Explicit power laws in analytic continuation problems via reproducing kernel Hilbert spaces.

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Abstract

The need for analytic continuation arises frequently in the context of inverse problems. Notwithstanding the uniqueness theorems, such problems are notoriously ill-posed without additional regularizing constraints. We consider several analytic continuation problems with typical global boundedness constraints that restore well-posedness. We show that all such problems exhibit a power law precision deterioration as one moves away from the source of data. In this paper we demonstrate the effectiveness of our general Hilbert space-based approach for determining these exponents. The method identifies the "worst case" function as a solution of a linear integral equation of Fredholm type. In special geometries, such as the circular annulus or upper half-plane this equation can be solved explicitly. The obtained solution in the annulus is then used to determine the exact power law exponent for the analytic continuation from an interval between the foci of an ellipse to an arbitrary point inside the ellipse. Our formulas are consistent with results obtained in prior work in those special cases when such exponents have been determined.

1 Introduction

Many inverse problems reduce to analytic continuation questions when solutions of direct problems are known to possess analyticity in a domain in the complex plane but can be measured only on a subset (often a part of the boundary) of this domain. For example, if one wants to recover a signal corrupted by a low-pass convolution filter, then one needs to recover an entire function from its measured values on an interval [11, 2]. Another large class of inverse problems can be termed "Dehomogenization" [7, 26], where one wants to reconstruct some details of microgeometry from measurements of effective properties of the composite. The idea of reconstruction is based on the analytic properties of effective moduli [3, 24, 17] of composites. See e.g. [25] for an extensive bibliography in this area.

The method of recovery via analytic continuation is a tempting proposition in view of the uniqueness properties of analytic functions. Unfortunately, analyticity is a local property "stored" at an infinite depth within the continuum of function values and can be represented by delicate cancellation properties responsible for the validity of Carleman and Carleman type extrapolation formulas [6, 18, 1]. Adding small errors to the exact values of analytic functions destroys these local properties. Instead we want to accumulate the remnants of analyticity and use global properties of analytic functions to achieve analytic continuation. This is only possible under some additional regularizing constraints, such as global boundedness [12, 5, 33, 16, 35]. Taking this idea to the extreme, any bounded entire function is a constant by Liouville's theorem, so that the effect of boundedness depends strongly on the geometry of the domain of analyticity.

In order to quantify the degree to which analytic continuation is possible, consider an analytic function F in a domain Ω. Assume that F is measured on a curve $\Gamma \Subset \Omega$ with a relative error ϵ, with respect to some norm $\|F\|_\Gamma$. Can one perform an analytic continuation of F from Γ to Ω in the presence of measurement errors? Without discussing specific analytic continuation algorithms we would like to examine theoretical feasibility of such a procedure. For example, if two different algorithms are deployed matching F on Γ with relative precision ϵ how far their outputs could possibly differ at a given point $z \in \Omega \setminus \Gamma$? To answer this question we consider the difference f of the two purported analytic continuations. Such a difference will be small on Γ, and we want to quantify how large such a function can possibly be at some point $z \in \Omega$ relative to its global size on Ω.

Based on established upper and lower bounds, exact and numerical results [9, 5, 8, 23, 30, 14, 36, 15, 10, 35] a general *power law principle* emerges, whereby the relative precision of analytic continuation decays as power law $\epsilon^{\gamma(z)}$, where the exponent $\gamma(z) \in (0, 1)$ decreases to 0, as we move further away from the source of data. How fast $\gamma(z)$ decays depends strongly on the geometry of the domain and the data source [35, 20]. In [20] we considered an example, where Ω is the complex upper half-plane and Γ is the interval $[-1, 1]$ on the real axis. We have proved that for z in the upper half-plane $\gamma(z)$ is the angular size of the interval $[-1, 1]$ as viewed from z, measured in units of π. Conformal mappings can also be used to relate the exponents for one geometry to the exponents for the conformally equivalent ones. We believe that such power law transition from well-posedness to practical ill-posedness is a general property of analytic continuation, quantifying the tug-of-war between their rigidity (unique continuation property) and flexibility (as in the Riesz density theorem [29]).

The lower bounds on $\gamma(z)$ can be obtained by exhibiting bounded analytic functions that are small on a curve Γ, but not quite as small at a particular extrapolation point. The upper bounds are harder to prove but there is ample literature where such results are achieved [9, 5, 8, 23, 30, 14, 36, 15, 10, 35]. In fact, it was observed in [35] that upper and lower bounds of the form $\epsilon^{\gamma(z)}$ on the extrapolation error do hold for all geometries. However, with few exceptions the upper and lower bounds do not match. In those examples where they do match [10, 35] the optimality of the bounds are concluded a posteriori.

In our recent work [20] we have developed a new method for characterizing analytic functions in the upper half-plane $\mathbb{H}_+$ attaining the optimal upper bound in terms of a solution of an integral equation of the second kind with compact, positive, self-adjoint operator on $L^2(\Gamma)$. In Section 3.1, we extend this result to reproducing kernel Hilbert spaces $\mathcal{H} = \mathcal{H}(\Omega)$ of analytic functions in a domain $\Omega \subset \mathbb{C}$. The error maximization problem is reformulated as a maximization of a linear objective functional subject to quadratic constraints, permitting us to use convex duality methods. The optimality conditions take the form of a linear integral equation of Fredholm type, where the positive, compact self-adjoint operator $\mathcal{K}$ is expressed in terms of the reproducing kernel of $\mathcal{H}(\Omega)$. The integral operator $\mathcal{K}$ occurs frequently in the context of reproducing kernel Hilbert spaces (e.g. [9]) and is related to the restriction operator $\mathcal{R} : \mathcal{H} \to L^2(\Gamma)$. Namely, $\mathcal{K} = \mathcal{R}^*\mathcal{R}$. The exponent $\gamma(z)$ in the power law asymptotics can then be expressed in terms of the rates of exponential decay of eigenvalues of the integral operator $\mathcal{K}$ and its eigenfunctions at the extrapolation point $z \in \Omega$. For certain classes of restriction operators the exponential decay of the eigenvalues of $\mathcal{K}$ has been known for a long time, and their exact asymptotics has been established in [28] (see also [37, 27, 21, 31]). Alternatively, the exponent $\gamma(z)$ can be read off the explicit solution of the integral equation in cases where such an explicit solution is available [20]. This allows us to compute $\gamma(z)$ explicitly in a number of special cases. For example, when Γ is a circle in the upper half-plane (Section 2.2) or a circle in an annulus (Section 2.1).

In Section 4.3 we present a somewhat unexpected application of the annulus result to the problem of analytic continuation in a Bernstein ellipse [4], studied in [10]. Since the annulus is not conformally equivalent to the ellipse one would not expect a direct relation. The trick we use, inspired by [10], is to map the Bernstein ellipse cut along $[-1, 1]$ onto the annulus using the inverse of the Joukowski function. Then, functions analytic in the ellipse are distinguished from functions analytic in the cut ellipse by their continuity across the cut. After the conformal transformation the image of functions analytic in the entire ellipse would consist of functions analytic in the annulus with a reflection symmetry on the unit circle. Our Hilbert space-based approach can easily incorporate linear constraints by making an appropriate choice of the underlying Hilbert space. However, the question is about the relation between the problems with and without such constraints. In the case of the Bernstein ellipse and the annulus, we discover that the subspace of functions analytic in the annulus corresponding to functions analytic in the Bernstein ellipse is invariant with respect to the integral operator $\mathcal{K}$. It is this invariance that permits us to solve the problem with additional linear constraints using the known solution of the original problem. This is discussed in Section 3.4. When the extrapolation point z lies on the real line inside the Bernstein ellipse we recover the optimal exponent $\gamma(z)$ obtained in [10]. However, our approach also gives the formula for the exponent $\gamma(z)$ for arbitrary points z inside the ellipse.

2 Main Results

Notation: We will write $A \lesssim B$, if there exists a constant c such that $A \leq cB$ and likewise the notation $A \gtrsim B$ will be used. If both $A \lesssim B$ and $A \gtrsim B$ are satisfied, then we will write $A \simeq B$. Throughout the paper all the implicit constants will be independent of the parameter ϵ.

2.1 The annulus

For $0 < \rho < 1$, $r > 0$ let

$$A_\rho = \{\zeta \in \mathbb{C} : \rho < |\zeta| < 1\}, \qquad \Gamma_r = \{\zeta \in \mathbb{C} : |\zeta| = r\}. \tag{2.1}$$

Consider the Hardy space (e.g. [13])

$$H^2(A_\rho) = \{f \text{ is analytic in } A_\rho : \|f\|_{H^2(A_\rho)} = \sup_{\rho<r<1} \|f\|_{L^2(\Gamma_r)} < \infty\}, \tag{2.2}$$

where for a curve $\Gamma \subset \mathbb{C}$ the space $L^2(\Gamma)$ denotes the space of square-integrable functions on Γ with respect to the arc length measure $|d\tau|$ on Γ.

