SUPERCONGRUENCES FOR
THE ALMKVIST-ZUDILIN NUMBERS

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ABSTRACT. Given a prime number $p$, the study of divisibility properties of a sequence $c(n)$ has two contending approaches: $p$-adic valuations and supercongruences. The former searches for the highest power of $p$ dividing $c(n)$, for each $n$; while the latter (essentially) focuses on the maximal powers $r$ and $t$ such that $c(p^r n)$ is congruent to $c(p^{r-1} n)$ modulo $p^t$. This is called supercongruence. In this paper, we prove a conjecture on supercongruences for sequences that have come to be known as the Almkvist-Zudilin numbers. Some other (naturally) related family of sequences will be considered in a similar vain.

1. Introduction

The Apéry numbers $A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ were valuable to R. Apéry in his celebrated proof [1] that $\zeta(3)$ is an irrational number. Since then these numbers have been a subject of much research. For example, they stand among a host of other sequences with the property

$$A(p^r n) \equiv_p A(p^{r-1} n)$$

now known as supercongruence — a term dubbed by F. Beukers [2].

At the heart of many of these congruences sits the classical example $\frac{\binom{p^b}{p c}}{\binom{p^b}{p}} \equiv_p \binom{b}{c}$ which is a stronger variant of the famous congruence $\frac{\binom{p^b}{p c}}{\binom{p^b}{p}} \equiv_p \binom{b}{c}$ of Lucas. For a compendium of references on the subject of Apéry-type sequences, see [9].

Let us begin by fixing notational conventions. Denote the set of positive integers by $\mathbb{N}^+$. For $m \in \mathbb{N}^+$, let $\equiv_m$ represent congruence modulo $m$. Throughout, assume $p \geq 5$ is a prime.

In this paper, true to tradition, we aim to investigate similar type of supercongruences for the following family of sequences. For integers $i \geq 0$ and $n \geq 1$, define

$$a_i(n) := \sum_{k=0}^{[(n-1)/3]} (-1)^{n-k} \binom{3k+i}{k} \binom{2k+i}{k} \binom{n}{3k+i} \binom{n+k}{k} 3^{n-3k-i}$$

In recent literature, $a_0(n)$ are referred to as the Almkvist-Zudilin numbers. Our motivation for the present work here emanates from the following claim found in [6] (see also [3], [7]).

Conjecture 1.1. For a prime $p$ and $n \in \mathbb{N}^+$, the Almkvist-Zudilin numbers satisfy

$$a_0(p n) \equiv_p a_0(n).$$

Date: June 28, 2015.

2010 Mathematics Subject Classification. Primary ??

Key words and phrases. ??
Our main results can be summed up as:

if $p$ is a prime and $n \in \mathbb{N}^+$, then $a_0(pn) \equiv_{p^3} a_0(n)$ and $a_i(pn) \equiv_{p^2} 0$ for $i > 0$.

The organization of the paper is as follows. Section 2 lays down some preparatory results to show the vanishing of $a_i(pn)$ modulo $p^2$, for $i > 0$. Section 3 sees the completion of the proof. Our principal approach in proving the main conjecture $a_0(pn) \equiv_{p^3} a_0(n)$ relies on a “machinery” we develop as a proof strategy which may be described schematically as:

reduction + $p$-identities.

Sections 4 and 5 exhibit its elaborate execution. The reduction brings in a tighter claim and it also offers an advantage in allowing to work with a single sum instead of a double sum. In Section 6, we complete the proof for Conjecture 1.1. The paper concludes with Section 7 where we declare an improvement on the results from Section 3 which states a congruence for the family of sequences $a_i(pn)$ modulo $p^3$, when $i > 0$. Furthermore, in this last section, the reader will find a proof outline guided by our “machinery”.

2. Preliminary results

Fermat quotients are numbers of the form $q_p(x) = \frac{x^{p-1} - 1}{p}$ and they played a useful role in the study of cyclotomic fields and Fermat’s Last Theorem, see [8]. The next three lemmas are known and we give their proofs for the sake of completeness.

Lemma 2.1. If $a \not\equiv_p 0$ then for $d \in \mathbb{Z}$,

\begin{equation}
q_p(a^d) \equiv_{p^2} d q_p(a) + p \left( \frac{d}{2} \right) q_p(a)^2.
\end{equation}

Proof. Since by Fermat’s little theorem $a^{p-1} \equiv_p 1$ then it follows that

\((a^{p-1})^d = (1 + (a^{p-1} - 1))^d \equiv_{p^3} 1 + d(a^{p-1} - 1) + \left( \frac{d}{2} \right) (a^{p-1} - 1)^2.\)

\[\Box\]

Lemma 2.2. Let $H_n = \sum_{j=1}^{n} \frac{1}{j}$ be the $n$-th harmonic number. Then, for $n \in \mathbb{N}^+$, we have

\begin{equation}
\sum_{k=1}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{1}{k} = -2H_n.
\end{equation}

Proof. For an indeterminate $y$, a simple partial fraction decomposition proves the identity (see [5, Lemma 3.1])

\begin{equation}
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{1}{k+y} = \frac{(-1)^n}{y} \prod_{j=1}^{n} \frac{y-j}{y+j}.
\end{equation}

Now, subtract $\frac{1}{y}$ from both sides and take the limit as $y \to 0$. The right-hand side takes the form

\[\frac{1}{n!} \lim_{y \to 0} \left[ \prod_{j=1}^{n} (j-y) - \prod_{j=1}^{n} (j+y) \right] = -2 \sum_{k=1}^{n} \frac{1}{k}.\]

The conclusion is clear. \[\Box\]
Lemma 2.3. Suppose $p$ is a prime and $0 \leq k < p/3$. Then,

$$(-1)^k \left( \frac{\lfloor p/3 \rfloor}{k} \right) \left( \frac{\lfloor p/3 \rfloor + k}{k} \right) \equiv_p \left( \frac{3k}{k,k,k} \right)^{3^{-3k}}.$$  

Proof. We observe that $\binom{n}{k} \binom{n+k}{k} = \binom{2k}{k} \binom{n+k}{2k}$. If $p \equiv 3 \pmod{3}$, then $\frac{p-1}{3} = \frac{p-1}{3}$ and hence

$$\binom{\frac{p-1}{3} + k}{2k} = \frac{\frac{p-1}{3} + k}{(2k)!} \prod_{j=1}^{k-1} \left( \frac{p-1}{3} \pm j \right) \equiv_p \frac{(-1)^k (3k-1)}{3^{2k} (2k)!} \prod_{j=1}^{k-1} (3j+1) = \frac{(-1)^k (3k)!}{3^{2k} (2k)! k !}.$$  

