0$, then $m \geq 2$. By scaling we let $a_m = 1$. Let us first show that $n \leq m$. For the sake of contradiction assume $n > m$. If also $s > n$, then (5.10) with $j = s - 1$ reads $c_s z k = 0$, which is a contradiction and therefore $s \leq n$. Now (5.10) with $j = n - 1$ reads

$$
z k' + [1 + c_n z] k = 0,
$$

with the convention that $c_n = 0$ if $s < n$. As in the previous case w.l.o.g. we assume $c_n = 0$ so that $k(z) = \frac{1}{z}$. Using these and looking at (5.10) for $j = n - 2$ and $j = n - 3$ we obtain exactly the same contradiction (5.12) as in the previous case (only with a different free constant).

Thus $n \leq m$, and it is easy to see that also $s \leq m$. The relation (5.10) for $j = m - 1$ reads

$$
z k'' + (2 + b_m z) k' + (b_m + c_m z) k = 0,
$$

(5.13)

whose solution is, with $\alpha_1, \alpha_2 \in \mathbb{C}$.
$$k(z) = e^{-\frac{b_m z}{z}} \cdot \begin{cases} \alpha_1 \sinh(\mu z) + \alpha_2 \cosh(\mu z), \\ \alpha_1 z + \alpha_2, \end{cases} \quad \mu^2 := \frac{b^2}{z^2} - c_m \neq 0, \quad (5.14)$$

Invoking Remark 3 let us w.l.o.g. assume $b_m = 0$. Then from (5.13)

$$k''(z) = -\frac{2k'(z) + c_m z k(z)}{z}. \quad (5.15)$$

The relation (5.10) for $j = m - 2$ (after dividing it by $m - 1$) is

$$\left[a_{m-1} z + \frac{m}{2} z^2\right] k'' + (b_{m-1} z + 2a_{m-1}) k' + \left[c_{m-1} z + \frac{m}{2} c_m z^2 + b_{m-1} - m\right] k = 0.$$  

Substituting $k''$ from (5.15) into this equation we obtain

$$(b_{m-1} - m) z k' + [(c_{m-1} - c_m a_{m-1}) z + b_{m-1} - m] k = 0. \quad (5.16)$$

Let us now consider the cases for different values of $m$:

a) let $m = 2$, then $e(y) = y^2 - 1$, $b_2 = 0$ further the boundary conditions imply $b_1 = 2$, $b_0 = 0$ and hence $\delta(y) = 2y$. Then (5.16) reads $c_1 k = 0$, hence $c_1 = 0$ and so $e(y) = c_2 y^2$. $k(z)$ is determined from (5.14), where $\mu^2 = -c_2$. This proves formulas (3.1) and (3.2) of Theorem 1 in the limiting case $\lambda = 0$.

b) let $m = 3$, then $e(y) = (y^2 - 1)(y - \sigma)$ and $b_3 = 0$. In particular we see that $a_2 = -\sigma$ and $a_1 = -1$. From the boundary conditions $b_0 = 2 - b_2$, $b_1 = -2\sigma = 2a_2$. The relation (5.10) with $j = m - 3 = 0$ reads

$$(z^3 + a_2 z^2 + a_1 z) k'' + (b_2 z^2 + b_1 z + 2a_1) k' + (c_3 z^3 + c_2 z^2 + c_1 z) k = 0.$$  

Substituting $k''$ from (5.15) this simplifies to

$$(b_2 - 2) z^2 k' + [(c_2 - c_3 a_2) z^2 + (c_1 + c_3) z] k = 0,$$

and combining this with (5.16) we obtain

$$zk' + (c_1 + c_3 - b_2 + 3) k = 0.$$  

But because $k$ has a simple pole at 0, we must have $c_1 + c_3 - b_2 + 3 = 1$, hence $c_3 = b_2 - c_1 - 2$. Then $k(z) = 1/z$, substituting this expression into (5.13) we conclude $c_1 = b_2 - 2$ and hence $c_3 = 0$. Next we substitute it into (5.16) to find $c_2 = 0$. Thus

$$\begin{cases} e(y) = (y^2 - 1)(y - \sigma) \\ \delta(y) = b_2 y^2 - 2\sigma y + 2 - b_2 \\ e(y) = (b_2 - 2) y \end{cases}$$

This proves item 4 of Theorem 1, when $\beta = b_2 - 3$ and $p$ is a first order polynomial.

c) let $m = 4$, then $e(y) = (y^2 - 1)(y - \sigma_1)(y - \sigma_2)$, $b_4 = 0$. Note that $a_3 = -\sigma_1 - \sigma_2$; $a_2 = \sigma_1 \sigma_2 - 1$. Further, from the boundary conditions on $\delta$ we get $b_1 = 2(a_2 + 2) - b_3$ and $b_0 = -b_2 + 2a_3$. From (5.14) $k$ has two possible forms, assume first $k(z) = \frac{1}{2}(\alpha_1 z + \alpha_2)$ in which case $c_4 = \frac{b_2^2}{4} = 0$. Since $k$ has a simple pole at the origin $\alpha_2 \neq 0$ and let us normalize $\alpha_2 = 1$. (5.16) in this case reads $(b_3 - 4) zk' + (c_2 z + b_3 - 4) k = 0$. Substituting the expression for $k$ into this equation we obtain

$$c_3 \alpha_1 z + c_3 + (b_3 - 4) \alpha_1 = 0,$$
which implies that $c_3 = 0$ and
\[ \alpha_1(b_3 - 4) = 0. \] (5.17)

The relations (5.10) with $j = m - 3$ and $j = m - 4$ read respectively as
\[ (4z^3 + 3a_3z^2 + 2a_2z)k'' + (3b_3z^2 + 2b_2z + 4a_2)k' + 2(c_2z - 3a_3 + b_2)k = 0, \] (5.18)
\[ (z^4 + a_3z^3 + a_2z^2 + a_1z)k'' + (b_3z^2 + b_2z + 2a_1)k' + (c_2z^2 + c_1z - 2a_2 + b_1)k = 0. \] (5.19)

Now, (5.17) implies that we should consider two cases:

- If $\alpha_1 = 0$, we substitute $k(z) = \frac{1}{z}$ into (5.18) and find $c_2 = \frac{3}{2}b_3 - 4$. Finally substitution into (5.19) gives
  \[ \frac{b_3 - 4}{2} z + 2a_3 - b_2 + c_1 = 0, \]
therefore $b_3 = 4$ and $c_1 = -2a_3 + b_2$. Putting everything together we obtain
  \[
  \begin{cases}
    \phi(y) = (y^2 - 1)(y - \sigma_1)(y - \sigma_2) \\
    \delta(y) = 4y^3 + b_2y^2 + 2(\sigma_1\sigma_2 - 1)y - b_2 - 2(\sigma_1 + \sigma_2) \\
    \epsilon(y) = 2y^2 + (b_2 + 2\sigma_1 + 2\sigma_2)y
  \end{cases}
  \]

This proves item 4 of Theorem 1, when $\beta = b_2 + 3(\sigma_1 + \sigma_2)$ and $\rho$ is a second order polynomial.

- If $\alpha_1 \neq 0$, we get $b_3 = 4$, substituting $k(z) = \alpha_1 + \frac{1}{z}$ into (5.18) we obtain
  \[ c_2\alpha_1z + (b_2 - 3a_3)\alpha_1 + c_2 - 2 = 0, \]

  hence we deduce $c_2 = 0$ and $\alpha_1(b_2 - 3a_3) = 2$. Finally, we substitute $k$ into (5.19) and obtain $c_1 = -3a_3 + b_2$ and
  \[ a_3(b_2 - 3a_3) = 0, \]
  but because $b_2 - 3a_3 \neq 0$ we get $a_3 = 0$, i.e. $\sigma_1 = -\sigma_2$. Then also $\alpha_1 = \frac{2}{b_2}$, $k(z) = \frac{2}{b_2} + \frac{1}{z}$ and
  \[
  \begin{cases}
    \phi(y) = (y^2 - 1)(y^2 - \sigma_1^2), \\
    \delta(y) = 4y^3 + b_2y^2 - 2(\sigma_1^2 + 1)y - b_2, \\
    \epsilon(y) = b_2y
  \end{cases}
  \]

This establishes item 3 of Theorem 1 with $\beta = b_2/2$.

Let now $k(z) = \frac{1}{z}(\alpha_1 \sinh(\mu z) + \alpha_2 \cosh(\mu z))$, with $\mu^2 = -c_4 \neq 0$. One can check by subsequent substitutions into (5.16), (5.18) and (5.19) that this case is impossible.

d) Subsequent substitutions show also that $m \geq 5$ is impossible.

6 Relations for coefficients

In this section we consider (1.3) with $L_1, L_2$ given by (2.2). We assume (A) holds, $k$ is analytic at 0, but not identically zero near 0 and finally $k$ is not of the form $e^{\alpha z}$. We aim to find the relations that the coefficient functions $\delta_j$, $\epsilon_j$ must satisfy. Write $k(z) = \sum_{n=0}^{\infty} \frac{k_n}{m^n} z^n$ near $z = 0$. The $n$-th derivative of (R) w.r.t. $z$ at $z = 0$ gives
\[ (-1)^n[\beta_1 k_{n+2} + \beta'_1 k_{n+1} + \beta_2 k_n] - \sum_{j=0}^{n} C_j^n \beta'_2^{(n-j)} k_{j+2} - \sum_{j=0}^{n} C_j^n \beta'_2^{(n-j+1)} k_{j+1} - \sum_{j=0}^{n} C_j^n \beta_2^{(n-j)} k_j = 0, \]  

where \( C_j^n = \binom{n}{j} \), when \( n = 0 \) we get

\[ k_1(\beta'_1 - \beta'_2) + k_2(\beta_1 - \beta_2) + k_0(\epsilon_1 - \epsilon_2) = 0. \]

- If \( k_0 = k_1 = 0 \), then let us show that \( k \) is trivial. Assume first \( \beta_1 \neq \pm \beta_2 \), then clearly \( k_2 = 0 \). Let us prove by induction that all \( k_j = 0 \), which contradicts to the assumption that \( k \) doesn’t vanish near 0. Assume \( k_j = 0 \) for \( j = 0, \ldots, m \), then (6.1) for \( n = m - 1 \) reads \([(-1)^{m-1}\beta_1 - \beta_2] k_{m+1} = 0 \), therefore \( k_{m+1} = 0 \). Let now \( \beta_1 = \beta_2 \), assume for the induction step that \( k_j = 0 \) for \( j = 0, \ldots, n \), then (6.1) reads

\[ \sum_{j=0}^{n} (-1)^j k_{n+2} + \sum_{j=0}^{n} (-1)^j k_{n+1} = 0. \]

When \( n \) is odd we immediately obtain \( k_{n+1} = 0 \). When \( n \) is even we get \( k_{n+2} = 0 \) and because of boundary conditions \( \beta_1(\pm 1) = 0 \) we deduce \( k_{n+1} = k_{n+2} = 0 \). Finally, the case \( \beta_1 = -\beta_2 \) can be done analogously.