THEOREM **2.1** (Annulus). *Let $\Gamma = \Gamma_r$ with $r \in (\rho, 1)$ fixed and $z \in A_\rho \backslash \Gamma$. Then there exists $C > 0$, such that for any $\epsilon > 0$ and any $f \in H^2(A_\rho)$ with $\|f\|_{H^2(A_\rho)} \leq 1$ and $\|f\|_{L^2(\Gamma)} \leq \epsilon$, we have*

$$|f(z)| \leq C\epsilon^{\gamma(z)}, \tag{2.3}$$

where

$$\gamma(z) = \begin{cases} \dfrac{\ln|z|}{\ln r}, & \text{if } r < |z| < 1 \\ \dfrac{\ln(|z|/\rho)}{\ln(r/\rho)}, & \text{if } \rho < |z| < r \end{cases} \tag{2.4}$$

Moreover, (2.3) *is asymptotically optimal in ϵ and the function attaining the bound is*

$$M(\zeta) = \epsilon^{2-\gamma(z)} \sum_{n\in\mathbb{Z}} \frac{(\overline{z}\zeta)^n}{r^{2n} + \epsilon^2(1+\rho^{2n})}, \qquad \zeta \in A_\rho. \tag{2.5}$$

In addition M is analytic in the closure of A_ρ and $\|M\|_{H^\infty(\overline{A_\rho})}$ is bounded uniformly in ϵ.

Remark 2.2. *The statement that M attains the bound in* (2.3) *means that $\|M\|_{H^2(A_\rho)} \lesssim 1$, $\|M\|_{L^2(\Gamma)} \lesssim \epsilon$ and $|M(z)| \simeq \epsilon^{\gamma(z)}$, with all implicit constants independent of ϵ.*

It is somewhat surprising that the worst case function, which was required to be analytic only in A_ρ is in fact analytic in a larger annulus $\{|z_\rho^*| < |\zeta| < |z_1^*|\}$, where $z_1^* = 1/\overline{z}$ is the point symmetric to z w.r.t the circle Γ_1 and $z_\rho^* = \rho^2/\overline{z}$ is the point symmetric to z w.r.t the circle Γ_ρ. In particular, $M \in H^\infty(A_\rho)$. Hence, $M(\zeta)$ also maximizes $|M(z)|$, asymptotically, as $\epsilon \to 0$, if the constraints were given in $H^\infty(A_\rho)$ and $L^\infty(\Gamma)$, instead of $H^2(A_\rho)$ and $L^2(\Gamma)$, respectively.

Remark 2.3. *The limiting case as $\rho \to 0$ corresponds to the analytic continuation from the circle Γ_r into the unit disk D. The limiting value of the exponent is $\gamma(z) = \frac{\ln|z|}{\ln r}$ for $|z| > r$, and $\gamma(z) = 1$, for $|z| < r$. The numerical stability of extrapolation inside Γ_r can be seen directly from Cauchy's integral formula. The same formula for $\gamma(z)$ has been obtained in [35] for $H^\infty(D)$.*

2.2 The upper half-plane

Let $\mathbb{H}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ denote the complex upper half-plane and consider the Hardy space

$$H^2(\mathbb{H}_+) := \{f \text{ is analytic in } \mathbb{H}_+ : \sup_{y>0} \|f(\cdot + iy)\|_{L^2(\mathbb{R})} < \infty\}.$$

It is well known [22] that these functions have L^2-boundary data, and that $\|f\| = \|f\|_{L^2(\mathbb{R})}$ defines a norm in $H^2(\mathbb{H}_+)$. Assume that the data curve $\Gamma \Subset \mathbb{H}_+$ is a circle. By considering affine automorphisms $z \mapsto az + b$, $a > 0$, $b \in \mathbb{R}$, of $\mathbb{H}_+$ we may "translate" Γ to be centered at i.

THEOREM **2.4.** *Let Γ be a circle centered at i of radius $r < 1$. Let $z \in \mathbb{H}_+$ be a point outside of Γ. Then there exists $C > 0$, such that for any $\epsilon > 0$ and any $f \in H^2(\mathbb{H}_+)$ with $\|f\|_{H^2(\mathbb{H}_+)} \leq 1$ and $\|f\|_{L^2(\Gamma)} \leq \epsilon$, we have*

$$|f(z)| \leq C\epsilon^{\gamma(z)}, \tag{2.6}$$

where

$$\gamma(z) = \frac{\ln |m(z)|}{\ln \rho}, \qquad \rho = \frac{1 - \sqrt{1 - r^2}}{r}, \tag{2.7}$$

and

$$m(\zeta) = \frac{\zeta - z_0}{\zeta + z_0}, \qquad z_0 = i\sqrt{1 - r^2}$$

is the Möbius map transforming the upper half-plane into the unit disc and the circle Γ into a concentric circle, whose radius has to be ρ. Moreover, (2.6) *is asymptotically optimal in ϵ and the function attaining the bound can be written as a convergent in the upper half-plane "power" series*

$$M(\zeta) = \frac{\epsilon^{2-\gamma(z)}}{\zeta + z_0} \sum_{n=1}^{\infty} \frac{\left(\overline{m(z)} m(\zeta)\right)^n}{\epsilon^2 + \rho^{2n}}, \qquad \zeta \in \mathbb{H}_+. \tag{2.8}$$

Remark 2.5. *When z is inside Γ we have complete stability, indeed Cauchy's integral formula implies that*

$$|f(z)| \leq c\epsilon$$

for a constant c independent on ϵ.

2.3 The Bernstein ellipse

Let E_R be the open ellipse with foci at ± 1 and the sum of semi-minor and semi-major axes equal to $R > 1$. The axes lengths of such an ellipse are therefore $(R \pm R^{-1})/2$. E_R is called the Bernstein ellipse [4, 34]. Its boundary is an image of a circle of radius R centered at the origin under the Joukowski map $J(z) = (z + z^{-1})/2$. Let $H^\infty(E_R)$ be the space of bounded analytic functions in E_R, with the usual supremum norm.

THEOREM **2.6.** *Let $z \in E_R \backslash [-1, 1]$. Then there exists $C > 0$, such that for every $\epsilon > 0$ and $F \in H^\infty(E_R)$ with $\|F\|_{H^\infty(E_R)} \leq 1$ and $\|F\|_{L^\infty(-1,1)} \leq \epsilon$, we have*

$$|F(z)| \leq C\epsilon^{\alpha(z)}, \tag{2.9}$$

where

$$\alpha(z) = 1 - \frac{\ln|J^{-1}(z)|}{\ln R} \in (0,1), \qquad J^{-1}(z) = z + (z-1)\sqrt{\frac{z+1}{z-1}} \tag{2.10}$$

Moreover, (2.9) *is asymptotically optimal in* ϵ *and function attaining the bound is*

$$M(\zeta) = \epsilon^{2-\alpha(z)} \sum_{n=1}^{\infty} \frac{(\overline{J^{-1}(z)})^n T_n(\zeta)}{1+\epsilon^2 R^{2n}}, \tag{2.11}$$

where T_n *is the Chebyshev polynomial of degree* n*:* $T_n(x) = \cos(n \cos^{-1} x)$ *for* $x \in [-1,1]$*.*

Several remarks are now in order.

(i) $J^{-1}(\zeta)$ is the branch of an inverse of the Joukowski map J, that is analytic in the slit ellipse $E_R \backslash [-1,1]$ and satisfies the inequalities $1 < |J^{-1}(\zeta)| < R$.

(ii) Chebyshev polynomials T_n play the same role in the ellipse as monomials ζ^n play in the annulus, i.e. they are the building blocks of analytic functions. In fact $J^{-1} \circ T_n \circ J = \zeta^n$.

(iii) The same bound (2.9) was obtained in [10] when $z \in E_R \cap \mathbb{R}$, where it was shown that the bound (up to logarithmic factors) could be attained by a polynomial

$$g(\zeta) = \epsilon T_{K(\epsilon)}(\zeta), \qquad K = K(\epsilon) = \lfloor \ln(1/\epsilon)/\ln R \rfloor. \tag{2.12}$$

We observe that the terms in (2.11) increase exponentially fast from $n = 1$ to $n = K(\epsilon)$ and then decrease exponentially fast for $n > K(\epsilon)$. Hence, asymptotically (up to logarithmic factors) we can say that

$$|M(\zeta)| \approx \epsilon^{2-\alpha(z)} \frac{|J^{-1}(z)|^{K(\epsilon)} |T_{K(\epsilon)}(\zeta)|}{1+\epsilon^2 R^{2K(\epsilon)}} \approx \epsilon |T_{K(\epsilon)}(\zeta)|,$$

in agreement with (2.12).