Therefore, we gather that

$$(-1)^k \left( \frac{\frac{p-1}{3}}{k} \right) \left( \frac{\frac{p-1}{3} + k}{k} \right) = (-1)^k \left( \frac{2k}{k} \right) \left( \frac{\frac{p-1}{3} + k}{2k} \right) \equiv_p \left( \frac{3k}{k,k,k} \right)^{3^{-3k}}.$$  

The case $p \equiv 3 \pmod{3}$ runs analogously. \hfill \qed

Corollary 2.4. For a prime $p$ and an integer $0 < i < \frac{p}{3}$, we have the congruences

$$\sum_{k=1}^{p-1} \left( \frac{3k}{k,k,k} \right) \frac{3^{-3k}}{k} \equiv_p \sum_{k=1}^{\lfloor p/3 \rfloor} \left( \frac{3k}{k,k,k} \right) \frac{3^{-3k}}{k} \equiv_p 3q_p(3),$$

$$\sum_{k=0}^{p-1} \left( \frac{3k}{k,k,k} \right) \frac{3^{-3k}}{k+i} \equiv_p \sum_{k=0}^{\lfloor p/3 \rfloor} \left( \frac{3k}{k,k,k} \right) \frac{3^{-3k}}{k+i} \equiv_p 0.$$  

Proof. For the first assertion, we combine (2.2), Lemma 2.3 and the congruence ([4, p. 358])

$$H_{\lfloor p/3 \rfloor} \equiv_p -3 \sum_{r=1}^{\lfloor p/3 \rfloor} \frac{1}{p-3r} \equiv_p -\frac{3q_p(3)}{2}.$$  

The second congruence follows from (2.3) with $y = i$ and Lemma 2.3. \hfill \qed

3. Main Results on the Sequences $a_i(n)$ for $i > 0$

Theorem 3.1. For a prime $p$ and $n, i \in \mathbb{N}^+$ with $i < \frac{p}{3}$, we have $a_i(pn) \equiv_p 0$.

Proof. Let $k = pm + r$ for $0 \leq r \leq p - 1$. Note: $3k + i = 3pm + 3r + i \leq pn$. Write

$$a_i(pn) = \sum_{m=0}^{\lfloor n/3 \rfloor} \sum_{r=0}^{p-1} (-1)^{p-n-r} \left( \frac{3pm + 3r + i}{pm + r} \right) \left( \frac{2pm + 2r + i}{pm + r} \right) \cdot \left( \frac{pn}{3pm + 3r + i} \right) \left( \frac{pm + pm + r}{pm + r} \right)^{3^{p-n-3pm-3r-i}}.$$  

If $t := 3r + i \geq p + 1$, it is easy to show that the following terms vanish modulo $p^2$:

$$\left( \frac{3pm + t}{pm + r} \right) \left( \frac{2pm + 2r + i}{pm + r} \right) \left( \frac{pn}{3pm + t} \right) = \left( \frac{3pm + t}{pm + r} \right) \left( \frac{pm + pm + r}{pm + r} \right) \left( \frac{pn}{3pm + t} \right).$$
Therefore, we may restrict to the remaining sum with $3r + i \leq p$:

$$a_i(pm) = \sum_{m=0}^{\lfloor n/3 \rfloor} \sum_{r=0}^{\lfloor (p-i)/3 \rfloor} (-1)^{n-m-r} \left( \binom{3pm + 3r + i}{pm + r} \binom{2pm + 2r + i}{pm + r} \right) \binom{pn}{3pm + 3r + i} \binom{pn + pm + r}{pm + r} 3^{pn-3pm-3r-i}.$$ 

We need Lucas’s congruence $(\frac{p^b+c}{pm}) \equiv_p (\frac{d}{a}) \binom{c}{a}$ to arrive at

$$a_i(pm) \equiv_p \sum_{m=0}^{\lfloor n/3 \rfloor} \sum_{r=0}^{\lfloor (p-i)/3 \rfloor} (-1)^{n-m-r} \binom{3m}{m} \binom{3r+i}{r} \binom{2m}{m} \binom{2r+i}{r} \binom{pn}{3pm + 3r + i} \binom{pn + pm + r}{pm + r} \binom{n+m}{m} 3^{pn-3pm-3r-i}.$$ 

For $0 < j < p$, we apply Gessel’s congruence $\binom{p}{j} \equiv_p (-1)^{j-1} \frac{j}{i}$ (if $p = 3r + i$, in this case, still the corresponding term properly absorbs into the sum below) so that

$$\binom{pn}{3pm + 3r + i} \equiv_p (-1)^{r+i-1} \frac{pm}{3r+i} \binom{n-1}{3m},$$

which leads to

$$a_i(pm) \equiv_p pn \sum_{m=0}^{\lfloor n/3 \rfloor} \sum_{r=0}^{\lfloor (p-i)/3 \rfloor} (-1)^{n-m-r} \binom{3m}{m} \binom{3r+i}{r} \binom{2m}{m} \binom{2r+i}{r} \binom{pn}{3pm + 3r + i} \binom{pn + pm + r}{pm + r} \binom{n+m}{m} 3^{pn-3pm-3r-i}.$$ 

Next, we use Fermat’s Little Theorem and decouple the double sum to obtain

$$a_i(pm) \equiv_p n \sum_{m=0}^{\lfloor n/3 \rfloor} \sum_{r=0}^{\lfloor (p-i)/3 \rfloor} (-1)^{n-m-i-1}3^{n-3m-i} \binom{3m}{m} \binom{2m}{m} \binom{n-1}{3m} \binom{n+m}{m} \frac{\binom{3m}{m} \binom{2m}{m} \binom{n-1}{3m} \binom{n+m}{m}}{3^{3r+i}}.$$ 

It suffices to verify the sum over $r$ vanishes modulo $p$. To achieve this, apply partial fraction decomposition and Corollary 2.4 (upgrading the sum to $\lfloor p/3 \rfloor$ is harmless here). Thus,

$$\sum_{k=0}^{\lfloor p/3 \rfloor} \binom{3k+i}{k} \binom{2k+i}{i} 3^{-3k} \frac{3}{3k+i} = \sum_{k=0}^{\lfloor p/3 \rfloor} \binom{3k}{k,k,k} 3^{-3k} \prod_{j=1}^{i-1}(3k+j) \prod_{j=1}^{i}(k+j)^{-1}$$

$$= \sum_{j=1}^{i} \alpha_j(i) \sum_{k=0}^{\lfloor p/3 \rfloor} \binom{3k}{k,k,k} 3^{-3k} \frac{3}{k+j} \equiv_p \sum_{j=1}^{i} \alpha_j(i) \cdot 0 = 0;$$

where $\alpha_j(i) \in \mathbb{Q}$ are some constants. We have enough reason to conclude the proof. \qed
4. The Reduction on the Sequence $a_0(n)$

Our proof of Conjecture 1.1 requires a slightly more delicate analysis than what has been demonstrated in the previous sections for the sequences $a_i(n)$, where $i > 0$. As a first major step forward, we state and prove the following somewhat stronger result. This will be crucial in scaling down a double sum, which emerges (see proof below) as an expression for the sequence $a_0(pn)$, to a single sum.

