Optimized Schwarz Method for Poisson's Equation in Rectangular Domains

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Abstract An analysis of the convergence properties of Optimized Schwarz methods applied as solvers for Poisson's Equation in a bounded rectangular domain with Dirichlet (physical) boundary conditions and transmission conditions on the artificial boundaries of the family $OO0$ is presented. To our knowledge this is the first time that this is done for multiple subdomains in a bounded domain.

1 Introduction

Classical Schwarz methods are Domain Decomposition (DD) methods in which the transmission conditions between subdomains are Dirichlet boundary conditions. Optimized Schwarz methods are DD methods in which the transmission conditions are chosen in such a way to minimize convergence bounds, and thus improve upon the classical method [1, 2, 4]. These transmission conditions are optimized approximations of the optimal transmission conditions, which are obtained by approximating the global Poincaré-Steklov operator by local differential operators. There is more than one family of transmission conditions that can be used for a given PDE (e.g, $OO0$ and $OO2$ for Poisson's equation), each of these families consisting of a particular approximation of the optimal transmission conditions.

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In this paper we analyze the convergence properties of Optimized Schwarz methods (OSM) applied as solvers for Poisson's Equation in a bounded rectangular domain with Dirichlet (physical) boundary conditions and transmission conditions of the family $OO0$. To our knowledge, this is the first time an analysis of convergence of Optimized Schwarz applied to a problem defined in a bounded domain and with arbitrary number of subdomains is presented.

2 Equations of OSM for Poisson's in rectangular domain for $OO0$ case

We want to solve Poisson's equation in a rectangular domain subject to nonhomogeneous Dirichlet boundary conditions, i.e,

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where $\Omega = [0, L1] \times [0, L2]$.

We divide the physical domain into $p \times q$ overlapping rectangular subdomains. To simplify the presentation, we consider square subdomains where each side is of length H and the same overlap on each side, but the analysis presented here is also valid for arbitrary rectangles and arbitrary ovelaps. Each of these subdomains is represented by a pair of indexes, (s, r), with $s \in \{1, ..., p\}$ and $r \in \{1, ..., q\}$. Let h be the length of the side of each subdomain as if it were a partition with no overlap. Let us now displace (outward) each of the boundaries of the nonoverlapping subdomains by a γ amount. We have then overlapping square subdomains with side $H = h + 2\gamma$ and can use γ as a parameter to quantify the amount of overlap between subdomains. The Optimized Schwarz iteration process associated with problem (1) and with $OO0$ transmission conditions is defined, for an interior subdomain (i.e, for $1 < s < p$, $1 < r < q$), by

$$\begin{cases} \Delta u_{n+1}^{s,r} = f & \text{in } \Omega^{s,r} \\ -\frac{\partial u_{n+1}^{s,r}}{\partial x} + \alpha u_{n+1}^{s,r} = -\frac{\partial u_n^{s-1,r}}{\partial x} + \alpha u_n^{s-1,r} & \text{for } x = (s-1)h - \gamma \\ \frac{\partial u_{n+1}^{s,r}}{\partial x} + \alpha u_{n+1}^{s,r} = \frac{\partial u_n^{s+1,r}}{\partial x} + \alpha u_n^{s+1,r} & \text{for } x = sh + \gamma \\ -\frac{\partial u_{n+1}^{s,r}}{\partial y} + \alpha u_{n+1}^{s,r} = -\frac{\partial u_n^{s,r-1}}{\partial y} + \alpha u_n^{s,r-1} & \text{for } y = (r-1)h - \gamma \\ \frac{\partial u_{n+1}^{s,r}}{\partial y} + \alpha u_{n+1}^{s,r} = \frac{\partial u_n^{s,r+1}}{\partial y} + \alpha u_n^{s,r+1} & \text{for } x = rh + \gamma. \end{cases} \tag{2}$$

where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are, in this instance, normal derivatives and $u_{n+1}^{s,r}$ is the solution of the local problem (2) at the $(n+1)$ iteration in $\Omega^{s,r}$. The parameter α is the one which we want to tune to optimize the convergence rate of the method. The exterior subdomains have one or two boundaries that are actually physical (not artificial) boundaries. The equations for the exterior subdomains are similar to (2) with the exception that one or two of the boundary conditions are Dirichlet, namely, the ones associated to the physical boundaries.

3 Recasting equations as a fix point iteration on the error coefficients

By linearity, we can see that the local error (of interior subdomains) of the iteration process is described by (2) with $f = 0$ and $g = 0$. Similar equations can be obtained for exterior subdomains. Using separation of variables, Sturm-Liouville theory and superposition principle, we can write the local errors in the form of a series [3]. Then, using the non-homogeneous boundary conditions in each local problem, we obtain a relationship between the error series coefficients at iteration $(n+1)$ and the ones at iteration n.

