SOLUTION TO PROBLEM #10663
PROPOSED BY E. DEUTCH

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PROBLEM: [**P**] The Pell sequence is defined by $p_0 = 1, p_1 = 2$, and $p_n = 2p_{n-1} + p_{n-2}$ for $n \geq 2$. Show that, for $n \geq 0$,

$$p_n = \sum \frac{(i+j+k)!}{i!j!k!},$$

where the summation extends over all nonnegative integers i, j, k satisfying $i + j + 2k = n$.

PROOF: Replacing $i + j + k = n - k$, $i = n - 2k - j$ and denoting the sum in above by a_n, we have

$$a_n = \sum_k \binom{n-k}{k} \sum_j \binom{n-2k}{j} = \sum_k \binom{n-k}{k} 2^{n-2k} \tag{1}$$

where the binomial identity $\sum_k \binom{x}{k} = 2^x$ has been used.

Now, we employ the amazing Maple package *EKHAD* on the second sum in (1). `(EKHAD available at http://www.math.temple.edu/zeilberg/programs.html)` The result: a_n satisfies *exactly the same* second order recurrence as p_n does! The Wilf-Zeilberger "certificate" being

$$\frac{-4(n+1-k)k}{(n+1-2k)(n+2-2k)}.$$

To prove that $a_n = p_n$ for *all* n, it suffices to check at $n = 0, 1$. $\square$

References:

[**P**] P #10663, *American Mathematical Monthly,* **(105) #5,** 1998.