**Theorem 4.1.** The congruence

\[
\sum_{r=1}^{p-1} (-1)^r \left( \frac{3pm + 3r}{pm + r} \right) \left( \frac{2pm + 2r}{pm + r} \right) \left( \frac{pn}{3pm + 3r} \right) \equiv p^3 \left( \frac{3m}{m} \right) \left( \frac{2m}{m} \right) \left( \frac{n + m}{m} \right) \cdot q_p(3^{-(n-3m)})
\]

or

\[
\sum_{r=0}^{p-1} (-1)^r \left( \frac{3pm + 3r}{pm + r} \right) \left( \frac{2pm + 2r}{pm + r} \right) \left( \frac{pn}{3pm + 3r} \right) \equiv p^3 \left( \frac{3m}{m} \right) \left( \frac{2m}{m} \right) \left( \frac{n + m}{m} \right) 3^{-(n-3m)(p-1)}
\]

implies $a_0(pn) \equiv_p a_0(n)$.

**Proof.** Let $k = pm + r$ for $0 \leq r < p$. Then, by using the new parameters,

\[
a_0(pn) = \sum_{m=0}^{n-1} 3^{p(n-3m)} (-1)^{n-m} \sum_{r=0}^{p-1} (-1)^r \left( \frac{3pm + 3r}{pm + r} \right) \left( \frac{2pm + 2r}{pm + r} \right) \cdot \left( \frac{pn}{3pm + 3r} \right) \left( \frac{p(n + m) + r}{pm + r} \right) 3^{3r}
\]

Let’s isolate the case $r = 0$, then, from $\binom{p^h}{p^j} \equiv_p \binom{h}{j}$ and the hypothesis we get

\[
a_0(pn) \equiv_p \sum_{m=0}^{n-1} 3^{p(n-3m)} (-1)^{n-m} \left( \frac{3m}{m} \right) \left( \frac{2m}{m} \right) \left( \frac{n + m}{m} \right) \cdot \left[ 1 + p_q(3^{-(n-3m)}) \right]
\]

\[
\equiv_p \sum_{m=0}^{n-1} (-1)^{n-m} \left( \frac{3m}{m} \right) \left( \frac{2m}{m} \right) \left( \frac{n + m}{m} \right) 3^{(n-3m)} = a_0(n).
\]

\[\square\]

5. Further Preliminary Results

In this section, we build a few valuable results aiming at the proof of Theorem 4.1 and hence that of Conjecture 1.1.

**Lemma 5.1.** If $a > b \geq 0$ and $0 < j < p$ then

\[
\binom{ap}{bp + j} \equiv p^2 \left( a - b \right) \binom{a}{b} \binom{p}{j} \quad \text{and} \quad \binom{ap}{bp - j} \equiv p^2 \left( a \right) \binom{a}{b} \binom{p}{j}.
\]
Moreover, for \(0 \leq r < p\),

\[
\binom{p(n + m) + r}{pm + r} \equiv_{p^2} \binom{n + m}{m} \left( 1 + n \left( \binom{p + r}{r} - 1 \right) \right)
\]

\[
\binom{2pm + 2r}{pm + r} \equiv_{p^2} \binom{2m}{m} \left( \binom{2r}{r} + 2m \binom{p + 2r}{r} - 2m \binom{2r}{r} \right),
\]

\[
\binom{3pm + 3r}{pm + r} \equiv_{p^2} \binom{3m}{m} \left( 2m \binom{p + 3r}{r} + m \binom{p + 3r}{2r} - (3m - 1) \binom{3r}{r} \right)
+ \left( \binom{3m}{m-1} \left( \binom{3r}{p+r} + (m-1) \binom{p + 3r}{2p + r} - 3m \binom{3r}{p+r} \right) \right).
\]

Also, \(\binom{pn}{3pn+3r} \equiv_{p^3} \frac{pn}{3pn+3r} U_r\) where

\[
U_r \equiv_{p^2} (3m+1) \left( \binom{n-1}{3m+1} \left( \binom{2p-1}{3r-1} - \binom{p-1}{3r-1} - \binom{p-1}{3r-1-p} \right) \right)
+ (3m+2) \left( \binom{n-1}{3m+2} \left( \binom{2p-1}{3r-1-p} - \binom{p-1}{3r-1} - \binom{p-1}{3r-1-2p} \right) \right)
\]

\[
+ (3m+3) \left( \binom{n-1}{3m+3} \left( \binom{2p-1}{p+3r-1} - \binom{p-1}{3r-1} \right) \right)
+ \left( \binom{n-1}{3m} \left( \binom{p-1}{3r-1} + \binom{n-1}{3m+1} \binom{p-1}{3r-1-p} + \binom{n-1}{3m+2} \binom{p-1}{3r-1-2p} \right) \right).
\]

**Proof.** For (5.1), we have

\[
\binom{ap}{bp+j} = \binom{ap}{bp} \prod_{k=1}^{j-1} \frac{(a-b)p-k}{bp+k} \equiv_{p^2} \frac{(a-b)(a)}{b} \binom{p(-1)^{j-1}}{j} \equiv_{p^2} \frac{(a-b)(a)}{b} \binom{p}{j},
\]

and therefore

\[
\binom{ap}{bp-j} = \binom{ap}{(a-b)p+j} \equiv_{p^2} b \binom{a}{b} \binom{p}{j}.
\]