- If \( k_0 = 0, k_1 \neq 0 \), by rescaling let \( k_1 = 1 \) and by considering \( e^{-\frac{k_2 z}{2}} k(z) \) instead of \( k(z) \) (see Remark 11) we may assume \( k_2 = 0 \). Now, \( \beta_2(y) = \beta_1(y) + \alpha \) for some \( \alpha \in \mathbb{C} \). From (6.1) with \( n = 1 \) we find \( \epsilon_2 = -\epsilon_1'' - 2k_3 \epsilon_1 - \epsilon_1 - k_3 \alpha \). Using the obtained expressions, from the relation corresponding to \( n = 2 \) we get

\[ \epsilon_1' = -\frac{1}{2} \epsilon_1'' - k_3 \epsilon_1' + \frac{k_4 \alpha}{2}. \]  

Now, (6.1) with \( n = 3 \) reads

\[ 2\epsilon_1^{(4)} + k_3 \epsilon_1'' + 5k_4 \epsilon_1' + 2(k_4^2 - k_5) \epsilon_1 + 3 \epsilon_1'' + \alpha(k_4^2 - k_5) = 0. \]

Let us now replace \( \epsilon_1'' \) using (6.2). The result becomes an ODE for \( \beta_1 \): for some constants \( \alpha_j \),

\[ \beta_1^{(4)} + \sum_{j=0}^{3} \alpha_j \beta_1^{(j)} = \alpha_4. \]

- If \( k_0 \neq 0 \), by rescaling let \( k_0 = 1 \) and by considering \( e^{-k_1 z} k(z) \) instead of \( k(z) \) (see Remark 11) we may assume \( k_1 = 0 \). Note that \( \epsilon_2 = \epsilon_1 + k_2(\beta_1 - \beta_2) \), using this in (6.1) with \( n = 1 \), we get

\[ \epsilon_1' = -k_3(\beta_1 + \beta_2) - k_2(2 \beta_1' + \beta_2'). \]  

The relation for \( n = 2 \) reads

\[ -k_2(\beta_1'' + 2 \beta_2'') + k_3(\beta_1' - 3 \beta_2') + (k_4 - k_5^2)(\beta_1 - \beta_2) - \epsilon_1'' = 0, \]

and replacing \( \epsilon_1'' \) using (6.3) we obtain

\[ k_2(\beta_1'' - \beta_2'') + 2k_3(\beta_1' - \beta_2') + (k_4 - k_5^2)(\beta_1 - \beta_2) = 0. \]

Consider the following cases:

1. If \( k_2 = k_3 = 0 \), then we are going to show that \( k \) is trivial. Assume first that \( \beta_1 \neq \pm \beta_2 \), so from the above equation \( k_4 = 0 \). Further, we see that in this case \( \epsilon_1 = \epsilon_2 = \text{const} \). Let now \( k_j = 0 \) for \( j = 1, \ldots, n + 1 \), then (6.1) reads

\[ k_{n+2} [(-1)^n \beta_1 - \beta_2] = 0, \]

so \( k_{n+2} = 0 \) and by induction \( k_j = 0 \) for any \( j \neq 0 \), i.e. \( k \) is trivial. When \( \beta_1 = \beta_2 \), or \( \beta_1 = -\beta_2 \) the result follows analogously.
2. If \( k_2 = 0 \) and \( k_3 \neq 0 \), then \( \delta_2(y) = \delta_1(y) + \alpha e^{\tau y} \) with \( \tau = -\frac{k_4}{2k_3} \) and some \( \alpha \in \mathbb{C} \). From (6.1) with \( n = 3 \) (by replacing \( e_1'' \) using (6.3)) we find

\[
\epsilon_1 = -\frac{\alpha}{2k_3}(5\tau^2k_3 + 4\tau k_4 + k_5)e^{\tau y} - 2\delta_1'' - \frac{5k_4}{2k_3}\delta_1' - \frac{k_8}{k_3}\delta_1.
\]

Finally we replace this and \( \delta_2 \) in (6.3) to obtain, for some other constants \( \alpha_j \)

\[
\delta_1^{(3)} + 2 \sum_{j=0}^{3} \alpha_j \delta_1^{(j)} = \alpha_3 e^{\tau y}.
\]

3. If \( k_2 \neq 0 \), then \( \delta_2(y) = \delta_1(y) + f(y) \) and \( f \) solves \( k_2f'' + 2k_3f' + (k_4 - k_2^2)f = 0 \), so either \( f(y) = \lambda_1 e^{\tau_1 y} + \lambda_2 e^{\tau_2 y} \) or \( f(y) = (\lambda_1 y + \lambda_2) e^{\tau y} \). Using the ODE for \( f \), (6.1) for \( n = 3 \) can be written as

\[
4k_2\delta_1'''' + 6k_3\delta_1'' + 5k_4\delta_1' + 2k_5\delta_1 + \epsilon_1''' + 3k_2\epsilon_1' + 2k_3\epsilon_1 = -k_4f' + (k_2k_3 - k_5)f.
\]

Let us now replace \( \epsilon_1'' \) and \( \epsilon_1' \) in the above relation using (6.3). The result becomes

\[
2k_3\epsilon_1 = -k_2\epsilon_1'''' - 4k_3\epsilon_1'' + (9k_2^2 - 5k_4)\epsilon_1' + (6k_3k_4 - 2k_5)\epsilon_1 + (4k_2^2 - 2k_4 + 2k_5)\epsilon_1' + (3k_2k_4 - k_5 + \frac{k_4}{k_3})f,
\]

but because \( f' \) has the same form as \( f \) we can rewrite the above relation as

\[
2k_3\epsilon_1(y) = -k_2\epsilon_1'''' + \sum_{j=0}^{3} \gamma_j \epsilon_1^{(j)}(y) + f(y),
\]

with different constants \( \lambda_j \) in \( f \) and \( \gamma_j \) are some constants. Now if \( k_3 = 0 \) we got an ODE for \( \delta_1 \), otherwise divide by it and substitute the obtained expression and the expression of \( \delta_2 \) into (6.3), the result is (with different constants)

\[
\delta_1^{(4)} + \sum_{j=0}^{3} \gamma_j \delta_1^{(j)} = f(y).
\]

7 Reduction of the general case

In this section we prove Theorem 3, i.e. if \( k \) is nontrivial, then \( L_1 = L_2 \) or \( L_1 = -L_2 \). Analysis of the previous section shows that \( \delta_j, \epsilon_j \) are linear combinations of polynomials multiplied with an exponential, moreover the polynomials have degree at most five. So let us consider a typical such term:

\[
\delta_1(y) \leftrightarrow \left( \sum_{j=0}^{5} b_j y^j \right) e^{\lambda y}, \quad \epsilon_1(y) \leftrightarrow \left( \sum_{j=0}^{5} c_j y^j \right) e^{\lambda y},
\]

and analogous terms in \( \delta_2, \epsilon_2 \) only with possibly different coefficients \( \tilde{b}_j, \tilde{c}_j \) respectively. Set \( k(z) = \kappa(z)e^{-\frac{\pi}{2}z} \) and let

\[
\kappa_+(z) = \frac{1}{2}[\kappa(z) + \kappa(-z)], \quad \kappa_-(z) = \frac{1}{2}[\kappa(z) - \kappa(-z)].
\]

Substituting the expressions for \( \delta_j, \epsilon_j \) and \( k(z) = e^{-\frac{\pi}{2}z}[\kappa_+(z) + \kappa_-(z)] \) into (R), we obtain that a linear combination of terms \( y^j e^{\lambda y} \) is zero. From linear independence we conclude that each coefficient must vanish. In particular, the relation corresponding to \( y^5 e^{\lambda y} \) reads

\[
(b_5 - \tilde{b}_5)\kappa'' + \left( (b_5 - \tilde{b}_5)\frac{\lambda^2}{4} + \tilde{c}_5 - c_5 \right) \kappa - (b_5 + \tilde{b}_5)\kappa'' + \left( (b_5 + \tilde{b}_5)\frac{\lambda^2}{4} - \tilde{c}_5 - c_5 \right) \kappa = 0.
\]
Because $\kappa_+$ is even, and $\kappa_-$ is odd we can add the above relation, with $z$ replaced by $-z$, to itself. Like this we separate the above relation into two ODEs one for $\kappa_+$ and the other for $\kappa_-:

\begin{align*}
(b_5 - \tilde{b}_5)\kappa''_+ - \left((b_5 - \tilde{b}_5)\frac{\lambda^2}{4} + \tilde{c}_5 - c_5\right)\kappa_+ &= 0, \\
(b_5 + \tilde{b}_5)\kappa''_- - \left((b_5 + \tilde{b}_5)\frac{\lambda^2}{4} - \tilde{c}_5 - c_5\right)\kappa_- &= 0.
\end{align*}

If $b_5 \neq \pm \tilde{b}_5$, then $\kappa_+ = \cosh(\mu z)$ and $\kappa_-$ is either $z$ or $\sinh(\mu z)$ for some $\mu \in \mathbb{C}$, therefore $k$ is trivial. Therefore, we consider the following cases:

- **$b_5 = b_5$, then obviously $c_5 = \tilde{c}_5$ and we get $b_5\kappa_''_+ - \left(b_5\frac{\lambda^2}{4} - c_5\right)\kappa_+ = 0$. Assume $b_5 \neq 0$, then by normalization we can make $b_5 = 1$, now with $\mu^2 = \frac{\lambda^2}{4} - c_5$**

$$\kappa_-(z) = \begin{cases} \alpha z, & \mu = 0, \\
\alpha \sinh(\mu z), & \mu \neq 0. \end{cases}$$

Using the ODE that $\kappa_-$ solves, the even part of the relation corresponding to $y^4e^{\lambda y}$ reads

$$(b_4 - \tilde{b}_4)\kappa''_+ - \left((b_4 - \tilde{b}_4)\frac{\lambda^2}{4} + \tilde{c}_4 - c_4\right)\kappa_+ = 0,$$

which immediately implies $b_4 = \tilde{b}_4$, and hence $c_4 = \tilde{c}_4$. Odd part of that relation is

$$z\kappa''_+ + 2\kappa'_+ - \mu^2 z\kappa_+ = -\frac{2b_4}{\lambda} \kappa''_+ + \left(\frac{b_4\lambda^2}{10} - \frac{2c_4}{\lambda} + \lambda\right) \kappa_-.$$  

Making the change of variables $\kappa_+(z) = \frac{u(z)}{\lambda}$, the left-hand side of the above relation becomes $u'' - \frac{\mu^2}{\lambda^2} u$, therefore using the expression for $\kappa_-$ and the evenness of $\kappa_+$ we find

$$\kappa_+(z) = \begin{cases} \alpha_1 z^2 + \alpha_0, & \mu = 0, \\
\alpha_1 \cosh(\mu z) + \alpha_0 \frac{\sinh(\mu z)}{\mu}, & \mu \neq 0. \end{cases}$$

If $\kappa_+$ is given by the first formulas, then $k$ is trivial. Therefore, we assume $\mu \neq 0$ and the second formula holds. The even part of the relation for $y^4e^{\lambda y}$ is

$$(-10z^2 + b_3 - \tilde{b}_3)\kappa''_+ - 20z\kappa'_+ + \left(\frac{5\lambda^2}{2} - 10c_5\right)z^2 - (b_3 - \tilde{b}_3)\frac{\lambda^2}{4} + c_3 - \tilde{c}_3\right)\kappa_+ = 4b_4 z\kappa''_+ - (b_4\lambda^2 - 4c_4 + 10\lambda)z\kappa_-.$$  

When we substitute the formulas for $\kappa_{\pm}$ and multiply the relation by $z^3$, the result has the form

$$p(z)e^{\mu z} - p(-z)e^{-\mu z} = 0,$$

where $p(z) = \sum_{j=0}^{1} p_j z^j$, therefore by linear independence we conclude that all the coefficients of $p$ vanish, in particular one can compute that $p_0 = -2\alpha_0(b_3 - \tilde{b}_3)$ and $p_2 = \alpha_0 \left(-b_3 \tilde{b}_3\mu^2 + (b_3 - \tilde{b}_3)\frac{\lambda^2}{4} + \tilde{c}_3 - c_3\right)$, if $\alpha_0 = 0$, then obviously $k$ is trivial, so $p_0 = 0$ implies $b_3 = \tilde{b}_3$, but then $p_2 = 0$ implies $c_3 = \tilde{c}_3$. Looking at the even part of the relation coming from $y^2e^{\lambda y}$ we obtain an analogous equation, where the polynomial $p$ may be of 5th order, but expressions of $p_0, p_2$ stay the same, only the subscripts of $b_3, \tilde{b}_3, c_3, \tilde{c}_3$ change to two. And we conclude $b_2 = \tilde{b}_2$ and $c_2 = \tilde{c}_2$. Likewise looking at the even parts of the relations coming from $ye^{\lambda y}, e^{\lambda y}$ we find $b_j = \tilde{b}_j$ and $c_j = \tilde{c}_j$ for $j = 1, 0$.

When we look at another term with $\left(\sum_{j=0}^{5} b_j' y^j\right) e^{\lambda y}$ in the coefficient $c_1$ (and similar terms for other coefficient functions) we must have $b_0' = \tilde{b}_0'$, otherwise $k$ is trivial.

If $b_5 = 0$, the same procedure applies, we only need to relabel the coefficients in the above equations. Thus our conclusion is that $L_1 = L_2$.

- **$b_5 = -b_5$, this case is analogous to the previous one and the conclusion is $L_1 = -L_2$.**

20
In this section we aim to prove Theorem 4. Item 1 (in the limiting case \( \gamma = 0 \)) and item 2 of Theorem 4 are derived in Corollary 13. Item 1 (in the case \( \gamma \neq 0 \)) and item 3 are derived in Sections 8.3, 8.4. So let us assume the setting of Theorem 4.