3 Quantifying stability of analytic continuation

3.1 Reproducing kernel Hilbert spaces

Our goal is to characterize how large a function f analytic in a domain Ω can be at a point $z \in \Omega$, provided that it is small on a curve $\Gamma \Subset \Omega$, relative to its global size in Ω. If some norms $\|f\|_\Gamma$ and $\|f\|_{\mathcal{H}}$ are used to measure the magnitude of f on Γ and on Ω, respectively, then we are looking at the problem

$$\begin{cases} |f(z)| \to \max \\ \|f\|_{\mathcal{H}} \le 1 \\ \|f\|_\Gamma \le \epsilon \end{cases} \tag{3.1}$$

Assume that the global norm is induced by an inner product $(\cdot,\cdot)$ and that the point evaluation functional $f \mapsto f(z)$ is continuous (for any point $z \in \Omega$), then by the Riesz representation theorem, there exists an element $p_z \in \mathcal{H}$ such that $f(z) = (f, p_z)$. Now inner products with the function $p(\zeta, z) := p_z(\zeta)$ reproduce values of a function in $\mathcal{H}$. In this case $\mathcal{H}$ is called a a reproducing kernel Hilbert space (RKHS) with kernel p. Examples of such spaces include the Hardy spaces H^2 over unit disk, annulus or upper half-plane. From now on we will drop the subscript $\mathcal{H}$ for the Hilbert space norm in $\mathcal{H}$.

Lemma **3.1.** *Suppose that $\mathcal{H}$ is a RKHS whose elements are continuous functions on a metric space Ω. Then the function $\Omega \ni \tau \mapsto \|p_\tau\|$ is bounded on compact subsets of Ω.*

Proof. Assume the contrary. Suppose $S \subset \Omega$ is compact, but there exists a sequence $\{\tau_k\}_{k=1}^{\infty} \subset S$, such that $\|p_{\tau_k}\| \to \infty$ as $k \to \infty$. Since S is compact we can extract a convergent subsequence (without relabeling it) $\tau_k \to \tau_*$, then for any $f \in \mathcal{H}$ we have $f(\tau_k) = (f, p_{\tau_k}) \to f(\tau_*) = (f, p_{\tau_*})$, by continuity of f. Thus, $p_{\tau_k} \rightharpoonup p_{\tau_*}$ in $\mathcal{H}$, but this implies boundedness of $\|p_{\tau_k}\|$, leading to a contradiction. □

Corollary 3.2. *Under the assumption of Lemma 3.1 the function $p(\zeta, \tau)$ is bounded on compact subsets of $\Omega \times \Omega$, since $|p(\zeta, \tau)| = |(p_\tau, p_\zeta)| \le \|p_\tau\|\|p_\zeta\|$.*

Assume that the smallness on Γ is measured in $L^2 := L^2(\Gamma, |\mathrm{d}\tau|)$-norm (where $|\mathrm{d}\tau|$ is the arc length measure). Then, there is a constant $c > 0$ such that

$$\|f\|_\Gamma \le c\|f\|, \qquad \forall f \in \mathcal{H}. \tag{3.2}$$

Indeed, for all $\tau \in \Gamma$ we have $|f(\tau)| = |(f, p_\tau)| \le \|p_\tau\|\|f\|$. Since Γ lies in a compact subset of Ω and has finite length we conclude by Lemma 3.1 that (3.2) holds.

In order to analyze problem (3.1) we consider a Hermitian symmetric form

$$B : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \qquad B(f, g) = (f, g)_\Gamma.$$

By (3.2) $B(f, g)$ is continuous, and thus there exists a positive, self-adjoint and bounded operator $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ with $B(f, g) = (\mathcal{K}f, g)$. Moreover we can write an explicit formula for $\mathcal{K}$ in terms of the kernel p:

$$(\mathcal{K}f, g) = (f, g)_\Gamma = \int_\Gamma f(\tau)(p_\tau, g)|\mathrm{d}\tau| = \left(\int_\Gamma f(\tau)p_\tau|\mathrm{d}\tau|, g\right). \tag{3.3}$$

Thus, for every $f \in \mathcal{H}$

$$(\mathcal{K}f)(\zeta) = \int_\Gamma p(\zeta, \tau) f(\tau)|\mathrm{d}\tau|, \qquad \zeta \in \Omega. \tag{3.4}$$

This formula permits to define a new operator $\mathcal{K} : L^2(\Gamma) \to \mathcal{H}$. However, in doing so we may lose injectivity, which underlies uniqueness of analytic continuation[1]. Therefore, we restrict the domain of $\mathcal{K}$ to a closed subspace of $L^2(\Gamma)$

$$\mathcal{W} = \mathrm{cl}_{L^2}\left(\mathcal{H}|_\Gamma\right) \subset L^2(\Gamma). \tag{3.5}$$

[1]It is this property that forces us to restrict attention to reproducing kernel Hilbert spaces of analytic functions.

In fact, in many cases $\mathcal{W} = L^2(\Gamma)$. The density in the context of Hardy spaces is known as the Riesz theorem (see e.g. [29]). If Ω is bounded it is usually proved using density of polynomials in $L^2(\Gamma)$, which always holds if all polynomials are in $\mathcal{H}$ (and Γ is not a closed curve).

We note that the operator $\mathcal{K} : \mathcal{W} \to \mathcal{H}$ is bounded. Indeed, by Corollary 3.2 the function $\Gamma \ni \tau \mapsto p(\zeta, \tau)$ is bounded for each $\zeta \in \Omega$ and by (3.3) we have

$$\|\mathcal{K}f\|^2 = (\mathcal{K}f, \mathcal{K}f) = (f, \mathcal{K}f)_\Gamma \leq \|\mathcal{K}f\|_\Gamma \|f\|_\Gamma \leq c\|\mathcal{K}f\| \|f\|_\Gamma, \tag{3.6}$$

where we have used (3.2) in the last inequality. It follows that $\|\mathcal{K}f\| \leq c\|f\|_\Gamma$.

The outcome of our constructions is the ability to write the two inequalities in (3.1) as quadratic constraints for $f \in \mathcal{H}$:

$$(f, f) \leq 1, \qquad (\mathcal{K}f, f) \leq \epsilon^2. \tag{3.7}$$

The final observation is that the objective functional $|f(z)|$ in (3.1) can be replaced by a (real) linear functional $\Re(f, p_z)$. Indeed,

$$|f(z)| = \sup_{|\lambda|=1} \Re(\lambda f(z)) = \sup_{|\lambda|=1} \Re(\lambda f, p_z).$$

It remains to notice that if f satisfies (3.7) then so does λf for every $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Thus we arrive at the problem

$$\begin{cases} \Re(f, p_z) \to \max \\ (f, f) \leq 1 \\ (\mathcal{K}f, f) \leq \epsilon^2 \end{cases} \tag{3.8}$$

Lemma **3.3.** *The operator $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ is compact, positive definite and self-adjoint.*

Proof. Self-adjointness and positivity of $\mathcal{K}$ on $\mathcal{H}$ are immediate consequences of (3.3). To prove compactness, let $\{f_k\}_{k=1}^\infty \subset \mathcal{H}$ be a bounded sequence. Extract a weakly convergent subsequence (without relabeling it) $f_k \rightharpoonup f$. Then for every $\tau \in \Omega$ we have $f_k(\tau) = (f_k, p_\tau) \to (f, p_\tau) = f(\tau)$. In addition, for every $\tau \in \Gamma$ we have $|f_k(\tau)| = |(f_k, p_\tau)| \leq \|f_k\| \|p_\tau\|$. The sequence $\|f_k\|$ is bounded, since f_k is weakly convergent, while $\|p_\tau\|$ is bounded on Γ by Lemma 3.1. Thus, $f_k(\tau)$ is uniformly bounded on Γ. Then $f_k|_\Gamma \to f|_\Gamma$ in the L^2 norm. But then by the estimate $\|\mathcal{K}(f_k - f)\| \leq c\|f_k - f\|_\Gamma$ (see (3.6)) we conclude that $\mathcal{K}f_k \to \mathcal{K}f$ in $\mathcal{H}$.

□

Theorem **3.4.** *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS of functions analytic in domain Ω, with kernel p and norm $\|\cdot\|$. Let $\Gamma \Subset \Omega$ be a rectifiable curve of finite length and $\|\cdot\|_\Gamma$ be the $L^2 := L^2(\Gamma, |d\tau|)$ norm. Fix a point $z \in \Omega \backslash cl(\Gamma)$ and assume $f \in \mathcal{H}$ with $\|f\| \leq 1$ and $\|f\|_\Gamma \leq \epsilon$, then*

$$|f(z)| \leq \frac{3}{2} u_{\epsilon,z}(z) \min\left\{\frac{1}{\|u_{\epsilon,z}\|}, \frac{\epsilon}{\|u_{\epsilon,z}\|_\Gamma}\right\} \tag{3.9}$$

where $u_{\epsilon,z} \in \mathcal{H}$ *is the unique solution of*

$$\mathcal{K}u + \epsilon^2 u = p_z. \tag{3.10}$$

Moreover, (3.9) *is optimal since it is attained (up to the factor* $3/2$*) by*

$$M_{\epsilon,z}(\zeta) = u_{\epsilon,z}(\zeta) \min\left\{ \frac{1}{\|u_{\epsilon,z}\|}, \frac{\epsilon}{\|u_{\epsilon,z}\|_\Gamma} \right\}. \tag{3.11}$$

Before we prove this theorem several remarks need to be made.