For (5.2), use Vandermonde-Chu’s identity and (5.1) so that

\[
\binom{p(n + m) + r}{pm + r} = \sum_{j=0}^{r} \binom{p(n + m)}{pm + j} \binom{r}{r-j}
\]

\[
\equiv_{p^2} \binom{n + m}{m} + n \binom{n + m}{m} \sum_{j=1}^{r} \binom{p}{j} \binom{r}{r-j}
\]

\[
\equiv_{p^2} \binom{n + m}{m} \left( 1 + n \left( \binom{p + r}{r} - 1 \right) \right).
\]
In a similar way, we prove (5.3) as follows:

\[
\binom{2pm + 2r}{pm + r} = \sum_{j=-r}^{r} \binom{2pm}{pm + j} \binom{2r}{r - j}
\]

\[
= \binom{2pm}{pm} \binom{2r}{r} + \sum_{j=1}^{r} \binom{2pm}{pm + j} \binom{2r}{r - j} + \sum_{j=1}^{r} \binom{2pm}{pm - j} \binom{2r}{r + j}
\]

\[
\equiv_p^2 \binom{2m}{m} \binom{2r}{r} + m \sum_{j=1}^{r} \binom{p}{j} \binom{2r}{r - j} + m \sum_{j=1}^{r} \binom{p}{p - j} \binom{2r}{r + j}
\]

Moreover,

\[
\binom{3pm + 3r}{pm + r} = \sum_{j=-2r}^{r} \binom{3pm}{pm + j} \binom{3r}{r - j}
\]

\[
= \binom{3pm}{pm} \binom{3r}{r} + \sum_{j=1}^{r} \binom{3pm}{pm + j} \binom{3r}{r - j} + \sum_{j=1}^{r} \binom{3pm}{pm - j} \binom{3r}{r + j}
\]

\[
\equiv_p^2 \binom{3m}{m} \binom{3r}{r} + 2m \left( \binom{p + 3r}{r} - \binom{3r}{r} \right) + \sum_{j=1}^{2r} \binom{3pm}{pm - j} \binom{3r}{r + j}
\]

Now, (5.4) is equal to

\[
\sum_{j=1}^{p-1} \binom{3pm}{pm - j} \binom{3r}{r + j} + \binom{3pm}{pm - p} \binom{3r}{r + p} + \sum_{j=p+1}^{2r} \binom{3pm}{pm - j} \binom{3r}{r + j}
\]

\[
= \sum_{j=1}^{p-1} \binom{3pm}{pm - j} \binom{3r}{r + j} + \binom{3pm}{pm - p} \binom{3r}{r + p} + \sum_{j=1}^{2r-p} \binom{3pm}{p(m-1) - j} \binom{3r}{r + p + j}
\]

\[
\equiv_p^2 m \binom{3m}{m} \left( \binom{p + 3r}{p + r} - \binom{3r}{r} - \binom{3r}{r + p} \right) + \binom{3m}{m-1} \binom{3r}{r + p}
\]

\[
+ (m - 1) \binom{3m}{m-1} \sum_{j=1}^{2r-p} \binom{p}{p - j} \binom{3r}{r + p + j}
\]

\[
\equiv_p^2 m \binom{3m}{m} \left( \binom{p + 3r}{p} - \binom{3r}{r} - \binom{3r}{p + r} \right) + \binom{3m}{m-1} \binom{3r}{p + r}
\]

\[
+ (m - 1) \binom{3m}{m-1} \left( \binom{p + 3r}{2p + r} - \binom{3r}{p + r} \right).
\]

The proof of the last congruence in (5.5) is analogous and hence is omitted here. \(\square\)

\textbf{Proof.} We provide an alternative proof of Lemma 5.1 by reviving certain results found in [10] as equations (26) and (27), respectively. These are stated follows. If \(n = n_1p + n_0\) and
$k = k_1 p + k_0$ where $0 < n_0, k_0 < p$ then

\[(5.6) \quad \binom{np}{k} \equiv_{p^2} n\binom{n-1}{k_1}\binom{p}{k_0},\]

\[(5.7) \quad \binom{n}{k} \equiv_{p^2} \binom{n_1}{k_1} \left[ (1 + n_1) \binom{n_0}{k_0} - (n_1 + k_1) \binom{n_0 - p}{k_0} - k_1 \binom{n_0 - p}{k_0 + p} \right].\]

For (5.1) of the lemma, apply (5.6) with $n_1 = a, n_0 = 0, k_1 = b, k_0 = j$. So,

\[
\binom{ap}{bp + j} \equiv_{p^2} a\binom{a - 1}{b}\binom{p}{j} = (a - b)\binom{a}{b}\binom{p}{j}.
\]

For (5.2), apply (5.7) with $n_1 = n + m, n_0 = r = k_0, k_1 = m$. So,

\[
\binom{p(n + m) + r}{pm + r} \equiv_{p^2} \binom{n + m}{m} \left[ (1 + n + m) \binom{r}{r} - (n + 2m) \binom{r - p}{r} - m \binom{r - p}{r + p} \right].
\]

To put this in the desired format consider applying (5.7) to \binom{p + r}{r} \equiv_{p^2} 2 - \binom{r - p}{p} (with $n_1 = 1, n_0 = k_0 = r, k_1 = 0$); to \binom{r + p}{r} = \binom{r + p}{r} \equiv_{p^2} -3 + 2\binom{r - p}{r} (with $n_1 = -1, n_0 = r, k_1 = -2, k_0 = 0$). After substitution and simplifications, the desired outcome is reached.

For (5.3), apply (5.7) with $n_1 = 2m, n_0 = 2r, k_1 = m, k_0 = r$. So,

\[
\binom{2pm + 2r}{pm + r} \equiv_{p^2} \binom{2m}{m} \left[ (1 + 2m) \binom{2r}{r} - 3m \binom{2r - p}{r} - m \binom{2r - p}{r + p} \right].
\]

Let’s reformulate this to get the result as stated in the lemma. To this end, employ (5.7) to \binom{p + 2r}{r} \equiv_{p^2} 2\binom{2r}{r} - \binom{2r - p}{r} (with $n_1 = 1, n_0 = 2r, k_1 = 0, k_0 = r$); to \binom{p + 2r}{r} = \binom{p + 2r}{r} \equiv_{p^2} 2\binom{2r}{r} - 2\binom{2r - p}{r} - \binom{2r - p}{r + p} (with $n_1 = k_1 = 1, n_0 = 2r, k_0 = r$). Routine substitution completes the argument.

The congruence (5.4) demands a careful analysis. The setup begins by expressing $3r = \epsilon p + d$ where $0 < d < p$ and $\epsilon \in \{0, 1, 2\}$ which correspond to $0 < 3r < p, p < 3r < 2p$ and $2p < 3r < 3p$, respectively. Here, $\epsilon = \left[ \frac{3r}{p} \right]$.