The analysis in the beginning of Section 6 shows that \( \delta \) solves a linear homogeneous ODE with constant coefficients of order at most 4. Hence \( \delta(y) \) is a linear combination of terms like \( y^j e^{\lambda_j y} \), where \( \lambda_j \) (called also a mode) is a root of fourth order polynomial. We will see that there are two major cases: \( \text{Re} \lambda_j = 0 \) (type 1) or \( \text{Re} \lambda_j \neq 0 \) (type 2). In the former case \( k(z) \) is given in three possible forms featuring a free real-valued and even function (cf. (8.8)). In the latter case \( k(z) \) is determined and has two possible forms (cf. (8.9)).

In Section 8.1 we analyze the multiplicity of the mode \( \lambda_j \), in particular type 2 mode cannot have multiplicity larger than one, as is shown in Lemma 15, while type 1 mode can have multiplicity at most 3 as established in Lemma 14.

Finally, in Section 8.2 we turn to the question of analyzing possibilities of having multiple modes, i.e. distinct roots \( \lambda_j \). In Corollary 17 we show that having three distinct type 1 modes is impossible. In Corollary 21 we show that having three distinct type 2 modes is impossible. In Lemma 18 we show that two type 2 and one type 1 mode all with multiplicity one analyzed in Section 8.3; and two type 1 modes with multiplicity 1 analyzed in Section 8.4.

Throughout this section, until Section 9 we will be working with \( k(-z) \) and with an abuse of notation it will be denoted by \( k(z) \). We will remember about this notational abuse when collecting the results in Theorem 4. In particular (R) becomes

\[
\delta(y)k''(z) - \delta(y+z)k''(-z) - \delta'(y)k'(z) + \delta'(y+z)k'(-z) + e(y)k(z) - e(y+z)k(-z) = 0. \quad (8.1)
\]

The analysis in the beginning of the Section 6 shows that \( \delta \) solves a linear homogeneous ODE with constant coefficients of order at most 4, and that

\[
-k_0 \delta'(y) + 2k_1 \delta(y) + k_1 \delta''(y) - 3k_2 \delta'(y) + 2k_3 \delta(y) = 0. \quad (8.2)
\]

So \( \delta \) has the following form

\[
\delta(y) = \sum_{j=1}^{\nu} p_{d_j}(y)e^{\lambda_j y}, \quad (8.3)
\]

where \( \lambda_1, ..., \lambda_{\nu} \) are distinct complex numbers and \( p_{d_j} \) are polynomials of degree \( d_j \), so that

\[
\nu + \sum_{j=1}^{\nu} d_j \leq 4.
\]

Then \( e(y) \) satisfying (8.2) must also have the same form, except the polynomials are different and there could be an extra exponential term \( e^{\frac{2k_1}{k_0} y} \), if \( \frac{2k_1}{k_0} \notin \{ \lambda_1, ..., \lambda_{\nu} \} \). Because we also require \( \delta(\pm 1) = 0 \), then either

I. \( \nu = 1, \ d_1 \geq 1 \);

II. \( \nu = 2, \ d_1 \geq 1 \);

III. \( \nu = 2, \ d_1 = d_2 = 0, \ \delta(y) = e^{i\beta y} \sin(\pi n (y-1)/2) \) for some \( \beta \in \mathbb{R} \) and \( n \geq 1 \);

IV. \( \nu \geq 3 \).
8.1 Single mode and multiplicities

In this section we concentrate on the single mode \( \lambda \) and analyze its multiplicity. So suppose \( p(y)e^{\lambda y} \) is one of the terms in (8.3), while \( q(y)e^{\lambda y} \) is one of the terms in \( e(y) \). Where \( p(y) = \sum_{j=0}^{4} p_j y^j \) and \( q(y) = \sum_{j=0}^{4} q_j y^j \).

We are going to show that type 2 mode cannot have multiplicity larger than one (see Lemma 15), while type 1 mode cannot have multiplicity larger than 3 (see Lemma 14). Finally, here we also derive item 1 (in the limiting case \( \gamma = 0 \)) and item 2 of Theorem 4 (see Corollary 13).

After substitution of the corresponding expressions for \( \delta, \epsilon, \phi \) into (8.1), we collect the coefficients of \( y^j e^{\lambda y} \) and from linear independence conclude that they must be zero. Like this we obtain 5 relations involving \( \kappa \).

Let us first change the variables \( k(z) = k(z)e^{\lambda z/2} \), then the relation corresponding to \( y^j e^{\lambda y} \) can be conveniently written as

\[
p_j \kappa''(z) - \frac{\nu^{(j)}(z)}{2} \kappa''(-z) + \frac{\nu^{(j+1)}(z)}{2} \kappa'(-z) - (j + 1) p_{j+1} \kappa'(z) + \frac{\nu^{(j)}(z)}{2} \kappa(-z) - \varepsilon_j \kappa(z) = 0, \quad j = 0, \ldots, 4, \tag{8.4}
\]

with the convention that \( p_5 = 0 \), and the notation

\[
\varepsilon(z) = \sum_{j=0}^{4} \varepsilon_j z^j, \quad \varepsilon_j = \frac{\lambda^2 p_j}{4} - q_j + \frac{(j+1)}{2} \lambda p_{j+1}.
\]

Let \( \deg(p) = m \) and \( \deg(q) = n \), and \( \kappa_+, \kappa_- \) be the even and odd parts of \( \kappa \), respectively. If \( n > m \) the relation in (8.4) for \( j = n \) reads \( q_n, \kappa_-(z) = 0 \), so \( k(z) = \kappa_+(z)e^{\lambda z/2} \), the symmetry (A) implies \( \lambda = 2i\beta \) for some \( \beta \in \mathbb{R} \) and that \( \kappa_+ \) is real valued.

Let now \( n \leq m \), then (8.4) for \( j = m \) reads

\[
\kappa''_-(z) - \mu^2 \kappa_-(z) = 0, \quad \mu = \sqrt{\frac{\lambda^2}{4} - \frac{q_m}{p_m}}, \tag{8.5}
\]

hence there are two possibilities: if \( \mu = 0 \), then \( \kappa_-(z) = \alpha z + \beta \) and if \( \mu \neq 0 \), then \( \kappa_-(z) = \alpha e^{\mu z} + \beta e^{-\mu z} \), using that \( \kappa_- \) is an odd function we conclude

\[
\kappa_-(z) = \begin{cases}
\alpha z, & \mu = 0, \\
\alpha \sinh(\mu z), & \mu \neq 0.
\end{cases} \tag{8.6}
\]

Thus, \( k(z) = e^{\lambda z/2}(\kappa_+(z) + \kappa_-(z)) \), where \( \kappa_+ \) is a free even function. Now the symmetry condition (A) says

\[
e^{\lambda z/2} (\kappa_+(z) + \kappa_-(z)) = e^{-\lambda z/2} (\kappa_+(z) - \kappa_-(z)) \tag{8.7}
\]

This equation can be solved uniquely for \( \kappa_+ \) if and only if \( \text{Re} \lambda \neq 0 \).

If \( \lambda = 2i\beta \), then \( \kappa_+ \) can be arbitrary real and even function, while solvability implies that

\[
k(z) = e^{i\beta z} \left( \kappa_+(z) + \begin{cases} \alpha z, & \mu = 0, \\
\alpha \sinh(\mu z), & \mu \neq 0 \end{cases} \right), \tag{8.8}
\]

where \( \alpha, \mu \in \mathbb{R} \). Observe that the case \( n > m \) is included here when we take \( \alpha = 0 \), therefore we may assume \( m \geq n \).

**Remark 14.** When \( \kappa_- \) is given by the second formula of (8.6), then (8.7) implies that there are two cases, either \( \alpha \in i\mathbb{R} \) and \( \mu \in \mathbb{R} \) which gives the second formula of (8.8), or \( \alpha \in \mathbb{R} \) and \( \mu \in i\mathbb{R} \), which gives the third one, where with the abuse of notation we denoted the imaginary part of \( \mu \) again by \( \mu \).

If \( \lambda = 2\gamma + 2i\beta \) with \( \gamma \neq 0 \), then

\[
k(z) = \begin{cases} 2e^{i\beta z} \frac{\alpha e^{-\gamma z} + \bar{\alpha} e^{\gamma z}}{\sinh(2\gamma z)}, & \mu = 0, \\
e^{i\beta z} \frac{\alpha e^{-\gamma z} \sinh(\mu z) + \bar{\alpha} e^{\gamma z} \sinh(\bar{\mu} z)}{\sinh(2\gamma z)}, & \mu \neq 0.
\end{cases} \tag{8.9}
\]
where $\alpha, \mu \in \mathbb{C}$.

So far we have analyzed only one of the relations from (8.4) and deduced the possible forms of $k$. When the mode $\lambda$ has multiplicity at least two we have $m \geq 1$, and therefore there are more relations in (8.4) that $k$ has to satisfy (in particular the one corresponding to $j = m - 1$). In the two subsections below we analyze these possibilities.

### 8.1.1 Type 1 mode and multiplicities

**Proposition 11.** Let $\text{Re} \lambda = 0$ and $m \geq 1$, then with $\lambda = 2i\beta$ and $\alpha, \mu, \kappa_0 \in \mathbb{R}$ we have (in fact $\kappa = i\alpha \omega$ with $\omega$ defined in (8.11) below)

$$
k(z) = e^{i\beta z}, \begin{align*}
\sinh(\kappa z) + \kappa_0 \sinh(\mu z), & \quad \mu = 0, \\
\sinh(\mu z) + \kappa_0 \sin(\mu z) - 2 \kappa \cosh(\mu z), & \quad \mu \neq 0, \\
\alpha e^{i\mu z} + \kappa_0 \sin(\mu z), & \quad \mu \neq 0.
\end{align*}
$$

(8.10)

**Proof.** So we see that the function $\kappa_+$ in (8.8) is not arbitrary and we are going to find it from the relation (8.4) with $j = m - 1$ (because $m \neq 0$ we can consider the index $m - 1$). Recall that w.l.o.g. we assumed $m \geq n$, note that $p^{(m-1)}(z) = m!p_m z + (m-1)!p_{m-1}$, $\varepsilon_m = \lambda^2 - q_m$ and $\varepsilon_{m-1} = \lambda^2 - q_{m-1} - \frac{m}{2} \lambda p_m$ so we obtain

$$
p_{m-1} \kappa''(z) - (mp_m z + p_{m-1}) \kappa''(-z) + mp_m \kappa'(-z) - \kappa'(z) + [m\varepsilon_m z + \varepsilon_{m-1}] \kappa(-z) - \varepsilon_{m-1} \kappa(z) = 0.
$$

Now using (8.5) we can rewrite the above relation as

$$
z \kappa'' + 2\kappa' - \mu^2 z \kappa = \omega \kappa, \quad \omega = -\lambda + \frac{2}{mp_m} \left( q_{m-1} - \frac{q_m p_{m-1}}{p_m} \right),
$$

(8.11)

where $\kappa_-$ appears in the three formulas from (8.8).

According to Remark 14, when $\kappa_-(z) = i\alpha \sin(\mu z)$, in the above relation $\mu$ should be replaced by $i\mu$, which changes the sign of the last term on LHS from negative to positive. This explains the difference of the sign in the second and third formulas of (8.10). Solving the obtained ODE, recalling that $\kappa_+$ is even and real valued, we find (8.10) with $\kappa = i\alpha \omega$.

When $m \geq 2$, we can consider (8.4) with $j = m - 2$, moreover we know that (8.5) and (8.11) also hold, and using these and $p^{(m-2)}(z) = \frac{m!}{2}p_m z^2 + (m-1)!p_{m-1} z + (m-2)!p_{m-2}$, the relation with $j = m - 2$ can be simplified to

$$
z \kappa_+ + \eta_1 \kappa_+ = \eta_2 z \kappa_+, \quad \eta_2 = \frac{\omega}{2},
$$

(8.12)

where $\omega$ is defined in (8.11) and $\eta_1$ is a constant whose precise expression is not important.