1. An obvious thing to do is to set $\epsilon = 0$ in (3.10). If $p_z \in \mathcal{K}(\mathcal{W})$, where $\mathcal{W}$ is given by (3.5), then $u_{\epsilon,z} \to u_0 = \mathcal{K}^{-1}p_z$, as $\epsilon \to 0$. In which case the upper bound (3.9) is simply

$$|f(z)| \le C\epsilon, \qquad C = \frac{3u_0(z)}{2\|u_0\|_\Gamma}. \tag{3.12}$$

   In other words we have numerically stable analytic continuation. Examples where this happens are mentioned in Remarks 2.3 and 2.5. This case will be referred to as the trivial case.

2. The function on the right-hand side of (3.11) is obviously in $\mathcal{H}$ and obviously satisfies the constraints in (3.1). Hence, the attainability of the bound (3.9) is trivial. Only the bound itself requires a proof.

3. The upper bound (3.9) is not an explicit function of ϵ and z. Its asymptotics as $\epsilon \to 0$ depends on fine properties of the operator $\mathcal{K}$. This will be discussed in Section 3.3. In specific examples in Section 4 equation (3.10) is solved explicitly and the power law behavior $M_{\epsilon,z}(z) \sim \epsilon^{\gamma(z)}$ is exhibited.

4. The precise asymptotics of the exponential decay of eigenvalues of $\mathcal{K}$ is known for certain classes of spaces. For example, assume $\mathcal{H}$ coincides with the Smirnov class $E^2(\Omega)$ [13]. If the domain Ω is bounded and simply connected and $\Gamma \Subset \Omega$ is a closed Jordan rectifiable curve of class $C^{1+\epsilon}$ for $\epsilon > 0$, with Ω' denoting the domain bounded by it, then the eigenvalues of $\mathcal{K}$ satisfy the asymptotic relation [28]

$$\lambda_n(\mathcal{K}) \sim \rho^{2n+1}, \qquad \text{as } n \to +\infty, \tag{3.13}$$

   where $\rho < 1$ is the Riemann invariant, whereby the domain $\Omega \backslash cl(\Omega')$ is conformally equivalent to the annulus $\{\omega \in \mathbb{C} : \rho < |\omega| < 1\}$.

The proof of Theorem 3.4 in the more general context of RKHS follows without much change from the proof of the same theorem for the Hardy space H^2 of analytic functions in the upper half-plane given in [20]. For the sake of completeness we give a short recap of the argument.

3.2 Proof of Theorem 3.4

We start by analyzing the trivial case.

Lemma **3.5.** *Assume the setting of Theorem 3.4, let* $p_z \in \mathcal{K}(\mathcal{W})$*, then*

$$|f(z)| \leq c\epsilon.$$

Proof. Let $v \in \mathcal{W} \subset L^2$ satisfy $\mathcal{K}v = p_z$, (note that v does not depend on ϵ), then using (3.3) we have

$$f(z) = (f, p_z) = (f, \mathcal{K}v) = (f, v)_\Gamma.$$

It remains to use the Cauchy-Schwartz inequality to conclude the desired inequality with $c = \|v\|_\Gamma$.

□

Let us now turn to the case $p_z \notin \mathcal{K}(\mathcal{W})$. For every f, satisfying (3.7) and for every nonnegative numbers μ and ν ($\mu^2 + \nu^2 \neq 0$) we have the inequality

$$((\mu + \nu\mathcal{K})f, f) \leq \mu + \nu\epsilon^2. \tag{3.14}$$

Applying convex duality to the quadratic functional on the left-hand side of (3.7) we get

$$\Re(f, p_z) - \frac{1}{2}\left((\mu + \nu\mathcal{K})^{-1}p_z, p_z\right) \leq \frac{1}{2}((\mu + \nu\mathcal{K})f, f) \leq \frac{1}{2}\left(\mu + \nu\epsilon^2\right), \tag{3.15}$$

so that

$$\Re(f, p_z) \leq \frac{1}{2}\left((\mu + \nu\mathcal{K})^{-1}p_z, p_z\right) + \frac{1}{2}\left(\mu + \nu\epsilon^2\right), \tag{3.16}$$

which is valid for every f, satisfying (3.7) and all $\mu > 0$, $\nu \geq 0$. In order for the bound to be optimal we must have equality in (3.15), which holds if and only if

$$p_z = (\mu + \nu\mathcal{K})f,$$

giving the formula for optimal vector f:

$$f = (\mu + \nu\mathcal{K})^{-1}p_z. \tag{3.17}$$

The goal is to choose the Lagrange multipliers μ and ν so that the constraints in (3.8) are satisfied by f, given by (3.17).

• if $\nu = 0$, then $f = \frac{p_z}{\mu}$ and optimality implies that the first inequality constraint of (3.8) must be attained, i.e. $\|f\| = 1$. Thus, $f = \frac{p_z}{\|p_z\|}$ does not depend on the small parameter ϵ, which leads to a contradiction, because the second constraint $(\mathcal{K}f, f) \leq \epsilon^2$ is violated if ϵ is small enough.

• if $\mu = 0$, then $\mathcal{K}f = \frac{1}{\nu}p_z$. But this equation has no solutions in $\mathcal{H}$ according to the assumption $p_z \notin \mathcal{K}(\mathcal{W})$.

Thus we are looking for $\mu > 0$, $\nu > 0$, so that equalities in (3.8) hold. (These are the complementary slackness relations in Karush-Kuhn-Tucker conditions.), i.e.

$$\begin{cases} ((\mu + \nu\mathcal{K})^{-1}p_z, (\mu + \nu\mathcal{K})^{-1}p_z) = 1, \\ (\mathcal{K}(\mu + \nu\mathcal{K})^{-1}p_z, (\mu + \nu\mathcal{K})^{-1}p_z) = \epsilon^2. \end{cases} \tag{3.18}$$

Let $\eta = \frac{\mu}{\nu}$, we can solve either the first or the second equation in (3.18) for ν

$$\nu^2 = \|(\mathcal{K} + \eta)^{-1}p_z\|^2, \tag{3.19}$$

or

$$\nu^2 = \epsilon^{-2}\left(\mathcal{K}(\eta + \mathcal{K})^{-1}p_z, (\eta + \mathcal{K})^{-1}p_z\right). \tag{3.20}$$

The two analysis paths stemming from using one or the other representation for ν lead to two versions of the upper bound on $|f(z)|$, optimality of neither we can prove. However, the minimum of the two upper bounds is still an upper bound and its optimality is then apparent. At first glance both expressions for ν should be equivalent and not lead to different bounds. Indeed, their equivalence can be stated as an equation

$$\Phi(\eta) := \frac{(\mathcal{K}(\mathcal{K} + \eta)^{-1}p_z, (\mathcal{K} + \eta)^{-1}p_z)}{\|(\mathcal{K} + \eta)^{-1}p_z\|^2} = \epsilon^2 \tag{3.21}$$

for η. Equation (3.21) has a unique solution $\eta_* = \eta_*(\epsilon) > 0$, because $\Phi(\eta)$ is monotone increasing (since its derivative can be shown to be positive), $\Phi(+\infty) = (\mathcal{K}p_z, p_z)/\|p_z\|^2$ and $\Phi(0^+) = 0$. (See [20] for technical details.)

In the examples in this paper the eigenvalues and eigenfunctions of $\mathcal{K}$ exhibit exponential decay. We have shown in [20] that such behavior implies that $\eta_*(\epsilon) \simeq \epsilon^2$, as $\epsilon \to 0$. However, *any* choice of η gives two valid upper bounds: one via (3.19), the other, via (3.20). In the anticipation that the exponential decay of eigenvalues and eigenfunctions holds we simply set $\eta = \epsilon^2$ and obtain, setting $u = (\mathcal{K} + \epsilon^2)^{-1}p_z$,

$$\Re(f, p_z) \le \frac{(u, p_z)}{2\|u\|} + \epsilon^2\|u\|, \qquad \Re(f, p_z) \le \frac{\epsilon(u, p_z)}{2\|u\|_\Gamma} + \epsilon\|u\|_\Gamma$$

Definition of u implies $u(z) = (u, p_z) = (u, \mathcal{K}u + \epsilon^2 u) = (u, \mathcal{K}u) + \epsilon^2(u, u)$, i.e.

$$u(z) = \|u\|_\Gamma^2 + \epsilon^2\|u\|^2, \tag{3.22}$$

which implies the inequalities

$$\epsilon^2\|u\| \le \frac{u(z)}{\|u\|}, \qquad \|u\|_{L^2} \le \frac{u(z)}{\|u\|_\Gamma}.$$

Therefore, we have both

$$|f(z)| = \Re(f, p_z) \le \frac{3}{2}\frac{u(z)}{\|u\|}, \qquad |f(z)| \le \frac{3\epsilon}{2}\frac{u(z)}{\|u\|_\Gamma}.$$

Inequality (3.9) is now proved. We remark that a possibly suboptimal choice $\eta = \epsilon^2$ still delivers asymptotically optimal upper bound (3.9), since it is attained by the function (3.11).