Let $n_1 = 3m + \epsilon, n_0 = d, k_1 = m, k_0 = r$ and implement (5.7). So,

\[
\binom{p(3m + \epsilon) + d}{pm + r} \equiv_{p^2} \binom{3m + \epsilon}{m} \left[ (3m + \epsilon + 1) \binom{d}{r} - (4m + \epsilon) \binom{d - p}{r} - m \binom{d - p}{r + p} \right].
\]

Next, engage (5.6) with (with $n_1 = \epsilon, n_0 = d, k_1 = 0, k_0 = r$ to get

\[
\binom{3r}{r} = \binom{\epsilon p + d}{r} \equiv_{p^2} (\epsilon + 1) \binom{d}{r} - \epsilon \binom{d - p}{r};
\]

with $n_1 = \epsilon + 1, n_0 = d, k_1 = 0, k_0 = r$ to get

\[
\binom{p + 3r}{r} = \binom{(\epsilon + 1)p + d}{r} \equiv_{p^2} (\epsilon + 2) \binom{d}{r} - (\epsilon + 1) \binom{d - p}{r};
\]

with $n_1 = \epsilon + 1, n_0 = d, k_1 = 1, k_0 = r$ to get

\[
\binom{p + 3r}{2r} = \binom{(\epsilon + 1)p + d}{p + r} \equiv_{p^2} (\epsilon + 1)(\epsilon + 2) \binom{d}{r} - (\epsilon + 1)(\epsilon + 2) \binom{d - p}{r} - (\epsilon + 1) \binom{d - p}{r + p}.
\]
After proper substitutions, the result becomes
\[
\binom{3pm + 3r}{pm + r} \equiv_p \left( 3m + \epsilon \right) \binom{3r}{m} \binom{3r}{r} + \binom{3m + \epsilon}{m} \left( \frac{1}{\epsilon + 1} \binom{p + 3r}{2r} - \binom{3r}{r} \right) + 2m \left( \binom{p + 3r}{r} - \binom{3r}{r} \right).
\]
For (5.5), apply (5.6) with \(n_1 = n - 1, n_0 = p - 1, k_1 = 3m + \epsilon, k_0 = d - 1\). Follow this through using \((-1)^j = (-1)^j\). The outcome is:
\[
\binom{pm}{3pm + 3r} \equiv_p \binom{pm}{3pm + 3r} \left( \frac{p(n - 1) + p - 1}{3pm + 3\epsilon} \right) \equiv_p \binom{pm}{3pm + 3r} \binom{n - 1}{3m + \epsilon} \left( \binom{p - 1}{3r - 1 - \epsilon} \right) + (-1)^{r-\epsilon}(n - 1). \tag{5.8}
\]
Although doable, we opt to leave this congruence in its present form instead of committing to transform it into (5.5) because (5.8) will be more convenient for our subsequent calculations.

\[\square\]

**Corollary 5.2.** For \(p > 3\) a prime and an integer \(0 \leq r < p\), we have the congruence
\[
\binom{3pm + 3r}{pm + r} \equiv_p^2 \binom{3m}{m, m, m} \binom{3r}{r, r, r} \left[ 1 + 3pm(H_{3r} - H_r) \right].
\]

**Proof.** This is a consequence of Lemma 5.1 and (5.7). However, we offer a more direct approach. Since \((pm + k)^{-1} \equiv_p \frac{1}{k} (1 - \frac{pm}{k})\), we obtain \((pm + k)^{-3} \equiv_p \frac{1}{k^3} \left( 1 - \frac{pm}{k} \right)^3 \equiv_p \frac{1}{k^3} \left( 1 - \frac{3pm}{k} \right) = \frac{1}{k^3} (k - 3pm)\). For notational simplicity, denote \(\binom{3j}{j, j, j} = \binom{3j}{2j} \binom{3j}{j} \). We consider the expansion \(\prod_{i=1}^{n}(\lambda_i + x) = \sum_{j=0}^{n} e_j(\lambda) x^{n-j}\) as our running theme, where \(e_j\) is the \(j\)-th elementary symmetric function in the parameters \(\lambda = (\lambda_1, \ldots, \lambda_n)\). In particular, \(e_n = 1\) and \(e_{n-1}(1, \ldots, n) = n!H_n\). The claim then follows from
\[
\binom{3pm + 3r}{(pm + r)^3} = \binom{3pm}{(pm)^3} \prod_{j=1}^{3r} \binom{j + 3pm}{pm + k} \prod_{k=1}^{r} \left( k - 3pm \right) \equiv_p \binom{3pm}{(pm)^3} \frac{1}{r!^4} \prod_{j=1}^{3r} \binom{j + 3pm}{pm + k} \prod_{k=1}^{r} \left( k - 3pm \right) \equiv_p \binom{3pm}{(pm)^3} \frac{1}{r!^4} \binom{3r}{3r} [1 + 3pmH_{3r} - 3pmH_r].
\]

\[\square\]

This fact is even more general as stated below but its proof is left to the interested reader.

**Exercise 5.3.** If \(A > 0, 0 \leq r < p\) are integers and \(p > 3\) a prime, then
\[
\binom{Apm + Ar}{pm + r, \ldots, pm + r} := \binom{Apm + Ar}{pm + r}!^A \equiv_p \binom{Am}{m, \ldots, m} \binom{Ar}{r, \ldots, r} \left[ 1 + Apm(H_{Ar} - H_r) \right].
\]

**Corollary 5.4.** For \(p > 3\) a prime and an integer \(0 \leq r < p\), we have
\[
\binom{p(n + m) + r}{pm + r} \equiv_p \binom{n + m}{m} \left[ 1 + pnH_r \right].
\]
Proof. It is easy to check that \((p+r)^{r} = \frac{1}{r!} \prod_{j=1}^{r} (p+j) \equiv p^2 \mod 1 + pH_r\). The rest follows from (5.2) of Lemma 5.1. \(\square\)

**Corollary 5.5.** Let \(N = n - 3m\). For \(p > 3\) a prime and an integer \(0 < r < p\), it holds that

\[
\left(\frac{pn}{3pm + 3r}\right) \equiv_{p^3} \left(\frac{p}{3r} - \frac{p^2 m}{3r^2}\right) (-1)^r \left(\frac{n}{3m}\right) \cdot \begin{cases} N(-1 + pmH_{3r-1}), & \text{if } 0 < r < \frac{p}{3} \\ \frac{N}{2} \frac{2(1 - pmH_{3r-1-p})}{3m + 1}, & \text{if } \frac{p}{3} < r < \frac{2p}{3} \\ \frac{N}{3} \frac{6(-1 + pmH_{3r-1-2p})}{(3m + 1)(3m + 2)}, & \text{if } \frac{2p}{3} < r < p. \end{cases}
\]