**Proposition 12.** Let $\text{Re} \lambda = 0$ and $m \geq 2$, then with $\lambda = 2i\beta$ and $\alpha, \kappa_0, \mu \in \mathbb{R}$

$$
k(z) = e^{i\beta z} . \begin{align*}
\sinh(\mu z) + \kappa_0 \sin(\mu z) / z, & \quad \mu = 0, \\
\alpha e^{i\mu z} + \kappa_0 \sin(\mu z) / z, & \quad \mu \neq 0.
\end{align*}
$$

(8.13)

Moreover, in the second case the following relations between the involved parameters must be satisfied

$$
\kappa_0 \eta_2 = i\alpha \eta_1, \quad \eta_2 = \pm i\mu.
$$

(8.14)

**Proof.** By Proposition 11 we know what are the functions $\kappa_-$ and $\kappa_+$ that satisfy the two relations (8.4) with $j = m, m - 1$ (they are given in the three formulas in (8.10), with $\kappa = i\alpha \omega$). Here we want to see which of these satisfy the third relation (8.12). First note that $\kappa \in \mathbb{R}$ implies $\omega$ and hence also $\eta_2 = \frac{\omega}{2}$ are purely imaginary. The case (8.10)a implies that $k$ has rank at most three and so, is trivial.

If (8.10)b holds, then (8.12) after multiplying by $2\mu$ reads
By linear independence we conclude that the two coefficients must vanish: \(2i\alpha\mu^2 - \eta_2\kappa = 0\) and \(i\alpha\eta_1 - \eta_2\kappa_0 = 0\).

Let us ignore the second equation (it just gives some restrictions on \(q_j\)'s), using the expression for \(\kappa\) the first one becomes \(\alpha(\mu^2 - \eta_2^2) = 0\). If \(\alpha \neq 0\), because \(\eta_2 \in i\mathbb{R}\), we conclude \(\mu = \eta_2 = 0\) which is a contradiction. Thus \(\alpha = 0\), which gives the first formula of (8.13).

If (8.10)c holds, then (8.12) reads

\[
z(2i\alpha\mu^2 + \eta_2\kappa)\cos(\mu z) + 2\mu(i\alpha\eta_1 - \eta_2\kappa_0)\sin(\mu z) = 0.
\]

Again the two coefficients must be zero, the second one implies the first relation of (8.14) and the first one gives \(\alpha(\mu^2 + \eta_2^2) = 0\). One possibility is \(\alpha = 0\), another one: when \(\alpha \neq 0\), then \(\Im \eta_2 = \pm \mu\), hence we may write \(\kappa(z) = \pm \alpha(\cos\mu z \pm i\sin\mu z) + \kappa_0 \frac{\sin\mu z}{z} \pm \alpha e^{\pm i\mu z} + \kappa_0 \frac{\sin\mu z}{z}\). These cases can be unified in the second formula of (8.13).

**Corollary 13.** When there is one type 1 root with multiplicity three (i.e. \(\nu = 1\), \(m = 2\) and \(\lambda = 2i\beta\)), we obtain item 1 (in the limiting case \(\gamma = 0\)) and item 2 of Theorem 4.

**Proof.** Using the boundary conditions \(\theta(y) = (y^2 - 1)e^{\nu y}\), we know \(k\) from the above proposition so it only remains to find \(c\). Before that let us invoke Remark 11 and w.l.o.g. assume that \(\beta = 0\), or equivalently \(\lambda = 0\).

From (8.2) we know that \(c(y) = \sum_{j=0}^{3} c_j y^j + c_4 e^{\nu y}\) with \(\tau \neq 0\). Clearly \(\mu \neq 0\), otherwise \(k\) is trivial (see (8.13)). We substitute these expressions into (8.1) and obtain that a linear combination of \(e^{\nu y}\) and monomials \(y^j\) is zero, hence by linear independence each of the coefficients must vanish. The equation coming from the term \(e^{\nu y}\) reads

\[
c_4 [k(z) - e^{\nu z}k(-z)] = 0.
\]

Equations coming from the terms \(y^3, ..., 1\), respectively are

\[
c_3 [k(z) - k(-z)] = 0,
\]

\[
k''(z) - k''(-z) + c_2 k(z) - (3c_3 z + c_2) k(-z) = 0,
\]

\[
2zk''(-z) + 2k'(-z) - 2k'(z) + c_1 k(z) + (3c_3 z^2 + 2c_2 z + c_1) k(-z) = 0,
\]

\[
k''(z) + (z^2 - 1)k''(-z) - 2zk'(-z) - c_0 k(z) + (c_3 z^3 + c_2 z^2 + c_1 z + c_0) k(-z) = 0.
\]

Assume \(k\) is given by the first formula of (8.13), in particular it is even and (8.15) implies \(c_4 = 0\). The first equation of (8.16) is identity, the second one implies \(c_3 \sin(\mu z) = 0\) and hence \(c_3 = 0\). Third one reads \((c_2 + \mu^2) \sin(\mu z) = 0\), hence \(c_2 = -\mu^2\). Finally, the fourth relation simplifies to \(c_1 \sin(\mu z) = 0\), so that \(c_1 = 0\). We note that \(c_0\) remains free. Thus, we conclude that \(c(y) = -\mu^2 y^2 + c_0\) and since we are free to choose \(c_0\), we can rewrite \(c\) as \(c(y) = -\mu^2 \theta(y) + c_0\), which proves item 1 of Theorem 4 in the case \(\gamma = 0\) and \(\mu \in \mathbb{R}\).

Assume \(k\) is given by the second formula of (8.13). Because \(\kappa_0 \neq 0\), we may normalize it to be one. (8.15) reads

\[
c_4 \left[e^{-i\mu z} - e^{i\mu z} + e^{i(\mu + \tau)z} - e^{-(i\mu + \tau)z} + i\alpha z(e^{-(i\mu + \tau)z} - e^{i\mu z})\right] = 0,
\]

and from the linear independence of the involved exponentials we get \(c_4 = 0\). The first equation of (8.16) reads \(c_3 \alpha \sin(\mu z) = 0\), and there are two cases to consider.

If \(\alpha = 0\), the second equation reads \(c_3 \sin(\mu z) = 0\), so \(c_3 = 0\). The third equation becomes \((c_2 - \mu^2) \sin(\mu z) = 0\), hence \(c_2 = \mu^2\). Finally, the fourth equation implies \(c_1 = 0\) and again \(c_0\) is free. So we find \(c(y) = \mu^2 \theta(y) + c_0\), which proves item 1 of Theorem 4 in the case \(\gamma = 0\) and \(\mu \in i\mathbb{R}\).

If \(\alpha \neq 0\), then \(c_3 = 0\). The second equation of (8.16) implies \(c_2 = \mu^2\), the third one: \(c_1 = 2i\mu\) and finally the fourth one implies \(c_0 = -\mu^2 + \frac{\mu}{\alpha}\). Thus, \(c(y) = \mu^2(y^2 - 1) + 2i\mu y + \frac{2\mu}{\alpha}\), which proves item 2 of Theorem 4.

**Lemma 14.** Let \(\Re \lambda = 0\) and \(m \geq 3\), then \(k\) is trivial.
Proof. By the previous proposition we know that \( \kappa(z) \) has two possible forms coming from (8.13). The goal is to show that it cannot solve (8.4) with \( j = m - 3 \). Using the equations (8.5), (8.11) and (8.12) we can rewrite the relation for \( j = m - 3 \) as

\[
(\eta_2 z^2 + \eta_3)\kappa_- = z^2 \kappa'_+ + 3\eta_1 z \kappa_+,
\]

where \( \eta_1, \eta_2 \) are the same as in (8.12) and the expression for \( \eta_3 \) is not important.

When \( k \) is given by the first formula of (8.13), \( \kappa_-(z) = 0 \) and \( \kappa_+(z) = \kappa_0 \frac{\sinh(\mu z)}{z} \) so (8.17) implies \( \mu = 0 \) and hence \( k = 0 \).

When \( k \) is given by the second formula of (8.13), let us w.l.o.g. take \( \kappa_0 = 1 \). As we saw in the previous proposition \( \kappa_-(z) = i \alpha \sin(\mu z) \) and \( \kappa_+(z) = \frac{\sin(\mu z)}{z} - i \alpha \eta_2 \cos(\mu z) \) with \( \eta_2 = \pm \mu \) and \( i \alpha \eta_1 = \eta_2 \). Let first \( \eta_2 = i \mu \), then substituting \( \kappa_\pm \) into (8.17) we get

\[
[i \alpha \eta_3 + \kappa_0 (1 - 3 \eta_1)] \sin(\mu z) - z (\mu + 3 \alpha \eta_1) \cos(\mu z) = 0.
\]

But then \( \mu + 3 \alpha \eta_1 = 4 \mu \) which must be zero, hence \( k \) is trivial. The case \( \eta_2 = -i \mu \) is done analogously. \( \square \)

8.1.2 Type 2 mode and multiplicities

**Lemma 15.** Let \( \Re \lambda \neq 0 \) and \( m \geq 1 \), then \( k = 0 \).

**Proof.** Let \( \lambda = \gamma + i \beta \), with \( \gamma \neq 0 \), (8.7) implies

\[
\begin{cases}
\kappa_+ - \kappa_+ e^{\gamma z} = \kappa_- e^{\gamma z} + \kappa_- \kappa_+, \\
\kappa_- - \kappa_+ e^{\gamma z} = \kappa_- e^{\gamma z} + \kappa_- 
\end{cases}
\]

where the second equation was obtained by conjugating the first one, then

\[
\kappa_+ = - \coth(\gamma z) \kappa_- - \csc(h(\gamma z)) \kappa_-.
\]

We know that both of the relations (8.5) and (8.11) hold. When \( \mu = 0 \), we have \( \kappa_-(z) = \kappa z \), hence \( \kappa_+(z) = \frac{\omega z^2}{\beta} + \kappa_0 \) and comparing this with (8.18) we conclude \( k = 0 \). So let us assume \( \mu \neq 0 \), then from (8.6), \( \kappa_-(z) = \alpha \sinh(\mu z) \), hence solving the ODE (8.11) we get

\[
\kappa_+(z) = c_2 \frac{\sinh(\mu z)}{z} + \frac{\omega \alpha}{2 \mu} \cosh(\mu z),
\]

substitute this into (8.18) divide the result by \( \sinh(\mu z) \) to get

\[
\frac{c_2}{z} + \frac{\omega \alpha}{2 \mu} \coth(\mu z) = - \alpha \coth(\gamma z) - \frac{\omega}{\sinh(\mu z)} \csc(h(\gamma z)).
\]

Assume \( \gamma > 0 \) (otherwise negate \( (\gamma, \alpha, \omega) \)), write \( \mu = \mu_1 + i \mu_2 \), assume \( \mu_1 \neq 0 \), then we may assume \( \mu_1 > 0 \), otherwise multiply the equation by \(-1 \). Now consider the asymptotics as \( z \to +\infty \),

\[
\frac{c_2}{z} + \frac{\omega \alpha}{2 \mu} = - \alpha - 2 \mu e^{-\gamma z} e^{-2 i \mu_2 z},
\]

clearly this implies \( \alpha = c_2 = 0 \), so \( k = 0 \). Let now \( \mu_1 = 0 \), then the relation reads

\[
\frac{c_2}{z} - \frac{\omega \alpha}{2 \mu_2} \cot(\mu_2 z) = - \alpha \coth(\gamma z) + \frac{\omega}{\sinh(\mu_2 z)} \csc(h(\gamma z)),
\]

and asymptotics at \( +\infty \) gives \( \frac{c_2}{z} - \frac{\omega \alpha}{2 \mu_2} \cot(\mu_2 z) = - \alpha + 2 \mu e^{-\gamma z} \) which again implies \( \alpha = c_2 = 0 \). \( \square \)
8.2 Multiple modes

Before we start to analyze the possibilities of having multiple distinct modes \( \lambda_j \) in (8.3), we state that in view of Lemmas 14 and 15 the cases I and II can be rewritten

I. \( \nu = 1, \ d_1 = 2, \ \Re \lambda_1 = 0; \)

IIa. \( \nu = 2, \ d_1 \geq 1, \ \Re \lambda_1 = \Re \lambda_2 = 0; \)

IIb. \( \nu = 2, \ d_1 \geq 1, \ \Re \lambda_1 = 0, \ \Re \lambda_2 \neq 0. \)

The case I was analyzed in Corollary 13, so it remains to consider cases IIa,b and III, IV. We will see in Lemmas 18 and 22 that the cases IIa,b lead to trivial kernels \( k \). Case III will be analyzed in Section 8.4. We will show that case IV is only possible when there are exactly three modes: two type 1 and one type 2, all with multiplicity one. This case will then be analyzed in Section 8.3.