3.3 Solving the integral equation

We begin by making several observations about a priori properties of the solution u_ϵ of (3.10) in the non-trivial case $p_z \notin \mathcal{K}(\mathcal{W})$. The most immediate consequence of the non-triviality is that $\|u_\epsilon\|_\Gamma$ blows up as $\epsilon \to 0$. If it did not, we would be able to extract a weakly convergent subsequence $u_{\epsilon_k} \rightharpoonup u_0 \in \mathcal{W}$ and passing to the weak limits in (3.10) obtained that $(\mathcal{K}u_0)(\zeta) = p_z(\zeta)$, for $\zeta \in \Gamma$. However, since $\mathcal{K}(\mathcal{W}) \subset \mathcal{H}$ we get a contradiction with the non-triviality.

Next let us show that equation (3.22) implies that $M_{\epsilon,z}(z) \gg \epsilon$. On the one hand, dividing equation (3.22) by $\|u_\epsilon\|_\Gamma$ we obtain

$$\frac{u_\epsilon(z)}{\|u_\epsilon\|_\Gamma} \geq \|u_\epsilon\|_\Gamma.$$

On the other, we have $\|u_\epsilon\|_\Gamma^2 + \epsilon^2\|u_\epsilon\|^2 \geq 2\epsilon\|u_\epsilon\|_\Gamma\|u_\epsilon\|$ and therefore

$$\frac{u_\epsilon(z)}{\epsilon\|u_\epsilon\|} \geq 2\|u_\epsilon\|_\Gamma,$$

proving that $\epsilon^{-1}M_{\epsilon,z}(z) \geq \|u_\epsilon\|_\Gamma \to +\infty$. This means that one cannot expect full numerical stability of analytic continuation.

Finally, we prove the "mathematical well-posedness" of analytic continuation: $M_{\epsilon,z}(z) \to 0$ as $\epsilon \to 0$. This is a consequence of the weak convergence of $u_\epsilon/\|u_\epsilon\|$ to 0. If we divide (3.10) by $\|u_\epsilon\|$ and pass to weak limits, using the fact that $\|u_\epsilon\| \geq c^{-1}\|u_\epsilon\|_\Gamma \to +\infty$ we obtain that the weak limit $\widehat{u}$ of $u_\epsilon/\|u_\epsilon\|$ satisfies $\mathcal{K}\widehat{u} = 0$. But if $\mathcal{K}\widehat{u} = 0$, then $\|\widehat{u}\|_\Gamma^2 = (\mathcal{K}\widehat{u}, \widehat{u}) = 0$. It follows that the analytic function $\widehat{u} = 0$ on Γ and hence must vanish everywhere in Ω. This shows that the operator $\mathcal{K}$ has a trivial null-space and that $M_{\epsilon,z}(z) = (u_\epsilon/\|u_\epsilon\|, p_z) \to 0$, as $\epsilon \to 0$.

A consequence of the just established strict positivity of $\mathcal{K}$ is separability of the Hilbert space $\mathcal{H}$. This should not be surprising, since $\mathcal{H}$ consists of analytic functions each of which can be completely described by a countable set of numbers.

LEMMA **3.6.** *The Hilbert space $\mathcal{H}$ is always separable.*

Proof. We saw that $\mathcal{K} : \mathcal{H} \to \mathcal{H}$ given by (3.4) is a self-adjoint, compact operator. We have just seen that $\mathcal{K}$ has a trivial null-space. In this case the Hilbert space $\mathcal{H}$ is the orthogonal sum of countable number of finite dimensional eigenspaces of $\mathcal{K}$ with positive eigenvalues. Thus, $\mathcal{H}$ has a countable complete orthonormal set and is therefore separable. □

In applications of our theory in Section 4 we solve equation (3.10) exactly by finding all eigenvalues and eigenfunctions of $\mathcal{K}$. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal eigenbasis of $\mathcal{H}$ with $\mathcal{K}e_n = \lambda_n e_n$. In this basis the equation (3.10) diagonalizes:

$$\lambda_n(u, e_n) + \epsilon^2(u, e_n) = (p_z, e_n),$$

therefore we find

$$u_\epsilon(\zeta) = \sum_n \frac{\overline{e_n(z)}}{\lambda_n + \epsilon^2} e_n(\zeta). \tag{3.23}$$

Using this expansion, formula $\|u\|_\Gamma^2 = (\mathcal{K}u, u)$, and (3.22) we find that

$$u_\epsilon(z) = \sum_n \frac{|e_n(z)|^2}{\lambda_n + \epsilon^2}, \qquad \|u_\epsilon\|^2 = \sum_n \frac{|e_n(z)|^2}{(\lambda_n + \epsilon^2)^2}, \qquad \|u_\epsilon\|_\Gamma^2 = \sum_n \frac{\lambda_n |e_n(z)|^2}{(\lambda_n + \epsilon^2)^2}. \tag{3.24}$$

It follows that

$$\sum_n \frac{|e_n(z)|^2}{\lambda_n} = \infty, \tag{3.25}$$

since if the series had a finite sum then formula (3.24) for $\|u_\epsilon\|_\Gamma$ would imply

$$\|u_\epsilon\|_\Gamma^2 \le \sum_n \frac{|e_n(z)|^2}{\lambda_n},$$

contradicting the blow up of $\|u_\epsilon\|_\Gamma$.

In our examples where the eigenvalues λ_n and eigenfunctions $e_n(\zeta)$ can be found explicitly they are seen to decay exponentially fast to 0 (see also (3.13)). As we have shown in [20] this implies the power law principle

$$M_{\epsilon,z}(z) \simeq \epsilon^{\gamma(z)}, \qquad \text{as} \qquad \epsilon \to 0, \tag{3.26}$$

where $\gamma(z) \in (0, 1)$ can be expressed in terms of the rates of exponential decay of spectral data for $\mathcal{K}$.

THEOREM **3.7.** *Let $\{e_n\}_{n=1}^\infty$ be orthonormal eigenbasis of $\mathcal{H}$ with $\mathcal{K}e_n = \lambda_n e_n$. Let $u = u_{\epsilon,z}$ and $M_{\epsilon,z}$ be given by* (3.10) *and* (3.11) *respectively. Assume*

$$\lambda_n \simeq e^{-\alpha n}, \qquad |e_n(z)|^2 \simeq e^{-\beta n}, \qquad 0 < \beta < \alpha, \tag{3.27}$$

with implicit constants independent of n (so that (3.25) *holds). Then,*

$$\|u_{\epsilon,z}\|_\Gamma \simeq \epsilon \|u_{\epsilon,z}\| \simeq \epsilon^{\frac{\beta}{\alpha} - 1} \qquad \text{and} \qquad u_{\epsilon,z}(z) \simeq \epsilon^{2\left(\frac{\beta}{\alpha} - 1\right)},$$

with implicit constants independent of ϵ. In particular, this implies the power law principle (3.26) with exact exponent:

$$M_{\epsilon,z}(z) \simeq \epsilon^{\frac{\beta}{\alpha}}.$$

The proof of Theorem 3.7 immediately follows from (3.24) and Lemma A.1.

3.4 Linear constraints

In one of our examples we encounter a situation where additional linear constraints are imposed on a previously solved problem. In general all linear constraints on analytic functions will simply be incorporated into the definition of the RKHS $\mathcal{H}$. The question is whether we can use the already found solution of a problem if additional linear constraints are imposed. Let $L \subset \mathcal{H}$ be a closed, $\mathbb{C}$-linear subspace. Then L with the inner product from $\mathcal{H}$ is still a RKHS with the reproducing kernel $\mathcal{P}_L p_z$, where $\mathcal{P}_L$ denotes the orthogonal projection onto L. If we restrict f and g in (3.3) to elements from L, then the operator $\mathcal{K}$ can be written as $\mathcal{P}_L \mathcal{K} \mathcal{P}_L$. Then equation (3.10) can be written (in the language of the original RKHS $\mathcal{H}$) as

$$\mathcal{P}_L \mathcal{K} \mathcal{P}_L u + \epsilon^2 u = \mathcal{P}_L p_z, \qquad u \in \mathcal{H}, \tag{3.28}$$

whose unique solution u necessarily belongs to L. In general, one's ability to solve the original problem (3.10) would be of little help for solving (3.28), except in the special case when L is an invariant subspace of $\mathcal{K}$. In this case $\mathcal{P}_L$ commutes with $\mathcal{K}$ and if u solves (3.10), then $\mathcal{P}_L u$ solves (3.28).

The requirement that L be a $\mathbb{C}$-linear subspace is important, because the linearization argument taking the objective functional $|f(z)|$ in (3.1) to the one in (3.8) requires all the constraints to be invariant under multiplication by a phase factor $\lambda \in \mathbb{C}$, $|\lambda| = 1$. In some applications, like the analytic continuation of the complex electromagnetic permittivity the constraints may be just $\mathbb{R}$-linear, in which case other techniques have to be applied [19].