Proof. We continue where we left off (5.8) with \(\epsilon = \lfloor \frac{3r}{p} \rfloor\). That is,

\[
\left(\frac{pn}{3pm + 3r}\right) \equiv_{p^3} \frac{pn}{3pm + 3r} \left(\frac{n - 1}{3m + \epsilon}\right) \left[n \left(\frac{p - 1}{3r - 1 - \epsilon p}\right) + (-1)^r \epsilon(n - 1)\right].
\]

Combining this step and the easy facts \(\frac{1}{3pm + 3r} \equiv_{p^2} \frac{1}{3r} - \frac{pm}{3r^2}\), \((p^{-1}) \equiv_{p^2} (-1)^j[1 - pH_j]\), we reach the desired conclusion. \(\square\)

**Lemma 5.6.** If \(p > 3\) is a prime then

\[
\sum_{r=1}^{p-1} \left(\frac{3r}{r, r, r}\right) 3^{-3r} \equiv_{p^2} -3q_p(1/3) + \frac{3p}{2} q_p(1/3)^2,
\]

\[
\sum_{r=1}^{p-1} \left(\frac{3r}{r, r, r}\right) \frac{3^{-3r}}{r} \equiv_{p^2} -9q_p(1/3)^2,
\]

\[
\sum_{r=1}^{p-1} \left(\frac{3r}{r, r, r}\right) (H_{3r} - H_r) 3^{-3r} \equiv_{p^2} 0.
\]

Proof. By (2.1), \(q_p(1/27) \equiv_{p^2} 3q_p(1/3) + 3pq_p(1/3)^2\). Therefore, by (5) in [11, Theorem 4],

\[
\sum_{r=1}^{p-1} \left(\frac{3r}{r, r, r}\right) 3^{-3r} \equiv_{p^2} -3q_p(1/3) + \frac{3p}{2} q_p(1/3)^2.
\]

In a similar way, by (6) in [11, Theorem 4],

\[
\sum_{r=1}^{p-1} \left(\frac{3r}{r, r, r}\right) \frac{3^{-3r}}{r^2} \equiv_{p^2} -q_p(1/27) + \frac{p}{2} q_p(1/27)^2
\]

\[
\equiv_{p} -9q_p(1/3)^2.
\]

By (1) in [11, Theorem 1],

\[
\frac{(1/3)_k(2/3)_k}{(1)_k^2} \sum_{j=0}^{r-1} \left(\frac{1}{1/3 + j} + \frac{1}{2/3 + j}\right) = \sum_{k=0}^{r-1} \frac{(1/3)_k(2/3)_k}{(1)_k^2} \cdot \frac{1}{r - k}.
\]
Hence (5.11) is implied by the following
\[
\sum_{r=1}^{p-1} \binom{3r}{r,r,r} \frac{(3H_{3r} - H_r)3^{-3r}}{r} = \sum_{r=1}^{p-1} \frac{(1/3)^r(2/3)_r}{(1)_r^2} \cdot \frac{1}{r} \sum_{j=0}^{r-1} \left( \frac{1}{1/3 + j} + \frac{1}{2/3 + j} \right)
\]
\[
= \sum_{r=1}^{p-1} \frac{1}{r} \sum_{k=0}^{r-1} \frac{(1/3)_k(2/3)_k}{(1)_k^2} \cdot \frac{1}{r-k}
\]
\[
= \sum_{k=0}^{p-2} \frac{(1/3)_k(2/3)_k}{(1)_k^2} \sum_{r=k+1}^{p-1} \frac{1}{r(r-k)}
\]
\[
= \sum_{r=1}^{p-1} \frac{1}{r^2} + \sum_{k=1}^{p-2} \frac{(1/3)_k(2/3)_k}{(1)_k^2} \left( \frac{1}{k} \sum_{r=k+1}^{p-1} \left( \frac{1}{r-k} - \frac{1}{r} \right) \right)
\]
\[
\equiv_p \sum_{k=1}^{p-2} \frac{(1/3)_k(2/3)_k}{(1)_k^2} \cdot \frac{1}{k} (H_{p-1-k} - H_{p-1} + H_k)
\]
\[
\equiv_p \sum_{k=1}^{p-1} \left( \frac{3k}{k, k, k} \right) \frac{2H_k 3^{-3k}}{k},
\]
because \(H_{p-1-k} \equiv_p H_k\) and \(H_{p-1} \equiv_p \sum_{r=1}^{p-1} \frac{1}{r^2} \equiv_p \sum_{j=1}^{p-1} j \equiv_p 0\) as \(p \neq 2\). \(\square\)

6. Proof of Conjecture 1.1

In this section, we combine the results from the preceding sections to arrive at a proof for Theorem 4.1 (restated here for the reader’s convenience) and therefore for Conjecture 1.1.

**Theorem 6.1.** For a prime \(p > 3\) and \(m, n \in \mathbb{N}^+\), we have
\[
\sum_{r=1}^{p-1} (-1)^r \binom{3pm + 3r}{pm + r} \binom{2pm + 2r}{pm + r} \binom{pn}{3pm + 3r} \binom{p(n + m) + r}{pm + r} H_r 3^{-3r} \equiv_p p \left( \binom{3m}{m} \left( \frac{2m}{m} \right) \left( \binom{n}{3m} \right) \left( \frac{n + m}{m} \right) \right) q_p(3^{-(n-3m)}).
\]

**Proof.** Based on Corollaries 5.2, 5.4, 5.5 and the congruence (2.1), the assertion is equivalent to
\[
\sum_{r=1}^{p-1} \binom{3r}{r,r,r} (1 + 3pm(H_{3r} - H_r))(1 + pnH_r) \left( \frac{1}{3r} - \frac{pn}{3r^2} \right) B_r(p, n, m) 3^{-3r} \equiv_p q_p(1/3) + \frac{p(N - 1)}{2} q_p(1/3)^2;
\]
where
\[
B_r(p, n, m) = \begin{cases} 
-1 + pnH_{3r-1}, & \text{if } 0 < r < \frac{p}{3} \\
\frac{(N - 1)(1 - pnH_{3r-1-p})}{3m + 1}, & \text{if } \frac{p}{3} < r < \frac{2p}{3} \\
\frac{(N - 1)(N - 2)(-1 + pnH_{3r-1-2p})}{(3m + 1)(3m + 2)}, & \text{if } \frac{2p}{3} < r < p.
\end{cases}
\]
Now we split the sum on the left-hand side of (6.1) into three pieces according as

\[ S_1 = \sum_{r=1}^{\lfloor p/3 \rfloor} (\cdot), \quad S_2 = \sum_{r=\lfloor p/3 \rfloor}^{\lfloor 2p/3 \rfloor} (\cdot), \quad \text{and} \quad S_3 = \sum_{r=\lfloor 2p/3 \rfloor}^{p-1} (\cdot). \]