When \( \lambda_j = 2t\beta_j \) (of course \( \beta_1 \neq \beta_2 \)) then (8.8) holds true for both of the modes \( \lambda_j \) and we determine the free functions and conclude

\[
k(z) = \frac{\alpha_1 k_x(\mu_1 z)e^{i\beta_1 z} + \alpha_2 k_r(\mu_2 z)e^{i\beta_2 z}}{\sin(\beta_1 - \beta_2)z}, \quad r, s \in \{1, 2, 3\},
\]

(8.19) where all the constants are real, \( \mu_j \neq 0 \) and \( k_r \) is given by

\[
k_1(t) = t, \quad k_2(t) = \sin t, \quad k_3(t) = \sinh t.
\]

(8.20)

**Proposition 16.** Let \( k \) be given by (8.19), then \( \beta_1 \) and \( \beta_2 \) are determined by \( k \).

**Proof.** W.l.o.g. let \( \beta_1 - \beta_2 > 0 \), otherwise swap \( \beta_1 \) with \( \beta_2 \); \( r \) with \( s \); \( \mu_1 \) with \( \mu_2 \) and replace \( (\alpha_1, \alpha_2) \) by \( (-\alpha_2, -\alpha_1) \). There are six cases to consider.

- **If** \( (s, r) = (3, 3) \); we have

\[
k(it) = e^{-\beta_1 t} \cdot \frac{\alpha_1 \sin(\mu_1 t) + \alpha_2 \sin(\mu_2 t)e^{(\beta_1 - \beta_2)t}}{\sinh(\beta_1 - \beta_2)t},
\]

therefore

\[
k(it) \sim \begin{cases} 
2\alpha_1 \sin(\mu_1 t)e^{(\beta_2 - 2\beta_1)t} + 2\alpha_2 e^{-\beta_1 t} \sin(\mu_2 t), & t \to +\infty, \\
2\alpha_1 \sin(\mu_1 t)e^{-\beta_2 t} + 2\alpha_2 e^{(\beta_1 - 2\beta_2)t} \sin(\mu_2 t), & t \to -\infty.
\end{cases}
\]

When \( (s, r) = (1, 1) \) the same formulas hold with \( \sin(\mu_1 t) \) replaced by \( t \) for \( j = 1, 2 \). And when \( (s, r) = (1, 3) \) the same formulas hold with \( \sin(\mu_1 t) \) replaced by \( t \). The above asymptotics immediately conclude the proof in this case.

- **If** \( (s, r) = (2, 3) \), we may assume \( \mu_1 > 0 \), otherwise negate \( \alpha_1 \), so

\[
k(it) = e^{-\beta_1 t} \cdot \frac{\alpha_1 \sin(\mu_1 t) + \alpha_2 \sin(\mu_2 t)e^{(\beta_1 - \beta_2)t}}{\sinh(\beta_1 - \beta_2)t},
\]

and therefore

\[
k(it) \sim \begin{cases} 
\alpha_1 e^{(\mu_1 + \beta_2 - 2\beta_1)t} + 2\alpha_2 e^{-\beta_1 t} \sin(\mu_2 t), & t \to +\infty, \\
\alpha_1 e^{-(\mu_1 + \beta_2)t} + 2\alpha_2 e^{(\beta_1 - 2\beta_2)t} \sin(\mu_2 t), & t \to -\infty.
\end{cases}
\]

If \( \alpha_2 \neq 0 \) clearly \( \beta_1 \) and \( \beta_2 \) are determined. So assume \( \alpha_2 = 0 \), then from the above asymptotics we conclude that \( \alpha_3, \mu_1 + \beta_2 \) and \( \beta_1 \) are determined. But note that \( k_0 := k(0) = \frac{\alpha_1}{\beta_1 - \beta_2} \), so we have a system \( (k_1 \) denotes a parameter determined by \( k)\)

\[
\begin{cases} 
\alpha_1 \mu_1 + k_0 \beta_2 = k_0 \beta_1 \\
\mu_1 + \beta_2 = k_1
\end{cases}
\]
These values then the function in (8.19) is not entire, while all functions (8.10) are entire. Thus it must hold

\[ k(it) = e^{-\beta_1 t} \cdot \frac{\alpha_1 \sinh(\mu_1 t) + \alpha_2 \sinh(\mu_2 t)e^{(\beta_1 - \beta_2)t}}{\sinh(\beta_1 - \beta_2)t}, \]

therefore

\[ k(it) \sim \begin{cases} 
\alpha_1 e^{(\mu_1 + \beta_2 - 2\beta_1)t} + \alpha_2 e^{(\mu_2 - \beta_1)t}, & t \to +\infty, \\
\alpha_1 e^{-(\mu_1 + \beta_2)t} + \alpha_2 e^{-(\mu_2 - \beta_1 + 2\beta_2)t}, & t \to -\infty.
\end{cases} \]

If \( \alpha_1, \alpha_2 \neq 0 \), clearly \( \beta_1 \) and \( \beta_2 \) are determined. Assume \( \alpha_1 = 0 \), then from the above asymptotics we conclude that \( \alpha_2, \mu_2 - \beta_1 \) and \( \beta_2 \) are determined. Next, as above we look at \( k(0) = \frac{\mu_2}{\beta_1 - \beta_2} \), and conclude that \( \beta_1, \mu_2 \) are not determined if and only if \( \mu_2 = \beta_1 - \beta_2 \) in which case \( k \) is trivial. Analogous conclusion holds in the case \( \alpha_2 = 0 \).

\[ \square \]

**Corollary 17.** Having three distinct modes \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{i} \mathbb{R} \) is impossible.

**Lemma 18.** Having two distinct type 1 modes, one of them with multiplicity at least two leads to a trivial kernel. In other words, if \( k(z) \) can be written in the form (8.10) and (8.19), then \( k \) is trivial.

**Proof.** The denominator in (8.19) is zero when \( z = \pi n / (\beta_1 - \beta_2) \). If the numerator does not vanish at all of these values then the function in (8.19) is not entire, while all functions (8.10) are entire. Thus it must hold

\[ \alpha_1 k_s \left( \frac{\pi \mu n}{\beta_1 - \beta_2} \right) + (-1)^n \alpha_2 k_r \left( \frac{\pi \mu n}{\beta_1 - \beta_2} \right) = 0 \quad \forall n \in \mathbb{Z}. \]

This equation can hold in three cases \((r, s) = (2, 2), (2, 3) \) or \((1, 2) \). Let us consider the first one, the other two can be analyzed similarly, and in fact are simpler. The solutions of the above equation for \( r = s = 2 \) are

(a) \( \mu_j = m_j(\beta_1 - \beta_2) \) with \( m_j \in \mathbb{Z} \) for \( j = 1, 2 \),

(b) \( \alpha_1 = \pm \alpha_2 \), \( \mu_2 = (2m_1 + 1)(\beta_1 - \beta_2) + \mu_1 \).

In both of these cases \( k \) is a trigonometric polynomial. But if \( k \) is given by (8.10) and is a trigonometric polynomial, then \( k(z) = e^{i\beta z}(\alpha \sin \mu z + \alpha' \cos \mu z) \) for some constants \( \alpha, \alpha', \beta \) and \( \mu \). Showing that \( k \) is trivial.

\[ \square \]

**Lemma 19.** Let \( k \) be given by (8.9), then the pair \( (|\gamma|, \beta) \) is determined by \( k \).

**Proof.** Let \( k \) be given by the first formula, assume \( \gamma > 0 \), otherwise replace \( (\gamma, \alpha) \) with \((-\gamma, -\alpha)\), then

\[ k(z) \sim 2\pi z e^{-\gamma z} e^{i\beta z}, \quad \text{as } z \to +\infty, \quad (8.21) \]

so \( \alpha, \gamma, \beta \) are determined by \( k \). But note that the sign of \( \gamma \) is not determined.

Let now \( k \) be given by the second formula, write \( \mu = \mu_1 + i\mu_2 \) and \( \alpha = \alpha_1 + i\alpha_2 \),

1. let \( \mu_1 \neq 0 \), we may assume \( \mu_1 > 0 \), otherwise we replace \( (\alpha, \mu) \) with \((-\alpha, -\mu)\). Also assume \( \gamma > 0 \), otherwise we replace \( (\gamma, \alpha, \mu) \) with \((-\gamma, -\alpha, -\mu)\), then

\[ k(z) \sim \pi e^{-(\gamma + \mu_1)z} e^{i(\beta - \mu_2)z}, \quad \text{as } z \to +\infty, \quad (8.22) \]

so \( \alpha, -\gamma + \mu_1 \) and \( \beta - \mu_2 \) are determined by \( k \). We then note that \( k(0) = \frac{\text{Re}(\alpha \mu)}{\gamma} \) and \( k'(0) = i\beta k(0) - i \text{Im}(\alpha \mu) \). Because of the symmetry of \( k \), we know that \( k(0) \in \mathbb{R} \) and \( k'(0) \in i\mathbb{R} \), so let us set \( k_0 = k(0) \) and \( k_1 = \frac{k'(0)}{i} \), then we obtain the system
\[
\begin{aligned}
\begin{cases}
\alpha_1\mu_1 - \alpha_2\mu_2 - k_0\gamma = 0, \\
-\alpha_2\mu_1 - \alpha_1\mu_2 + k_0\beta = k_1, \\
\mu_1 - \gamma = k_2, \\
-\mu_2 + \beta = k_3,
\end{cases}
A = \begin{pmatrix}
\alpha_1 & -\alpha_2 & -k_0 & 0 \\
-\alpha_2 & -\alpha_1 & 0 & k_0 \\
1 & 0 & -1 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix},
\end{aligned}
\]

where the unknowns are \(\mu_1, \mu_2, \gamma, \beta, \) and \(k_2, k_3\) are parameters determined by \(k\). The system is linear and one can compute \(\det(A) = (\alpha_1 - k_0)^2 + \alpha_2^2\). If \(\det(A) \neq 0\), then the system has a unique solution and all the constants \(\mu_1, \mu_2, \gamma, \beta\) are determined by the function \(k\). Of course we see that the signs of \(\gamma\) and \(\mu_1\) are not determined.

When \(\det(A) = 0\), we get \(\alpha_1 = k_0\) and \(\alpha_2 = 0\), then (note that \(k_0 \neq 0\), because otherwise \(k = 0\)). Now we must have \(k_2 = 0\) and \(k_3 = \frac{k_1}{k_0}\) and the above system reduces to

\[
\begin{aligned}
\begin{cases}
\mu_1 - \gamma = 0, \\
-\mu_2 + \beta = k_3.
\end{cases}
\end{aligned}
\]

So \(\alpha\) is real and \(\mu_1 = \gamma\), and in this case one can check that the formula reduces to \(k(z) = \alpha e^{i(\beta + \mu_2)z}\) which is a trivial kernel.

2. \(\mu_1 = 0\), we may assume \(\gamma > 0\), otherwise replace \((\gamma, \alpha)\) by \((-\gamma, \alpha)\), then

\[
k(z) \sim \frac{\alpha_2 e^{-\gamma z}}{\sinh(2\gamma z)} \left[ e^{i(\beta - \mu_2)z} - e^{(\beta + \mu_2)z} \right] \quad \text{as } z \to +\infty,
\]

so \(\alpha, \gamma, \beta, \mu_2\) are determined by \(k\). And again we see that the sign of \(\gamma\) is not determined.

\[\square\]

**Corollary 20.** Let \(\lambda_j = 2\gamma_j + i2\beta_j\), with \(\gamma_j \neq 0\) for \(j = 1, 2\). Assume \(\lambda_1 \neq \lambda_2\), then \(\lambda_2 = -\overline{\lambda_1}\).

**Proof.** For each \(\lambda_j\), \(k\) can be given by two formulas from (8.9), let us refer to them as ”a” and ”b”. There are three cases to consider: (a,a); (b,b) and (a,b). By comparing the asymptotics (8.22) and (8.23) with (8.21) we see that they cannot be matched, hence the third case is impossible. Consider the first one, then

\[
k(z) = ze^{i\beta_j} \cdot \frac{\alpha_2 e^{-\gamma_j z} + \overline{\alpha_2} e^{\gamma_j z}}{\sinh(2\gamma_j z)}, \quad j = 1, 2.
\]

As we saw \(|\gamma_j|\) and \(\beta_j\) are determined by \(k\), hence we conclude \(|\gamma_1| = |\gamma_2|\) and \(\beta_1 = \beta_2\). Because \(\lambda_1 \neq \lambda_2\) we must have \(\gamma_1 = -\gamma_2\). The second case is done analogously.

\[\square\]

**Corollary 21.** Having three distinct modes \(\lambda_1, \lambda_2, \lambda_3 \notin i\mathbb{R}\) leads to trivial \(k\).