4 Applications

4.1 The annulus

Here we prove Theorem 2.1, so assume the setting of Section 2.1. Note that if we replace H^2-norm in Theorem 2.1 by another equivalent norm, this will only change the constant C in the inequality (2.3). In order to apply our theory we need a norm, induced by an inner product, with respect to which the reproducing kernel of the space H^2 is as simple as possible. To define such an inner product we use the Laurent expansion

$$f(\zeta) = \sum_{n \geq 0} f_n \zeta^n + \sum_{n < 0} f_n \zeta^n =: f_+(\zeta) + f_-(\zeta), \tag{4.1}$$

then $f \in H^2(A_\rho)$ if and only if $f_+ \in H^2(\{|\zeta| < 1\})$ and $f_- \in H^2(\{|\zeta| > \rho\})$ (cf. [32]). So we define

$$(f, g) = \tfrac{1}{2\pi}(f_+, g_+)_{L^2(\Gamma_1)} + \tfrac{1}{2\pi\rho}(f_-, g_-)_{L^2(\Gamma_\rho)}, \tag{4.2}$$

The norm in $H^2(A_\rho)$ induced by (4.2) is equivalent to the norm (2.2) (e.g. [32, 22]). Now the functions $\{\zeta^n\}_{n\in\mathbb{Z}}$ form a basis in $H^2(A_\rho)$, let us normalize them:

$$e_n(\zeta) = \begin{cases} \zeta^n, & n \geq 0 \\ (\zeta/\rho)^n, & n < 0, \end{cases} \tag{4.3}$$

then $\{e_n\}_{n\in\mathbb{Z}}$ is orthonormal basis of $H^2(A_\rho)$. Definition of the reproducing kernel implies that $p(\zeta,\tau) = \sum_n \overline{e_n(\tau)}e_n(\zeta)$. Computing this sum, or by adding kernels of the spaces $H^2(\{|\zeta| < 1\})$ and $H^2(\{|\zeta| > \rho\})$, we find the reproducing kernel of $H^2(A_\rho)$:

$$p(\zeta,\tau) = \frac{1}{1-\zeta\overline{\tau}} + \frac{\rho^2}{\zeta\overline{\tau}-\rho^2}. \tag{4.4}$$

Note that $p_z \notin \mathcal{K}(\mathcal{W})$. Indeed, the function p_z has simple poles at $\overline{z}^{-1}, \rho^2\overline{z}^{-1}$. At the same time, for any $f \in \mathcal{W} \subset L^2(\Gamma)$ the function $\mathcal{K}f$ may have singularities only in the set $S = \cup_{\tau\in\Gamma}\{\overline{\tau}^{-1}, \rho^2\overline{\tau}^{-1}\}$. If $\overline{z}^{-1} \in S$, then $z \in \Gamma \cup \rho^{-2}\Gamma$. If $\rho^2\overline{z}^{-1} \in S$, then $z \in \Gamma \cup \rho^2\Gamma$. But since $z \notin \Gamma$ and curves $\rho^{\pm 2}\Gamma$ are outside of the annulus A_ρ, the equation $\mathcal{K}f(\zeta) = p(\zeta, z)$ for $\zeta \in A_\rho$ cannot have any solutions in $\mathcal{W}$.

We observe that for any orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ of $\mathcal{H}$ we have, using (3.3),

$$\mathcal{K}f = \sum_{n\in\mathbb{Z}}(\mathcal{K}f, e_n)e_n = \sum_{n\in\mathbb{Z}}(f, e_n)_{L^2(\Gamma)}\, e_n. \tag{4.5}$$

It is easy to verify that when Γ is a circle centered at the origin, the functions $\{e_n\}$, given by (4.3) are also orthogonal in $L^2(\Gamma)$ and hence, taking $f = e_m$ in (4.5) we conclude that $\mathcal{K}e_m = \|e_m\|^2_{L^2(\Gamma)}e_m$. So we have proved

LEMMA **4.1.** *Let $\{e_n\}_{n\in\mathbb{Z}}$ be given by* (4.3) *and $\mathcal{K}$ given by* (4.5)*, then*

$$\mathcal{K}e_n = \lambda_n e_n, \qquad\qquad n \in \mathbb{Z},$$

where

$$\lambda_n = 2\pi r \begin{cases} r^{2n}, & n \geq 0 \\ (r/\rho)^{2n}, & n < 0 \end{cases} \tag{4.6}$$

We see that λ_n and $|e_n(z)|$ approach to zero along two different sequences and have two different asymptotic behaviors, which are distinguished by the location of z relative to Γ. Therefore, to apply Theorem 3.7 we need to consider two cases. Assume that z lies outside of Γ, i.e. $|z| \in (r, 1)$. The function u from (3.10) is given by

$$u(\zeta) = \sum_{n\in\mathbb{Z}} \frac{\overline{e_n(z)}e_n(\zeta)}{\lambda_n + \epsilon^2}. \tag{4.7}$$

Note that, for any $n \in \mathbb{Z}$

$$\frac{|e_n(z)|^2}{\lambda_n} = \frac{1}{2\pi r}\left(\frac{|z|}{r}\right)^{2n}.$$

By assumption the above quantity is summable over $n < 0$, this implies that in analyzing $u(z)$ the sum over negative indices is $O(1)$, as $\epsilon \to 0$, and hence can be ignored. The dominant part is the sum over $n \geq 0$. Analogously, in quantities $\|u\|_{H^2(A_\rho)}, \|u\|_{L^2(\Gamma)}$ as well, the sum can be restricted to $n \geq 0$. This determines the behaviors $\lambda_n \simeq r^{2n}$ and $|e_n(z)| \simeq |z|^n$, therefore Theorem 3.7 implies that the exponent is $\gamma(z) = \frac{\ln|z|}{\ln r}$. The case $|z| \in (\rho, r)$ is done analogously and (2.4) now follows.

Next, we can rewrite (4.7) as

$$u(\zeta) = \sum_{n\geq 0} \frac{\overline{z}^n \zeta^n}{2\pi r r^{2n} + \epsilon^2} + \sum_{n<0} \frac{\overline{z}^n \zeta^n}{2\pi r r^{2n} + \epsilon^2 \rho^{2n}}. \tag{4.8}$$

Let us consider the function

$$\tilde{u}(\zeta) = \sum_{n\in\mathbb{Z}} \frac{\overline{z}^n \zeta^n}{r^{2n} + \epsilon^2(1+\rho^{2n})}, \tag{4.9}$$

clearly for negative indices $\rho^{2n} \ll 1$ and hence can be ignored, and for positive indices 1 can be ignored from the denominator in the definition of $\tilde{u}$. Therefore, values of $\tilde{u}, u$ at z and their H^2 and L^2-norms have the same behavior in ϵ. Thus, we may consider $\tilde{u}$ instead, which then gives rise to the maximizer function M in (2.5). Finally, the fact that $\|M\|_{H^\infty(\overline{A_\rho})}$ is bounded uniformly in ϵ follows from the application of Lemma A.1.

4.2 The upper half-plane

Notation: Let $D(c,r)$ and $C(c,r)$ denote respectively the closed disk and the circle centered at c and of radius r in the complex plane.

In this section we prove Theorem 2.4. The Hardy space $H^2(\mathbb{H}_+)$ of functions analytic in the complex upper half-plane $\mathbb{H}_+$ is a RKHS with the inner product $(f,g) = (f,g)_{L^2(\mathbb{R})}$. By Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(x)\mathrm{d}x}{x-z}.$$

Therefore, the reproducing kernel p of $H^2(\mathbb{H}_+)$ is

$$p_\tau(\zeta) = p(\zeta,\tau) = \frac{i}{2\pi(\zeta - \overline{\tau})}, \qquad \{\zeta,\tau\} \subset \mathbb{H}_+.$$

In Theorem 2.4 the data is measured on $\Gamma = C(i,r)$ with $r \in (0,1)$. Using the definition of $\mathcal{K}$ (3.4) we have

$$\mathcal{K}u(\zeta) = \frac{1}{2\pi}\int_\Gamma \frac{iu(\tau)|\mathrm{d}\tau|}{\zeta - \overline{\tau}}.$$

Note that $p_z \notin \mathcal{K}(\mathcal{W})$. Indeed, the function p_z is analytic everywhere in $\mathbb{C}$, except at $\overline{z}$, where it has a pole. At the same time for any $f \in \mathcal{W} \subset L^2(\Gamma)$ the function $\mathcal{K}f$ is analytic everywhere in $\mathbb{C}$ outside of $\overline{\Gamma}$. But $\overline{z} \notin \overline{\Gamma}$, since z lies outside of Γ. Therefore, the equation $\mathcal{K}f = p_z$ has no solutions in $\mathcal{W}$.

LEMMA **4.2.** *Let $r \in (0,1)$ and $\Gamma = C(i,r)$. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal eigenbasis of $\mathcal{K}$ in $H^2(\mathbb{H}_+)$, with eigenvalues $\{\lambda_n\}_{n=1}^\infty$. Then*

$$\lambda_n = \frac{r\rho^{2n}}{1+\sqrt{1-r^2}}, \qquad e_n(\zeta) = \frac{\sqrt[4]{1-r^2}}{\sqrt{\pi}}\frac{m(\zeta)^n}{\zeta + z_0}, \tag{4.1}$$

where $\rho, z_0, m(\zeta)$ are as in Theorem 2.4.