As regards \( S_1 \),

\[ S_1 \equiv_p \frac{1}{3} \sum_{r=1}^{\lfloor p/3 \rfloor} \left( \frac{3r}{r, r, r} \right) \left( -\frac{1}{r} - \frac{pN(H_{3r} - H_r)}{3r^2} + \frac{pN(H_{3r} - H_r)}{r} \right) 3^{-3r}. \]

If \( \frac{2}{3} < r < \frac{2p}{3} \) then \( \binom{3r}{r, r, r} \equiv_p 0 \) and \( 1 + 3pm(H_{3r} - H_r) \equiv_p 1 + 3m \) with \( B_{p}(n, m) \equiv_p \frac{(N-1)}{(3m+1)} \). These imply that

\[ S_2 \equiv_p \frac{1}{3} \sum_{r=\lfloor p/3 \rfloor}^{\lfloor 2p/3 \rfloor} \left( \frac{3r}{r, r, r} \right) (1 + 3pm(H_{3r} - H_r))(1 + pnH_r) \left( \frac{1}{3r} - \frac{pm}{3r^2} \right) B_r(p, n, m) 3^{-3r} \]

\[ \equiv_p \frac{1}{3} \sum_{r=\lfloor p/3 \rfloor}^{\lfloor 2p/3 \rfloor} \left( \frac{3r}{r, r, r} \right) (1 + 3m) \left( \frac{1}{3r} \right) \frac{(N-1)}{(3m+1)} 3^{-3r} \]

\[ \equiv_p \frac{(N-1)}{3} \sum_{r=\lfloor p/3 \rfloor}^{\lfloor 2p/3 \rfloor} \left( \frac{3r}{r, r, r} \right) 3^{-3r}. \]

Finally, we have that \( S_3 \equiv_p 0 \) because obviously \( \binom{3r}{r, r, r} \equiv_p 0 \) as long as \( \frac{2p}{3} < r < p \).

Again \( \binom{3r}{r, r, r} \equiv_p 0 \) if \( \frac{p}{3} < r < \frac{2p}{3} \) and \( \binom{3r}{r, r, r} \equiv_p 0 \) if \( \frac{2p}{3} < r < p \). So, from Lemma 5.6 we know

\[ \frac{1}{3} \sum_{r=1}^{\lfloor 2p/3 \rfloor} \left( \frac{3r}{r, r, r} \right) 3^{-3r} \equiv_p \frac{1}{3} \sum_{r=1}^{p-1} \left( \frac{3r}{r, r, r} \right) 3^{-3r} - 3q_p(1/3) + \frac{3p}{2}q_p(1/3)^2, \]

\[ p \sum_{r=1}^{\lfloor 2p/3 \rfloor} \left( \frac{3r}{r, r, r} \right) 3^{-3r} \equiv_p \sum_{r=1}^{p-1} \left( \frac{3r}{r, r, r} \right) 3^{-3r} \equiv_p -\frac{9p}{2}q_p(1/3)^2. \]

As before \( \binom{3r}{r, r, r} \equiv_p 0 \) for \( \frac{2p}{3} < r < p \). As well as \( \binom{3r}{r, r, r} \equiv_p 0 \) and \( pH_{3r} - pH_r \equiv_p 1 \) for \( \frac{p}{3} < r < \frac{2p}{3} \). Therefore, by Lemma 5.6

\[ 0 \equiv_p p \sum_{r=1}^{p-1} \left( \frac{3r}{r, r, r} \right) \frac{(H_{3r} - H_r)3^{-3r}}{r} \equiv_p \sum_{r=1}^{\lfloor 2p/3 \rfloor} \left( \frac{3r}{r, r, r} \right) \frac{(H_{3r} - H_r)3^{-3r}}{r} \]

\[ \equiv_p \frac{1}{3} \sum_{r=\lfloor p/3 \rfloor}^{\lfloor 2p/3 \rfloor} \left( \frac{3r}{r, r, r} \right) (H_{3r} - H_r)3^{-3r} + \sum_{r=\lfloor p/3 \rfloor}^{\lfloor 2p/3 \rfloor} \left( \frac{3r}{r, r, r} \right) 3^{-3r}. \]
Putting all these together, we conclude that
\[
S_1 + S_2 + S_3 \equiv p^2 \frac{1}{3} \sum_{r=1}^{\lfloor p/3 \rfloor} \left( \frac{3r}{r, r, r} \right) \left( \frac{-1}{r} - \frac{pN}{3r^2} + \frac{pN(H_{3r} - H_r)}{r} \right) 3^{-3r} \\
+ \frac{(N - 1)}{3} \sum_{r=\lfloor p/3 \rfloor}^{\lfloor 2p/3 \rfloor} \left( \frac{3r}{r, r, r} \right) \frac{3^{-3r}}{r} + 0 \\
\equiv p^2 \frac{1}{3} \sum_{r=\lfloor p/3 \rfloor}^{\lfloor 2p/3 \rfloor} \left( \frac{3r}{r, r, r} \right) 3^{-3r} - N \frac{1}{9} \sum_{r=1}^{\lfloor p/3 \rfloor} \left( \frac{3r}{r, r, r} \right) 3^{-3r} + \frac{N}{3} \cdot 0 \\
\equiv p^2 \left( -3q_p(1/3) + \frac{3p}{2} q_p(1/3)^2 \right) - \frac{N}{9} \left( -\frac{9p}{2} q_p(1/3)^2 \right) \\
\equiv p^2 q_p(1/3) + \frac{p(N - 1)}{2} q_p(1/3)^2,
\]
which is exactly what we expect. The proof is complete. \(\square\)

7. Conclusions and Remarks

In this final section, we extend the congruence on \(a_i(n)\) (for \(i > 0\)), discussed in the earlier sections, from modulo \(p^2\) to modulo \(p^3\). While stating our claim in its generality, we only exhibit proof outlines for the case \(i = 1\) as a prototypical example. We believe the curious researcher would be able to account for the remaining cases.