**Lemma 22.** Having a type 2 mode and a type 1 mode of multiplicity at least two leads to a trivial kernel. In other words, if \(k(z)\) can be written in the form (8.10) and (8.9), then \(k\) is trivial.

**Proof.** So \(\lambda_1 = i2\beta_1\) and \(\lambda_2 = 2\gamma + i2\beta_2\) with \(\gamma \neq 0\). All the functions in (8.10) are entire, and one can easily check that the first function of (8.9) is entire if and only if \(\alpha = 0\), which leads to \(k = 0\). So let us consider the case when \(k\) is given by the second formula:

\[
k(z) = e^{i\beta_2} \cdot \frac{\alpha_2 e^{-\gamma z} \sinh(\mu z) + \overline{\alpha_2} e^{\gamma z} \sinh(\overline{\mu} z)}{\sinh(2\gamma z)} = e^{i\beta_2} \begin{cases}
\alpha_1 z + \kappa_0 + \frac{\kappa}{2} z^2, \\
\alpha_1 \sin \mu_0 z + \kappa_0 \sinh \mu_0 z, \\
\alpha_1 \sinh \mu_0 z + \kappa_0 \sin \mu_0 z \end{cases},
\]

where \(\mu_0 \neq 0\), \(\alpha_1, \kappa_0, \kappa \in \mathbb{R}\), and write \(\mu = \mu_1 + i\mu_2\).

**Case 1:** if \(\mu_1 \neq 0\), may assume \(\mu_1 > 0\) and \(\gamma > 0\). If \(k\) is given by the
1. 1st formula, then comparing the asymptotics we see that $\alpha_1 = \kappa = 0$, then for the LHS $k(z) \sim \kappa_0 e^{i\beta_1 z}$. Again comparing we find $\alpha_2 = \kappa_0$, $-\gamma + \mu_1 = 0$ and $\beta_2 - \mu_2 = \beta_1$. The last two conditions can be rewritten as $\lambda_2 - \lambda_1 = 2\mu$, and so $k(z) = \kappa_0 e^{i\beta_1 z}$, which is trivial.

2. 2nd formula, we may assume $\mu_0 > 0$, otherwise negate $(\alpha_1, \kappa_0, \kappa)$, then $k(z) \sim \frac{1}{2}(i\alpha_1 + \frac{\kappa}{2\mu_0}) e^{i\mu_0 z} e^{i\beta_1 z}$, comparing with (8.22) we conclude

$$-\gamma + \mu_1 = \mu_0, \quad \beta_2 - \mu_2 = \beta_1, \quad i\alpha_1 + \frac{\kappa}{2\mu_0} = 2\overline{\gamma}_2,$$

with these, in (8.24) we express sinh and cosh in terms of exponentials, by linear independence we conclude that $\kappa_0 = 0$, and obtain

$$-\overline{\gamma}_2 e^{(\gamma - \mu_1)z} + \alpha_2 e^{(\gamma - \mu_1)z} = e^{i2\mu_2 z} \left[ \alpha_2 e^{(-3\gamma + \mu_1)z} - \overline{\alpha}_2 e^{(-3\gamma - \mu_1)z} \right].$$

Hence $\mu_2 = 0$, then using that $\gamma, \mu_1 \neq 0$ we deduce that the above relation is possible (with $\alpha_2 \neq 0$) if and only if $\mu_1 = 2\gamma$. Thus $k(z) = e^{i\beta_1 z} \left[ i\alpha_1 \sinh \mu_0 z + \frac{\kappa}{2\mu_0} \cosh \mu_0 z \right]$ is trivial.

3. 3rd formula, we may assume $\mu_0 > 0$, otherwise negate $(\alpha_1, \kappa_0, \kappa)$, then $k(z) \sim e^{i\beta_1 z} \left[ (\alpha_1 - \frac{\kappa}{4\mu_0}) e^{i\mu_0 z} - (\alpha_1 + \frac{\kappa}{4\mu_0}) e^{-i\mu_0 z} \right]$, comparing this with (8.22) we conclude $-\gamma + \mu_1 = 0$ and

(a) $\beta_1 + \mu_0 = \beta_2 - \mu_2$, $\alpha_1 - \frac{\kappa}{4\mu_0} = \overline{\alpha}_2$ and $\alpha_1 + \frac{\kappa}{4\mu_0} = 0$, or
(b) $\beta_1 - \mu_0 = \beta_2 - \mu_2$, $\frac{\alpha_1}{2} - \frac{\kappa}{4\mu_0} = 0$ and $\frac{\alpha_1}{2} + \frac{\kappa}{4\mu_0} = -\overline{\alpha}_2$

Let us consider the first option, in that case (8.24) simplifies to $\kappa_0 e^{i\beta_1 z} \sinh \mu_0 z \overline{\kappa}_0 = 0$ which implies $\kappa_0 = 0$, and so $k(z) = \alpha_1 e^{i(\beta_1 + \mu_0)}$. The other case is done analogously.

**Case 2:** if $\mu_1 = 0$, we may assume $\gamma > 0$. If $k$ is given by the 1st or 3rd formulas, comparing the asymptotics of LHS with (8.23) we conclude $\gamma = 0$, which is a contradiction, so these cases lead to $k = 0$. Now let $k$ be given by the second formula, again w.l.o.g let $\mu_0 > 0$, then we see that the asymptotics cannot be matched because in (8.23) $e^{i(\beta_2 - \mu_2) z}$ are linearly independent, hence $k = 0$.

**Lemma 23.** Let $\lambda_1 = i2\beta_1$ and $\lambda_2 = 2\gamma + i2\beta_2$, with $\gamma \neq 0$, then $\beta_1 = \beta_2 =: \beta$ and

$$k(z) = \alpha e^{i\beta z} \frac{k_r(\mu z)}{\sinh \gamma z}, \quad r \in \{1, 2, 3\},$$

(8.25)

where $\alpha, \mu \in \mathbb{R}$ and $k_r$ is defined in (8.20).

**Proof.** So $k$ is given by both of the forms (8.9) and (8.8). Assume $k$ is given by the first formula of (8.9), then we can find

$$k_+^i(z) = \zeta e^{i(\Delta \beta) z} \frac{\alpha e^{-\gamma z} + \overline{\alpha} e^{\gamma z}}{\sinh(2\gamma z)} - i\alpha' k_r(\mu' z), \quad r \in \{1, 2, 3\},$$

where $\Delta \beta = \beta_2 - \beta_1$, $\mu', \alpha' \in \mathbb{R}$. It is easy to check that $k_+^i$ as above satisfies $k_+^i(-z) = \overline{k_+^i(z)}$, hence $k_+^i$ is real valued if and only if it is even, and with $\alpha = \alpha_1 + i\alpha_2$ the imaginary part of $k_+^i$ being zero reads

$$z\alpha_1 \sin(\Delta \beta z) \overline{\sinh(\gamma z)} - z\alpha_2 \cos(\Delta \beta z) \overline{\cosh(\gamma z)} = \alpha' k_r(\mu' z).$$

(8.26)

We may assume $\gamma > 0$, otherwise replace $(\gamma, \alpha_1)$ with $(-\gamma, -\alpha_1)$. Assume $k \neq 0$, note that

$$\text{LHS} \sim 2\zeta e^{-\gamma z} |\alpha_1 \sin(\Delta \beta z) - \alpha_2 \cos(\Delta \beta z)|, \quad \text{as } z \to +\infty.$$
Comparing this with the asymptotic of RHS for \( r = 1, 2, 3 \) we conclude that \( 8.26 \) is possible if and only if \( \Delta \beta = 0 \) and \( \alpha_2 = \alpha' = 0 \). And we see that \( k \) is given by \( 8.25 \) with \( r = 1 \).

Assume now \( k \) is given by the second formula of \( 8.9 \), then
\[
\kappa_+(z) = e^{i\Delta \beta z} \cdot \frac{\alpha e^{-\gamma z} \sinh(\mu z) + \bar{\alpha} e^{\gamma z} \sinh(\bar{\mu} z)}{\sinh(2\gamma z)} - i\alpha' k_r(\mu' z), \quad r \in \{1, 2, 3\}.
\]

Write \( \mu = \mu_1 + i\mu_2 \) and \( \alpha = \alpha_1 + i\alpha_2 \), w.l.o.g. let \( \gamma > 0 \), assume \( \mu_1 \neq 0 \) then we can assume \( \mu_1 > 0 \); again \( \kappa_+ \) being even and real valued are equivalent and \( \text{Im} \kappa_+ = 0 \) reads
\[
\frac{\sin(\Delta \beta z)}{\sinh(\gamma z)} \left[ (\alpha_1 \sinh(\mu_1 z) \cos(\mu_2 z) - \alpha_2 \cosh(\mu_1 z) \sin(\mu_2 z)) - \frac{\cos(\Delta \beta z)}{\cosh(\gamma z)} [\alpha_1 \cosh(\mu_1 z) \sin(\mu_2 z) + \alpha_2 \sinh(\mu_1 z) \cos(\mu_2 z)] \right] = \alpha' k_r(\mu' z). \tag{8.27}
\]

We note that as \( z \to \infty \)
\[
\text{LHS} \sim e^{(-\gamma + \mu_1 z)} \left[ \alpha_1 \sin(\Delta \beta - \mu_2 z) - \alpha_2 \cos(\Delta \beta - \mu_2 z) \right],
\]
comparing this with the asymptotic of RHS for \( r = 1, 2, 3 \) we conclude that \( 8.27 \) is possible for non-trivial \( k \) if and only if \( \Delta \beta = \mu_2 \) and \( \alpha_2 = \alpha' = 0 \). (For example when \( r = 2 \), \( 8.27 \) is also possible when \( \mu_1 = \gamma, \alpha_2 = 0, \alpha' = \alpha_1 \) and \( \Delta \beta - \mu_2 = \mu' \) but in this case one easily checks that \( k \) is trivial). Now \( 8.27 \) reduces to
\[
\sin(2\mu_2 z) \left[ \frac{\sinh(\mu_1 z)}{\sinh(\gamma z)} - \frac{\cosh(\mu_1 z)}{\cosh(\gamma z)} \right] = 0.
\]

If the second factor is zero, we must have \( \gamma = \mu_1 \) and in this case \( k \) reduces to a trivial kernel. So \( \mu_2 = 0 \), and \( k \) is given by \( 8.25 \) with \( r = 3 \). Let now \( \mu_1 = 0 \), then \( 8.27 \) becomes
\[
- \sin(\mu_2 z) \left[ \left( \frac{\sin(\Delta \beta z)}{\sin(\gamma z)} + \alpha_1 \frac{\cos(\Delta \beta z)}{\cosh(\gamma z)} \right) \right] = \alpha' k_r(\mu' z). \tag{8.28}
\]

We note that as \( z \to \infty \)
\[
\text{LHS} \sim -2e^{-\gamma z} \sin(\mu_2 z) \left[ \alpha_2 \sin(\Delta \beta z) + \alpha_1 \cos(\Delta \beta z) \right],
\]
comparing this with the asymptotics of RHS for \( r = 1, 2, 3 \) we find that \( 8.28 \) is possible for non-trivial \( k \) if and only if \( \Delta \beta = 0 \) and \( \alpha_1 = \alpha' = 0 \). And \( k \) is given by \( 8.25 \) with \( r = 2 \).

\[\square\]  

\textbf{Corollary 24.} Having three distinct modes \( \lambda_1, \lambda_2 \in i\mathbb{R} \) and \( \lambda_3 \notin i\mathbb{R} \) is impossible.