Before proving this lemma, let us see that it concludes the proof of Theorem 2.4 upon the application of Theorems 3.4 and 3.7. Indeed, $\lambda_n \simeq \rho^{2n}$ and $|e_n(z)| \simeq |m(z)|^n$, then the formula (2.7) for the exponent $\gamma(z)$ follows. The function u from (3.10) is given by

$$u(\zeta) = \frac{\pi^{-1}\sqrt{1-r^2}}{(\overline{z}+\overline{z}_0)(\zeta+z_0)} \sum_{n=1}^{\infty} \frac{\overline{m(z)}^n m(\zeta)^n}{\frac{r}{1+\sqrt{1-r^2}}\rho^{2n}+\epsilon^2}.$$

As in the case of annulus, ignoring the constants that don't affect the asymptotics of the function as $\epsilon \to 0$ we obtain the maximizer (2.8).

Proof of Lemma 4.2. Let $\mathcal{K}w(\zeta) = \lambda w(\zeta)$, then w must be analytic in the extended complex plane with the closed disk $D(-i,r)$ removed. In particular, it is analytic in $D(i,r)$. Thus, we can evaluate the operator $\mathcal{K}$ explicitly in terms of values of w.

$$\mathcal{K}w(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{irw(i+re^{it})dt}{\zeta+i-re^{-it}} = \frac{1}{2\pi}\int_{C(0,r)} \frac{rw(i+\tau)d\tau}{(\zeta+i)\tau-r^2}.$$

We note that $r^2/|\zeta+i| < r$ precisely when ζ is outside of the closed disk $D(-i,r)$. In addition $w(i+\tau)$ is analytic in $D(0,r)$, hence

$$\mathcal{K}w(\zeta) = \frac{ir}{\zeta+i} w\left(i + \frac{r^2}{\zeta+i}\right).$$

Next we note that the Möbius transformation

$$\sigma(\zeta) = i + \frac{r^2}{\zeta+i}$$

maps $D(-i,r)$ onto the exterior of $D(i,r)$. In particular there is a disk $D_1 \subset D(-i,r)$ such that $\sigma(D_1) = D(-i,r)$. Then $\mathcal{K}w$ is analytic in the exterior of D_1, since w is analytic outside of $D(-i,r)$. But w is an eigenfunction of $\mathcal{K}$, hence it must also be analytic outside of D_1. Repeating the argument using the fact that w is analytic in the larger domain $\mathbb{C} \setminus D_1$ we conclude that it must also be analytic outside of $D_2 \subset D_1$, such that $\sigma(D_2) = D_1$. We can continue like this indefinitely, showing that the only possible singularity of w must be at the fixed point $\zeta_0 \in D(-i,r)$ of $\sigma(\zeta)$. We find

$$\zeta_0 = -i\sqrt{1-r^2}.$$

Since w is analytic at infinity the transformation $\eta = 1/(\zeta-\zeta_0)$ will map the extended complex plane with ζ_0 removed to the entire complex plane (without the infinity). The eigenfunction w will then be an entire function in the η-plane. Let $v(\eta) = w(\eta^{-1}+\zeta_0)$. Then

$$w(\zeta) = v\left(\frac{1}{\zeta-\zeta_0}\right).$$

The relation $\mathcal{K}w = \lambda w$ now reads

$$\lambda v(\eta) = \frac{ir\eta}{\eta(\zeta_0 + i) + 1} v\left(\frac{\eta(\zeta_0 + i) + 1}{i - \zeta_0}\right).$$

One corollary of this equation is that $v(0) = 0$. Hence, $\phi(\eta) = \eta^{-1}v(\eta)$ is also an entire function, satisfying

$$\lambda\phi(\eta) = \frac{ir}{i - \zeta_0}\phi\left(\frac{\eta(\zeta_0 + i) + 1}{i - \zeta_0}\right).$$

We see that $\phi(\eta)$ is an entire function with the property that $\phi(a\eta+b)$ is a constant multiple of $\phi(\eta)$, with $b = \frac{1}{i-\zeta_0}$ and $a = \rho^2$, where ρ is given by (2.7). It remains to observe that such a property holds for functions $\phi_n(\eta) = (\eta - \eta_0)^n$, provided

$$\frac{\eta_0 - b}{a} = \eta_0 \quad \Longleftrightarrow \quad \eta_0 = \frac{b}{1-a}.$$

Indeed,

$$(a\eta + b - \eta_0)^n = a^n\left(\eta - \frac{\eta_0 - b}{a}\right)^n = a^n(\eta - \eta_0)^n.$$

In our case we get $\eta_0 = -\frac{1}{2\zeta_0}$ and conclude that $\phi_n(\eta) = \left(\eta + \frac{1}{2\zeta_0}\right)^n$ and λ_n is given by (4.1). Converting the formula back to $w_n(\zeta)$ we obtain (up to a constant multiple)

$$w_n(\zeta) = \frac{1}{\zeta - \zeta_0}\left(\frac{\zeta + \zeta_0}{\zeta - \zeta_0}\right)^n = \frac{m(\zeta)^n}{\zeta - \zeta_0}.$$

It remains to normalize the eigenfunctions w_n. For that we compute

$$\|w_n\|^2_{H^2(\mathbb{H}_+)} = \int_{\mathbb{R}} |w_n|^2 dx = \int_{\mathbb{R}} \frac{dx}{|x - \zeta_0|^2} = \frac{\pi}{\sqrt{1 - r^2}}.$$

□

4.3 The Bernstein ellipse

4.3.1 From the ellipse to the annulus

The ellipse E_R is conformally equivalent to a disk or the upper half-plane. The conformal mapping effecting the equivalence can be written explicitly in terms of the Weierstrass ζ-function, but the image of the interval $[-1, 1]$ will then be a curve that would not permit any kind of explicit solution of the resulting integral equation. Instead we use a much simpler Joukowski function $J(\omega) = \frac{\omega+\omega^{-1}}{2}$ that will convert the problem in the ellipse to the problem in an annulus with Γ being a concentric circle inside the annulus. We observe that $J(\omega)$ maps the annulus $\{R^{-1} < |\omega| < R\}$ onto the Bernstein ellipse E_R in 2-1 fashion, meaning that each point in E_R has exactly two (if we count the multiplicity) preimages in the annulus (note that $J(\omega) = J(\omega^{-1})$). Moreover, the unit circle gets mapped onto $[-1, 1] \subset E_R$ under

J. So given a function $F \in H^\infty(E_R)$, the function $f(\zeta) := F(J(R\zeta))$ is analytic in A_ρ defined in (2.1), with $\rho = R^{-2}$, has the same H^∞ norm, and satisfies the symmetry property

$$f(\overline{\zeta}) = f(\zeta) \qquad \forall |\zeta| = r = \frac{1}{R}. \tag{4.2}$$

Conversely, any function $f \in H^\infty(A_\rho)$, satisfying (4.2) defines an analytic function in a Bernstein ellipse (with the same H^∞ norm). This is so because (4.2) can also be written as

$$f\left(\frac{1}{R^2\zeta}\right) = f(\zeta) \qquad \forall |\zeta| = r. \tag{4.3}$$

The Schwarz reflection principle then guarantees that (4.3) holds for all $\zeta \in A_\rho$. This implies that $F(u) = f(R^{-1}J^{-1}(u))$ gives the same value for each of the two branches of J^{-1} and hence defines an analytic function in E_R. Thus, the analytic continuation problem in ellipse reduces to the one in the annulus, but with an additional symmetry constraint (4.2).

4.3.2 The annulus with symmetry

Let us now define

$$\mathcal{H} = \{f \in H^2(A_\rho) : f(\overline{\zeta}) = f(\zeta) \quad \forall |\zeta| = \sqrt{\rho}\}. \tag{4.4}$$

The curve Γ will be a circle Γ_r centered at the origin of radius $r = \sqrt{\rho}$.