**Conjecture 7.1.** For \(n, i \in \mathbb{N}^+\) and a prime \(p > 2i\),

\[
a_i(pn) \equiv p^3 (-1)^{i-1} \frac{a_1(pn)}{i^2 \binom{2i-1}{i-1}} \equiv p^3 \frac{(-1)^{i-1} p^2(n+2) a_1(n)}{i^2 \binom{2i-1}{i-1}}.
\]

**Proof.** Ingredients for \(a_1(pn) \equiv p^3 p^2 \binom{n+2}{2} a_1(n)\).

(A) By partial fraction decomposition

\[
a_i(n) = \frac{1}{3i} \sum_{k=0}^{n-1} (-1)^{n-k} \binom{3k}{k} \binom{2k}{k} \binom{n}{3k} \binom{n+k}{k} \binom{n-3k}{i} \binom{3n-3k}{k+i} \\
= (-1)^i a_0(n) + \frac{i}{3i} \sum_{j=1}^{i} (-1)^{j-1} \binom{i-1}{j-1} \binom{n+3j}{i} b_j(n)
\]

where for \(j \in \mathbb{N}^+\),

\[
b_j(n) := \sum_{k=0}^{n-1} (-1)^{n-k} (n-3k) \binom{3k}{k} \binom{2k}{k} \binom{n}{3k} \binom{n+k}{k} \frac{3^{n-3k}}{k+j}.
\]

Thus, \(a_0(np) \equiv p^3 a_0(n)\) implies

\[
a_1(np) = -a_0(np) + \frac{np+3}{3} b_1(np) \equiv p^3 -a_0(n) + \frac{np+3}{3} b_1(np) \\
\equiv p^3 a_1(n) + \frac{np+3}{3} b_1(np) - \frac{p+3}{3} b_1(p).
\]
By summing over $m$, it suffices to show that
\[ b_1(np) \equiv_{p^3} \frac{3}{np + 3} \left( p^2 \binom{n+2}{2} - 1 \right) a_1(n) + \frac{n+3}{np + 3} b_1(n), \]
or, since $a_1(n) = -a_0(n) + (n+3)b_1(n)/3$,
\[ b_1(np) \equiv_{p^3} p^2 \binom{n+3}{3} b_1(n) + \left( 1 - \frac{pm}{3} - \frac{p^2(n+3)(7n+6)}{18} \right) a_0(n). \]

Finally, we can modify a previous proof as follows:
\[ \sum_{r=0}^{p-1} (-1)^r \binom{3pn + 3r}{3pm + 3r} \binom{2pm + 2r}{3pm + 3r} \binom{pn}{3pm + 3r} \frac{3^{-3r}}{pm + r + 1} \equiv_{p^3} \left( \frac{p^2}{m+1} \binom{n+3}{3} + 1 - \frac{pm}{3} - \frac{p^2(n+3)(7n+6)}{18} \right) \cdot \binom{3m}{m} \binom{2m}{m} \binom{n+m}{m} 3^{-N(p-1)}. \]

By summing over $m$, it is immediate to recover (7.1).

In order to prove (7.2), we have the old machinery, $\frac{1}{pm+r+1} \equiv_{p^2} \frac{1}{r+1} - \frac{mp}{(r+1)^2}$, and
\[ \sum_{r=0}^{p-1} \binom{3r}{r,r,r} \frac{3^{-3r}}{r+1} = \frac{9p}{2} \binom{3p}{p,p,p} 3^{-3p} \equiv_{p^2} p - 3p^2 q_p(1/3), \]
\[ \sum_{r=0}^{p-1} \binom{3r}{r,r,r} \frac{3^{-3r}}{(r+1)^2} = \frac{9(9p+2)}{4} \binom{3p}{p,p,p} 3^{-3p} - \frac{9}{2} \equiv_p -\frac{7}{2}. \]

Finally, we can modify a previous proof as follows:
\[ \sum_{r=0}^{p-1} \binom{3r}{r,r,r} \frac{3H_{3r} - H_r}{r+1} 3^{-3r} = \sum_{r=1}^{p-1} \frac{(1/3)_r (2/3)_r}{(1)_r^2} \cdot \frac{1}{r+1} \cdot \sum_{j=0}^{r-1} \left( \frac{1}{1/3 + j} + \frac{1}{2/3 + j} \right) \]
\[ = \sum_{r=1}^{p-1} \frac{1}{r+1} \sum_{k=0}^{r-1} \frac{(1/3)_k (2/3)_k}{(1)_k^2} \cdot \frac{1}{r-k} \]
\[ = \sum_{k=0}^{p-2} \frac{(1/3)_k (2/3)_k}{(1)_k^2} \sum_{r=k+1}^{p-1} \frac{1}{(r+1)(r-k)} \]
\[ = \sum_{k=0}^{p-2} \frac{(1/3)_k (2/3)_k}{(1)_k^2} \left( \frac{1}{k+1} \sum_{r=k+1}^{p-1} \left( \frac{1}{r-k} - \frac{1}{r+1} \right) \right) \]
\[ = \sum_{k=0}^{p-2} \frac{(1/3)_k (2/3)_k}{(1)_k^2} \cdot \frac{H_{p-1-k} - H_p + H_{k+1}}{k+1} \]
\[ \equiv_p \sum_{k=0}^{p-1} \binom{3k}{k,k,k} \frac{H_k - H_p + H_{k+1}}{k+1} 3^{-3k}. \]
which implies that
\[ \sum_{r=0}^{p-1} \binom{3r}{r,r,r} (H_{3r} - H_r) 3^{-3r} \equiv_p \frac{1}{3} \sum_{k=0}^{p-1} \binom{3k}{k,k,k} \frac{(-1/p + 1/(k+1)) 3^{-3k}}{k+1} \equiv_p \frac{1}{3} \left( -1 - \frac{7}{2} \right) = -\frac{3}{2}. \]

\[ \square \]

Remark 7.2. We showed that the conjecture \( a_0(pn) \equiv p^3 a_0(n) \) holds true. Although it is not pursued here, the techniques established in this paper if combined with existing literature on supercongruences (see references below) for binomials of the type \( \binom{p^r n + k}{p^m + j} \), there is enough reliable verity to believe that \( a_0(p^r n) \equiv p^3 a_0(p^{r-1} n) \) should be within easy grasp.

Acknowledgements. The first-named author is grateful to A. Straub for bringing the conjecture on the Almkvist-Zudilin numbers to his attention.

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