\section*{8.3 Item 1, \( \gamma \neq 0 \)}

The previous analysis shows that case IV is only possible when we have exactly three modes \( \lambda_1, \lambda_2 \notin i\mathbb{R} \) and \( \lambda_3 \in i\mathbb{R} \) with multiplicities 1, that is \( d_j = 0 \) for \( j = 1, 2, 3 \). Moreover, by Corollary 20 and Lemma 23 we conclude that \( \lambda_1 = 2\gamma + 2i\beta, \lambda_2 = -2\gamma + 2i\beta, \lambda_3 = 2i\beta \) and \( k(z) \) is given by \( 8.25 \). Invoking Remark 11 let us w.l.o.g. assume \( \beta = 0 \). Thus,
\[
\lambda_1 = 2\gamma, \quad \lambda_2 = -2\gamma, \quad \lambda_3 = 0, \quad \text{and} \quad k(z) = \frac{k_r(\mu z)}{\sinh(\gamma z)}, \quad r \in \{1, 2, 3\},
\]
where \( k_r \) is defined in \( 8.20 \), moreover \( \delta(y) = \cosh(2\gamma y) - \cosh(2\gamma) \). Because of \( 8.2 \), \( \kappa \) has the following form
\[
\kappa(y) = (c_1 y + d_1) e^{\lambda_1 y} + (c_2 y + d_2) e^{\lambda_2 y} + (c_3 y + d_3) e^{\lambda_3 y} + c_4 e^{\tau y},
\]
where \( \tau \) is different from all \( \lambda_j \)’s. Substituting these expressions into \( 8.1 \) and looking at linearly independent parts it is easy to conclude that \( c_1 = c_2 = c_3 = c_4 = 0 \), and \( d_1 = \frac{\lambda_1^2 + 4\mu^2}{8}, \quad d_2 = \frac{\lambda_2^2 + 4\mu^2}{8} \) if in the formula for \( k \) we have \( r = 2 \). When \( r = 3 \) in the expressions of \( d_1, d_2 \); \( \mu \) should be replaced by \( i\mu \) and when \( r = 1 \), in those formulas \( \mu = 0 \). This concludes item 1 of Theorem 4 in the case \( \gamma \neq 0 \).
8.4 Item 3

Finally we consider the case III, because of the boundary conditions one can find that \( \lambda_2 - \lambda_1 = i \pi n \) with \( 0 \neq n \in \mathbb{Z} \), therefore \( \lambda_1, \lambda_2 \in i \mathbb{R} \) (otherwise by Corollary 20 and Lemma 23 the difference \( \lambda_2 - \lambda_1 \) is real). Let us now take \( \lambda_1 = 2i(\beta + \frac{\pi n}{4}) \) and \( \lambda_2 = 2i(\beta - \frac{\pi n}{4}) \) with some \( \beta \in \mathbb{R} \). In this case we find \( \delta(y) = e^{2i\beta y} \sin \left( \frac{\pi n(y - 1)}{2} \right) \) and by (8.19)

\[
k(z) = e^{i\beta z} \frac{\alpha_1 k_s(\mu_1 z) e^{i\pi n z/4} + \alpha_2 k_r(\mu_2 z) e^{-i\pi n z/4}}{\sin(\pi n z/2)}, \quad r, s \in \{1, 2, 3\}.
\]

From (8.2), \( e \) has the form

\[
e(y) = (c_1 y + d_1) e^{\lambda_1 y} + (c_2 y + d_2) e^{\lambda_2 y} + c_3 e^{\tau y},
\]

with \( \tau \neq \lambda_j \), note that also \( \tau = \frac{2 \nu_j(0)}{\nu(0)} \in i \mathbb{R} \). The denominator of \( k \) has zeros at \( z = \frac{2m}{n} \) for \( m \in \mathbb{Z} \), since we want \( k \) to be smooth in \([-2, 2]\), we need

\[
(-1)^{m} \alpha_1 k_s \left( \frac{2m_1 m}{n} \right) + \alpha_2 k_r \left( \frac{2m_2 m}{n} \right) = 0, \quad \forall m \in \mathbb{Z} \quad \text{s.t.} \quad \frac{m}{n} \in [-1, 1].
\]

1. \( r = s = 3 \), if \( n \neq \pm 1 \), then (8.30) must hold for \( m = 1, 2 \), one can easily see that this leads to a contradiction. Therefore \( n = \pm 1 \), in which case (8.30) implies \( \alpha_1 \sinh(2\mu_1) = \alpha_2 \sinh(2\mu_2) \). To find \( c \), we substitute these expressions into (8.1) and look at the coefficients of linearly independent parts, which must vanish. In particular the coefficient of \( e^{\tau y} \) gives

\[
c_3 \left\{ \alpha_2 \sinh(\mu_2 z) \left[ e^{-\lambda_1 - \pi \nu \pi z} - e^{\lambda_1} \right] + \alpha_1 \sinh(\mu_1 z) \left[ e^{-\lambda_2 - \pi \nu \pi z} - e^{\lambda_2} \right] \right\} = 0.
\]

The four exponentials in square brackets are linearly independent, moreover their exponents are purely imaginary, while \( \mu_1, \mu_2 \) are real, hence all the terms are linearly independent, therefore our conclusion is that \( c_3 = 0 \), otherwise \( k = 0 \). Using similar arguments and looking at coefficients of \( ye^{\lambda_1 y}, ye^{\lambda_2 y} \) we find \( c_1 = c_2 = 0 \) and

\[
d_1 = -\frac{i e^{-\frac{\pi}{4} \nu_1}}{8} [\lambda_1^2 - 4\mu_1^2], \quad d_2 = \frac{i e^{-\frac{\pi}{4} \nu_2}}{8} [\lambda_2^2 - 4\mu_2^2].
\]

2. \( s = 1, r = 3 \), we can absorb \( \mu_1 \) into \( \alpha_1 \) and relabel \( \mu_2 \) by \( \mu \), as in 1 we see \( n = \pm 1 \) and \( 2\alpha_1 = \alpha_2 \sinh(2\mu) \). Then one can find \( c_1 = c_2 = c_3 = 0 \) and (8.31) holds with \( \mu_2 = 0 \) and \( \mu_1 = \mu \).

3. \( r = s = 1 \), absorb \( \mu_1 \) into \( \alpha_j \), again \( n = \pm 1 \) and \( \alpha_1 = \alpha_2 \), in which case (up to a real multiplicative constant) \( k(z) = e^{i\beta z} \frac{z}{\sin(\pi z/4)} \), then we can conclude \( c_1 = c_2 = 0 \), \( \tau = 2i\beta \) and (8.31) holds with \( \mu_1 = \mu_2 = 0 \).

4. \( s = 1, r = 2 \), absorb \( \mu_1 \) into \( \alpha_1 \). If \( n = \pm 1 \) we get \( 2\alpha_1 = \alpha_2 \sin(2\mu_2) \), and following the strategy described in 1 we find \( c_1 = c_2 = c_3 = 0 \), and (8.31) holds with \( \mu_1 = 0 \) and \( \mu_2 \) replaced by \( i\mu_2 \). If \( |n| > 1 \), then (8.30) holds for at least \( m = 1, 2 \). It is easy to see that these two equations imply \( \alpha_1 = 0 \) and \( \sin \left( \frac{2\nu_2}{n} \right) = 0 \). But in that case (8.30) holds for any \( m \in \mathbb{Z} \). So \( \mu_2 = \frac{\pi n l}{2} \) for some \( l \in \mathbb{Z} \), hence we see that \( k \) is a trigonometric polynomial, and therefore is trivial.

5. \( s = 3, r = 2 \), again if \( |n| > 1 \) we get \( \alpha_1 = 0 \) and \( \sin \left( \frac{2\nu_2}{n} \right) = 0 \), which again implies \( k \) is trivial. So \( n = \pm 1 \), and we find \( \alpha_1 \sinh(2\mu_1) = \alpha_2 \sinh(2\mu_2) \)

6. \( s = r = 2 \), as we saw in Lemma 18 if \( n \neq \pm 1 \), then \( k \) is trivial. So \( n = \pm 1 \) and \( \alpha_1 \sinh(2\mu_1) = \alpha_2 \sinh(2\mu_2) \), one of \( \alpha_j \) is nonzero, assume it is \( \alpha_2 \). When \( \sin(2\mu_1) = 0 \), then \( \sin(2\mu_2) = 0 \) and again \( k \) is a trigonometric polynomial. So \( \sin(2\mu_1) \neq 0 \) and also \( \sin(2\mu_2) \neq 0 \), again because of the same reason. We then find \( c_1 = c_2 = 0 \), (8.31) holds with \( \mu_j \) replaced by \( i\mu_j \) for \( j = 1, 2 \). Finally the relation for \( e^{\tau y} \) reads

\[
c_3 \left\{ \tilde{\alpha}_1 \sin(\mu_1 z) \left[ e^{(\tau - \frac{\pi}{2})z} - e^{\frac{\lambda_1}{2} z} \right] + \tilde{\alpha}_2 \sin(\mu_2 z) \left[ e^{(\tau - \frac{\pi}{2})z} - e^{\frac{\lambda_2}{2} z} \right] \right\} = 0,
\]
where $\alpha_j = \sin(2\mu_j) \neq 0$, $\lambda_1 - \lambda_2 = \frac{2\pi}{L}$. Now $c_3 = 0$ or the function in curly brackets (denote it by $f(z)$) vanishes, looking at the asymptotics $f(iz)$ as $z \to \infty$, and also at $f'(0), f''(0), f^{(4)}(0)$ we can find that $f = 0$ if and only if $\mu_2 = \mu_1 \pm \frac{\pi}{L}$ (which implies $\alpha_1 = -\alpha_2$) and $\tau = 2i(\beta - \frac{\pi}{4} \pm \mu_1)$.

Choosing $\beta = 0$ (cf. Remark 11) we conclude item 3 of Theorem 4.

9 \ $L_2 = -L_1$

Assume the setting of Theorem 6, recall that $\delta := \delta_1$ and $\epsilon := e_1$. Now (R) reads

$$\delta(y)k''(-z) + \delta(y + z)k''(z) + \delta'(y)k'(z) - \delta'(y + z)k'(z) + \epsilon'(y)k(-z) + \epsilon(y + z)k(z) = 0. \tag{9.1}$$

The analysis in the beginning of Section 6 shows that (in the case $L_2 = -L_1$) $\delta(y)$ solves second order, linear homogeneous ODE with constant coefficients, and because of the boundary conditions it must be of the form

$$\delta(y) = b_1 e^{\lambda_1 y} + b_2 e^{\lambda_2 y}, \quad \epsilon(y) = c_1 e^{\lambda_1 y} + c_2 e^{\lambda_2 y} + c_0, \quad \lambda_1 \neq \lambda_2,$$

where $\epsilon$ is of the same form as $\delta$ because it satisfies $\epsilon' = -\frac{b_1}{k_0} \delta' - \frac{b_2}{k_0} \delta$. Clearly both $b_j$ are different from zero, and from boundary conditions

$$\lambda_1 - \lambda_2 = \pi i n, \quad n \in \mathbb{Z}. \tag{9.2}$$

With these formulas, (9.1) becomes a linear combination of functions $e^{\lambda y}$ with coefficients depending on $z$, hence each coefficient must vanish. Let us concentrate on the coefficient of $e^{\lambda_1 y}$, making the change of variables $k(z) = \kappa(z)e^{-\lambda_1 z/2}$ we rewrite it as

$$\kappa''(z) - \mu^2 \kappa(z) = 0, \quad \mu = \sqrt{\lambda_1^2 - \frac{c_1}{b_1}},$$

where $\kappa_+$ is the even part of $\kappa$, because it is an even function we get

$$\kappa_+(z) = \alpha \cosh(\mu z).$$

The symmetry of $k$ implies

$$e^{-\lambda_1 z/2} \left( \kappa_+(z) + \kappa_-(z) \right) = e^{\lambda_1 z/2} \left( \kappa_+(z) - \kappa_-(z) \right).$$

If $\lambda_1 = 2i\beta$ with $\beta \in \mathbb{R}$, then $\kappa_-$ is an arbitrary odd and purely imaginary function. Moreover, $\kappa_+$ must be real valued, hence

$$k(z) = e^{-i\beta z} \left( \kappa_-(z) + \begin{pmatrix} \alpha \cosh(\mu z) \\ \alpha \cos(\mu z) \end{pmatrix} \right), \tag{9.3}$$

where $\alpha, \mu \in \mathbb{R}.$

If $\lambda_1 = 2\gamma + 2i\beta$ with $\gamma \neq 0$, then (recalling that $k$ is smooth at 0), with $\kappa_0 \in \mathbb{R}$

$$k(z) = \alpha e^{-i\beta z} e^{\gamma z} \cosh(\mu z) - e^{-\gamma z} \cosh(\mu z) \overline{\sinh(\gamma z)}.$$

Now $k$ should come from two distinct modes $\lambda_1, \lambda_2$, and from (9.2) we see that $\Re \lambda_1 = \Re \lambda_2 = 2\gamma$, so if $\gamma \neq 0$ we must have

$$\alpha_1 e^{-i\beta_1 z} \left( e^{\gamma z} \cosh(\mu_1 z) - e^{-\gamma z} \sinh(\gamma z) \right) = \alpha_2 e^{-i\beta_2 z} \left( e^{\gamma z} \cosh(\nu z) - e^{-\gamma z} \cosh(\nu z) \right),$$

which implies $\beta_1 = \beta_2$, leading to a contradiction. Indeed, the function on LHS (denoted by $f(z)$) determines $\beta_1$, because with $\mu = \mu_1 + i\mu_2$

$$f(iz) = \kappa_0 e^{\beta_1 z} \left[ i e^{\mu_2 z} \sin((\gamma - \mu_1)z) + e^{-\mu_2 z} \cos((\gamma + \mu_1)z) \right].$$
Assume $\mu_2 > 0$, then $f(i z) \sim \kappa_0 e^{(\beta_1 + \mu_2) z} \sin((\gamma - \mu_1)z)$ as $z \to +\infty$, hence $\beta_1 + \mu_2$ is determined by $f$, but by looking at the asymptotics as $z \to -\infty$ we see that also $\beta_1 - \mu_2$ is determined, hence so is $\beta_1$. The case $\mu_2 \leq 0$ is done analogously.