Lemma **4.3** (Annulus with symmetry)**.** *Let $0 < \rho < 1$ and let $z \in \mathbb{C}$ be such that $r < |z| < 1$. Then there exists $C > 0$, such that for every $\epsilon > 0$ and every $f \in \mathcal{H}$ with $\|f\|_{H^2} \leq 1$ and $\|f\|_{L^2(\Gamma_r)} \leq \epsilon$ we have the bound*

$$|f(z)| \leq C\epsilon^{\gamma(z)}, \tag{4.5}$$

where the exponent $\gamma(z)$ is the same as in Theorem 2.1, i.e.

$$\gamma(z) = \frac{\ln|z|}{\ln r}. \tag{4.6}$$

Moreover, (4.5) *is asymptotically optimal as $\epsilon \to 0$ and the function attaining the bound is*

$$M(\zeta) = \epsilon^{2-\gamma(z)} \sum_{n=1}^{\infty} \frac{\overline{z}^n + (\rho/\overline{z})^n}{\rho^n + \epsilon^2}[\zeta^n + (\rho/\zeta)^n], \qquad \zeta \in A_\rho. \tag{4.7}$$

Proof. We note that the maximization problem in Lemma 4.3 differs from the one in Theorem 2.1 by the requirement of symmetry (4.2). Hence, following the theory in Section 3.4 we define the subspace

$$L = \{f \in H^2(A_\rho) : f(\zeta) = f(\overline{\zeta}) \qquad \forall \zeta \in \Gamma_r\}, \quad r = \sqrt{\rho}.$$

Then, the orthogonal projection onto L will be given by

$$\mathcal{P}_L f(\zeta) = \frac{f(\zeta) + f(\rho/\zeta)}{2}. \tag{4.8}$$

Lemma **4.4.** *The integral operator $\mathcal{K}$ with kernel (4.4) and $\Gamma = \Gamma_r$ commutes with $\mathcal{P}_L$.*

Proof. The commutation $\mathcal{P}_L\mathcal{K} = \mathcal{K}\mathcal{P}_L$ is then equivalent to

$$\int_{\Gamma_r} p(\zeta,\tau)u(r^2/\tau)|\mathrm{d}\tau| = \int_{\Gamma_r} p(r^2/\zeta,\tau)u(\tau)|\mathrm{d}\tau|$$

which, after change of variables on the left-hand side reduces to

$$p(\zeta,\rho/\tau) = p(\rho/\zeta,\tau) \qquad \forall\zeta\in A_\rho,\ \forall\tau\in\Gamma_r.$$

Substituting the definition of p from (4.4) into this formula we easily verify it. □

According to the theory in Section 3.4 the solution of (3.28) is $u_L = \mathcal{P}_L u$, where u is given by (4.7). We observe that in the case $r^2 = \rho$ we have $\lambda_n = \lambda_{-n}$ and $e_n(\rho/\zeta) = e_{-n}(\zeta)$, so that

$$u_L = \mathcal{P}_L u(\zeta) = \frac{1}{1+\epsilon^2} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{\overline{e_n(z)}+\overline{e_{-n}(z)}}{\lambda_n+\epsilon^2}[e_n(\zeta)+e_{-n}(\zeta)] \tag{4.9}$$

Substituting the expressions for λ_n, e_n from (4.6), (4.3), respectively, and ignoring the first $O(1)$ term and some constants, which affect the asymptotics of u_L by constant factors, we arrive at the function

$$u_L(\zeta) = \sum_{n=1}^{\infty}\frac{\overline{z}^n+(\rho/\overline{z})^n}{\rho^n+\epsilon^2}[\zeta^n+(\rho/\zeta)^n].$$

We note that

$$e_n^L = \frac{1}{2}\left(\zeta^n+(\rho/\zeta)^n\right), \quad n\geq 0,$$

is the orthonormal eigenbasis of L with respect to $\mathcal{P}_L\mathcal{K}\mathcal{P}_L$. The corresponding eigenvalues are $\lambda_n = 2\pi\sqrt{\rho}\rho^n$, and for $|z|\in(r,1)$ we have $|e_n^L(z)| \simeq |\overline{z}^n+(\rho/\overline{z})^n| \simeq |z|^n$. Then, Theorem 3.7 gives formula (4.6) as well as the maximizer function (4.7). □

4.3.3 From the annulus to the ellipse

In this section we will show that Theorem 2.6 follows from Lemma 4.3. Let $F\in H^\infty(E_R)$ be such that $\|F\|_{H^\infty}\leq 1$ and $|F(x)|\leq\epsilon$ for all $x\in[-1,1]$. As discussed in Section 4.3.1, the function $f(\zeta) := F(J(R\zeta))$ is analytic in A_ρ, with $\rho = R^{-2}$ and has the symmetry $f(\overline{\zeta}) = f(\zeta) \quad \forall|\zeta| = r,$ where $r = R^{-1}$. It also satisfies

$$\|f\|_{H^2(A_\rho)} \lesssim \|F\|_{H^\infty(E_R)} \leq 1$$

as well as

$$\|f\|^2_{L^2(\Gamma_r)} = \frac{1}{R}\int_0^{2\pi}|F(J(e^{it}))|^2\mathrm{d}t \leq \frac{2\pi\epsilon^2}{R}.$$

Let $z\in E_R\setminus[-1,1]$. Let $z_a\in A_\rho$ be the unique solution of $J(Rz_a) = z$, satisfying $|z_a| > r$. Then by Lemma 4.3 (with $\rho = R^{-2}$ and $r = R^{-1}$) we have

$$|F(z)| = |f(z_a)| \leq C\epsilon^{-\frac{\ln|z_a|}{\ln R}} = C\epsilon^{1-\frac{\ln\left|J^{-1}(z)\right|}{\ln R}} = C\epsilon^{\alpha(z)},$$

where $\alpha(z)$ is given by (2.10). This proves (2.9).

In order to prove the optimality of the bound (2.9) we use Lemma A.1 to show that $M(\zeta)$ given by (4.7) satisfies

$$\begin{cases} |M(\zeta)| \lesssim \epsilon, & |\zeta| = r, \\ |M(\zeta)| \lesssim 1, & r < |\zeta| < 1. \end{cases}$$

Using the Joukowski function to map this to a function on the Bernstein ellipse we obtain

$$M_{\text{ellipse}}(\omega) = M\left(R^{-1}J^{-1}(\omega)\right) = \epsilon^{2-\alpha(z)} \sum_{n=1}^{\infty} \frac{\overline{T_n(z)}T_n(\omega)}{1+\epsilon^2 R^{2n}}, \tag{4.10}$$

where T_n is the Chebyshev polynomial of degree n. Chebyshev polynomials are just monomials ζ^n in the annulus after the Joukowski transformation:

$$J^{-1} \circ T_n \circ J = \zeta \mapsto \zeta^n, \qquad \forall \zeta \neq 0.$$

We note that due to the choice of the branch of J^{-1} to correspond to a point in the exterior of the unit disk we can neglect $1/(J^{-1}(z))^n$ in

$$T_n(z) = \frac{1}{2}\left((J^{-1}(z))^n + \frac{1}{(J^{-1}(z))^n}\right).$$

Thus, the function in (2.11) is asymptotically equivalent to (4.10). Theorem 2.6 is now proved.

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A Appendix

LEMMA **A.1.** *Let $\{a_n, b_n\}_{n=1}^{\infty}$ be nonnegative numbers such that $a_n \simeq e^{-\alpha n}$ and $b_n \simeq e^{-\beta n}$ with $0 < \beta < \alpha$, where the implicit constants don't depend on n. Let $\eta > 0$ be a small parameter, then*

$$\sum_{n=1}^{\infty} \frac{b_n}{a_n+\eta} \simeq \eta^{\frac{\beta}{\alpha}-1}, \qquad \textit{and} \qquad \sum_{n=1}^{\infty} \frac{b_n}{(a_n+\eta)^2} \simeq \eta^{\frac{\beta}{\alpha}-2} \tag{A.11}$$

where the implicit constants don't depend on η.

Proof. Let us prove the first assertion of (A.11), the second one will follow analogously. Introduce the switchover index $J = J(\eta) \in \mathbb{N}$ defined by

$$\begin{cases} a_n \geq \eta & \forall\ 1 \leq n \leq J \\ a_n < \eta & \forall\ n > J \end{cases}$$

Below all the implicit constants in relations involving $\simeq$ or $\lesssim$ will be independent on η. It is clear that

$$\sum_{n=1}^{\infty} \frac{b_n}{a_n+\eta} \simeq \sum_{n\leq J} \frac{b_n}{a_n} + \frac{1}{\eta}\sum_{n>J} b_n.$$

Note that

$$\sum_{n>J} b_n \lesssim \sum_{n>J} e^{-\beta n} \lesssim e^{-\beta(J+1)},$$

therefore using our assumption on b_n we find

$$\sum_{n>J} b_n \simeq b_{J+1} \simeq b_J. \tag{A.12}$$

On the other hand

$$\sum_{n\leq J} \frac{b_n}{a_n} \lesssim \sum_{n\leq J} e^{(\alpha-\beta)n} = \frac{e^{\alpha}}{e^{\alpha}-e^{\beta}}(e^{(\alpha-\beta)J}-1) \lesssim e^{(\alpha-\beta)J} \simeq \frac{b_J}{a_J},$$

Thus we conclude

$$\sum_{n\leq J} \frac{b_n}{a_n} \simeq \frac{b_J}{a_J}. \tag{A.13}$$

Now $\eta \simeq a_J$ and $a_J \simeq e^{-\alpha J}$, therefore $e^{-J} \simeq \eta^{\frac{1}{\alpha}}$. Using these along with (A.12) and (A.13) we obtain

$$\sum_{n=1}^{\infty} \frac{b_n}{a_n+\eta} \simeq \frac{b_J}{a_J} + \frac{b_J}{\eta} \simeq \frac{b_J}{a_J} \simeq e^{(\alpha-\beta)J} \simeq \eta^{\frac{\beta}{\alpha}-1}.$$

□

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