INDEED SHALOSHABLE!

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In an article posted at the sci.math.research newsgroup and found at gopher://davinci.lfc.edu:70/OR373926-376715/MathRelItems/scimathArchive/scimathres.archive, it was mentioned that a certain identity was seemingly not provable by Ekhad. Here, we shall refute this and actually demonstrate how much shaloshable it is!

CLAIM: The alleged identity

\[
\frac{1}{4} + \sum_{n=0}^{\infty} \binom{2n-1}{n}^2 \frac{1}{24^n (n+1)!} = \frac{1}{\pi}
\]

is indeed shaloshable!!

Proof: Denote \((a)_k := a(a+1)\cdots(a+k-1)\), then we have

\[
s(n) := \sum_{k} \frac{(1/2)_k(-n)_k(n+1)(n+3/4)! (n+1/2)!}{k!(3/2+n)_k(2n+2k+3)(n+1/2)!} \equiv \text{CONSTANT}.
\]

The Maple package EKHAD supplies the recurrence \(s(n+1) - s(n) = 0\) and a WZ ”certificate”,

\[
-k(3n+4+4nk+6k)
4(n-k+1)(2n+3)(n+1)!
\]

Check at, say \(n = 0\) and determine the constant, which is \(\sqrt{2}/4\). To prove the claim, first rewrite equation (2) as

\[
\sum_{k} \frac{(1/2)_k(-n)_k(n+1)}{k!(3/2+n)_k(2n+2k+3)} = \frac{\sqrt{2}}{4} \frac{(n+1/2)!^2}{(n+3/4)! (n+1/4)!}
\]

and then “plug-in” \(n = -1/2\). The rest is trivial. □