Thus $\lambda_j = 2i/\beta_j \in i\mathbb{R}$ and $k$ is given by (9.3), then $\kappa_-$ is determined and we can find

$$k(z) = \frac{\alpha_1 k_1^r(\mu_1 z) e^{i\beta_1 z} + \alpha_2 k_1^s(\mu_2 z) e^{i\beta_2 z}}{i \sin(\beta_1 - \beta_2) z}, \quad r, s \in \{1, 2, 3\},$$

(9.4)

where all the constants are real, and $k_1^r$ is the derivative of function $k_r$ defined in (8.20). Moreover because $k$ is smooth at 0, we must have $\alpha_2 = -\alpha_1$. The denominator of the above function vanishes at $z = \frac{2m}{n}$ with $m \in \mathbb{Z}$, since $k$ is smooth in $[-2, 2]$ we should require

$$(-1)^m k_1^r \left( \frac{2m}{n} \right) - k_1^s \left( \frac{2m}{n} \right) = 0, \quad \forall m \in \mathbb{Z}, \text{ s.t. } \frac{m}{n} \in [-1, 1].$$

Because $n \neq 0$, this condition should hold at least for $m = 1$. One can easily check that this implies that the functions given by (9.4) are either zero, or trigonometric polynomials, and therefore: trivial.

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## 10 Appendix

Here we prove Lemma 8, stating that if the functions $\alpha, \beta, \sigma$ contain an exponential term, the polynomial multiplying it must be a constant. So let us concentrate on a typical exponential term in $\alpha$, $\beta$ and $\sigma$, namely

$$\alpha \leftrightarrow e^{\lambda y} \sum_{j=0}^{2} a_j y^j, \quad \beta \leftrightarrow e^{\lambda y} \sum_{j=0}^{3} b_j y^j, \quad \sigma \leftrightarrow e^{\lambda y} \sum_{j=0}^{3} c_j y^j.$$

The goal is to show that all the coefficients vanish, except possibly for $a_0, b_0, c_0$. We are going to substitute these expressions into (R1). The result becomes a linear combination of terms $y^j e^{\lambda y}$, hence the coefficient of each such terms must vanish. Below we analyze these coefficients, which are in fact ODEs for $k$.

1. First let us show that the polynomials in $\beta$ and $\sigma$ cannot be of higher order, than the polynomial in $\alpha$, i.e. $b_3 = c_3 = 0$. The equations corresponding to $y^3 e^{\lambda y}$ and $y^2 e^{\lambda y}$ are

$$\begin{cases}
 b_3 (e^{\lambda z} - 1) k' + [b_3 \lambda + c_3 (e^{\lambda z} - 1)] k = 0, \\
 3(b_3 k' + c_3 k) e^{\lambda z} + (a_2 k'' + b_2 k' + c_2 k) e^{\lambda z} + (2\lambda a_2 - b_2) k' - a_2 k'' - [\lambda^2 a_2 - b_2 \lambda + c_2 - 3b_3] k = 0.
\end{cases}$$

(10.1)

Assume $b_3 \neq 0$, from the first equation $k(z) = e^{\left(\lambda - \frac{a_2}{b_3}\right) z}/(e^{\lambda z} - 1)$. Invoking Remark 3 w.l.o.g. we assume $c_3 = \lambda b_3$ in which case $k(z) = 1/(e^{\lambda z} - 1)$. Substitute this into the second equation and multiplying the result by $(e^{\lambda z} - 1)^2$ we obtain

$$(a_2 \lambda^2 - b_2 \lambda + c_2) e^{2\lambda z} + (2b_2 \lambda - 2a_2 \lambda^2 + 3b_3 - 2c_2) e^{\lambda z} - 3b_3 \lambda e^{\lambda z} + a_2 \lambda^2 - b_2 \lambda + c_2 - 3b_3 = 0.$$\

The functions $e^{2\lambda z}, e^{\lambda z}, ze^{\lambda z}$ and 1 are linearly independent, hence the coefficient of each one must vanish. But we see that the coefficient of $ze^{\lambda z}$ is $3b_3 \lambda \neq 0$, which is a contradiction. Thus, $b_3 = 0$ and therefore also $c_3 = 0$.

2. We now show that $a_2 = 0$. The equations corresponding to $y^2 e^{\lambda y}$ and $ye^{\lambda y}$ are

$$\begin{align*}
 a_2 (e^{\lambda z} - 1) k'' + [2a_2 \lambda + b_2 (e^{\lambda z} - 1)] k' + [b_2 \lambda - a_2 \lambda^2 + c_2 (e^{\lambda z} - 1)] k = 0, \\
 2(a_2 k'' + b_2 k' + c_2 k) e^{\lambda z} z + (a_1 k'' + b_1 k' + c_1 k) e^{\lambda z} + (2\lambda a_1 + 4a_2 - b_1) k' - a_1 k'' - [\lambda^2 a_1 + (4a_2 - b_1) \lambda + c_1 - 2b_2] k = 0.
\end{align*}$$

(10.2)
Assume \( a_2 \neq 0 \), and by normalization let us assume \( a_2 = 1 \). Solving the first equation we get (as was done in (5.5))

\[
k(z) = \frac{e^{(\lambda - \frac{b_2}{2})z}}{e^{\lambda z} - 1} \cdot \begin{cases} \alpha_1 z + \alpha_2, \\ \alpha_1 e^{\mu z} + \alpha_2 e^{-\mu z}, \end{cases} \quad \mu := \sqrt{\frac{b_2^2}{4} - c_2} = 0
\]

Using Remark 3 let us w.l.o.g. assume \( b_2 = 2\lambda \).

Let \( k \) be given by the first formula. Since \( \alpha_2 \neq 0 \) we may normalize it to be one, so \( k(z) = \frac{\alpha_1 z + \alpha_2}{e^{\lambda z} - 1} \) and \( c_2 = \frac{b_2^2}{4} \). Substituting this expression into the second equation of (10.2) and multiplying the result by \((e^{\lambda z} - 1)^3\) we obtain

\[
(p_1 z + p_2)e^{3\lambda z} + \left[2\lambda^2 \alpha_1 z^2 + \left((2 - 3\alpha_1 a_1)\lambda^2 + (3b_1 - 8)\alpha_1 \lambda - 3c_1 a_1\right) z + p_3\right]e^{2\lambda z} + \left(p_4 z^2 + p_5 z + p_6\right)e^{\lambda z} + p_7 z + p_8 = 0,
\]

where \( p_j \) are constants depending on \( a_1, b_1, c_1, \alpha_1, \lambda \) and their particular expressions are not important. From linear independence the coefficient of \( z^2 e^{2\lambda z} \) must vanish, which implies \( \alpha_1 = 0 \), but then the coefficient of \( z e^{2\lambda z} \) becomes \( 2\lambda^2 \neq 0 \), which leads to a contradiction.

Let \( k \) be given by the second formula, then \( c_2 = \frac{b_2^2}{4} - \mu^2 \) and \( \mu \neq 0 \). Substituting \( k \) into the second equation of (10.2) and multiplying the result by \( e^{\mu z}(e^{\lambda z} - 1)^3 \) we obtain

\[
\alpha_1(\mu + \frac{\lambda}{2})z e^{(2\mu + \lambda)z} - \alpha_1(\mu - \frac{\lambda}{2}) z e^{(2\mu - 2\lambda)z} + \alpha_2(\mu + \frac{\lambda}{2}) z e^{2\lambda z} - \alpha_2(\mu - \frac{\lambda}{2}) z e^{\lambda z} =
\]

\[
= q_0 + q_1 e^{\lambda z} + q_2 e^{2\lambda z} + q_3 e^{3\lambda z} + q_4 e^{2\mu z} + q_5 e^{(2\mu + \lambda)z} + q_6 e^{(2\mu + 2\lambda)z} + q_7 e^{(2\mu + 3\lambda)z},
\]

where \( q_j \) are constants whose particular expressions are not important. Note that the functions on LHS of (10.3) are linearly independent from the ones on RHS. If all the exponents on LHS are distinct then the linear independence the coefficient of \( z e^{2\lambda z} \) becomes \( 2\lambda^2 \neq 0 \), which leads to a contradiction.

Let \( k \) be given by the second formula, then \( c_2 = \frac{b_2^2}{4} - \mu^2 \) and \( \mu \neq 0 \). Substituting \( k \) into the second equation of (10.2) and multiplying the result by \( e^{\mu z}(e^{\lambda z} - 1)^3 \) we obtain

\[
\alpha_1(\mu + \frac{\lambda}{2})z e^{(2\mu + \lambda)z} - \alpha_1(\mu - \frac{\lambda}{2}) z e^{(2\mu - 2\lambda)z} + \alpha_2(\mu + \frac{\lambda}{2}) z e^{2\lambda z} - \alpha_2(\mu - \frac{\lambda}{2}) z e^{\lambda z} =
\]

\[
= q_0 + q_1 e^{\lambda z} + q_2 e^{2\lambda z} + q_3 e^{3\lambda z} + q_4 e^{2\mu z} + q_5 e^{(2\mu + \lambda)z} + q_6 e^{(2\mu + 2\lambda)z} + q_7 e^{(2\mu + 3\lambda)z},
\]

where \( q_j \) are constants whose particular expressions are not important. Note that the functions on LHS of (10.3) are linearly independent from the ones on RHS. If all the exponents on LHS are distinct then the coefficients multiplying them must be zero. In particular \( \alpha_1(\mu + \frac{\lambda}{2}) = 0 \) and \( \alpha_1(\mu - \frac{\lambda}{2}) = 0 \), which imply \( \alpha_1 = 0 \). Analogously, \( \alpha_2 = 0 \) leading to \( k = 0 \). Now assume the exponents on LHS of (10.3) are not distinct, then there are two possibilities:

a) \( 2\mu + \lambda = 2\lambda \), hence \( \lambda = 2\mu \) and LHS of (10.3) becomes \( 2\mu(\alpha_1 + \alpha_2) z e^{4\mu z} \). Hence \( \alpha_1 = -\alpha_2 \), which then implies

\[
k(z) = \frac{2\alpha_1 \sinh(\mu z)}{e^{\lambda z} - 1}.
\]

This contradicts to the assumption that \( k \) has a simple pole at the origin.

b) \( 2\mu + 2\lambda = \lambda \), hence \( \lambda = -2\mu \). Similarly, this case also leads to a contradiction.

3. To show \( b_2 = c_2 = 0 \), we can apply the same argument of 1, because once we established \( a_2 = 0 \) the equations in (10.2) are exactly the ones in (10.1), the only difference is that in the latter we need to replace \( b_1, c_1 \) by \( \frac{b_1}{2}, \frac{c_1}{2} \), and \( a_2, b_2, c_2 \) by \( a_1, b_1, c_1 \) respectively. After this, in an analogous way to 2, we show that \( a_1 = 0 \), again the equations corresponding to \( y e^{\lambda y} \) and \( e^{\lambda y} \) are exactly the ones in (10.2) only \( a_2, b_2, c_2 \) need to be replaced by \( a_1, \frac{b_1}{2}, \frac{c_1}{2} \) and \( a_1, b_1, c_1 \) by \( a_0, b_0, c_0 \) respectively. Finally, again as in 1, we establish that also \( b_1 = c_1 = 0 \).

References