Fourier Analysis of solution of system of PDEs defining the local error

We analyze the local error of an interior subdomain, but the same analysis holds for exterior subdomains. Let $\eta_n^{s,r}$ be the local error in $\Omega^{s,r}$ at the iteration n. By superposition principle, we can write $\eta_n^{s,r} = \eta_{n,1}^{s,r} + \eta_{n,2}^{s,r} + \eta_{n,3}^{s,r} + \eta_{n,4}^{s,r}$, where $\eta_{n,i}^{s,r}$, $i = 1,\ldots,4$, is the solution of (2) with $f = 0$, $g = 0$, and with one non-homogeneous boundary condition and the rest homogeneous. Thus, using separation of variables, superposition principle and Sturm-Liouville theory, we can write each part of the local error $\eta_n^{s,r}$ as:

$$\begin{aligned}
\eta_{n,1}^{s,r}(x_\ell,y_\ell) &= \sum_{m=1}^{\infty} \left\{ A_{n,m,1}^{s,r} \left[\frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m x_\ell}{H}\right) + \cos\left(\frac{z_m x_\ell}{H}\right) \right] \times \right. \\
&\qquad \left. \left[\frac{-\bar{\alpha}}{z_m} \sinh\left(\frac{z_m (y_\ell - H)}{H}\right) + \cosh\left(\frac{z_m (y_\ell - H)}{H}\right) \right] \right\} && (3) \\
\eta_{n,2}^{s,r}(x_\ell,y_\ell) &= \sum_{m=1}^{\infty} A_{n,m,2}^{s,r} \left[\frac{\bar{\alpha}}{z_m} \sinh\left(\frac{z_m x_\ell}{H}\right) + \cosh\left(\frac{z_m x_\ell}{H}\right) \right] \left[\frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m y_\ell}{H}\right) + \cos\left(\frac{z_m y_\ell}{H}\right) \right] && (4) \\
\eta_{n,3}^{s,r}(x_\ell,y_\ell) &= \sum_{m=1}^{\infty} A_{n,m,3}^{s,r} \left[\frac{\bar{\alpha}}{z_m} \sinh\left(\frac{z_m y_\ell}{H}\right) + \cosh\left(\frac{z_m y_\ell}{H}\right) \right] \left[\frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m x_\ell}{H}\right) + \cos\left(\frac{z_m x_\ell}{H}\right) \right] && (5) \\
\eta_{n,4}^{s,r}(x_\ell,y_\ell) &= \sum_{m=1}^{\infty} \left\{ A_{n,m,4}^{s,r} \left[\frac{\bar{\alpha}}{z_m} \sin\left(\frac{z_m y_\ell}{H}\right) + \cos\left(\frac{z_m y_\ell}{H}\right) \right] \times \right. \\
&\qquad \left. \left[\frac{-\bar{\alpha}}{z_m} \sinh\left(\frac{z_m (x_\ell - H)}{H}\right) + \cosh\left(\frac{z_m (x_\ell - H)}{H}\right) \right] \right\} && (6)
\end{aligned}$$

where z_m satisfies the transcendental equation

$$\tan(z) = \frac{2z\bar{\alpha}}{\bar{\alpha}^2 - z^2}$$

with $\bar{\alpha} = \alpha H$, x_l and y_l are local coordinates related to the global coordinates x and y by

$$\begin{aligned}
x_\ell &= x - (s-1)h + \gamma \\
y_\ell &= y - (r-1)h + \gamma, && (7)
\end{aligned}$$

and $\left\{\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x_\ell}{H}\right)+\cos\left(\frac{z_m x_\ell}{H}\right)\right\}_{m\in\mathbb{N}}$ and $\left\{\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m y_\ell}{H}\right)+\cos\left(\frac{z_m y_\ell}{H}\right)\right\}_{m\in\mathbb{N}}$ are complete orthogonal sets in $[0,H]$. Therefore, equations (3) and (5) can be seen as Generalized Fourier series in x_l and equations (4) and (6) as Generalized Fourier series in y_l. Then, we have that

$$A^{s,r}_{n,m,1}=\frac{\int_0^H \eta^{s,r}_{n,1}(x_l,y_l)\left[\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x_\ell}{H}\right)+\cos\left(\frac{z_m x_\ell}{H}\right)\right]dx_l}{\left[\frac{-\bar{\alpha}}{z_m}\sinh\left(\frac{z_m(y_\ell-H)}{H}\right)+\cosh\left(\frac{z_m(y_\ell-H)}{H}\right)\right]\int_0^H\left[\frac{\bar{\alpha}}{z_m}\sin\left(\frac{z_m x_\ell}{H}\right)+\cos\left(\frac{z_m x_\ell}{H}\right)\right]^2 dx_l} \tag{8}$$

Let $S_m\subset[0,+\infty)$ be such that $\nu\in S_m$ implies that $z_m^\nu\sin(z_m)\le 1$. Let $\beta:\mathbb{N}\times\mathbb{R}\to\{-1\}\cup[0,1]$ such that

$$\beta(m,\bar{\alpha})=\begin{cases}-1, & \text{if } z_m<1\\ \max(S_m), & \text{if } z_m\ge 1\end{cases}$$

Then, with $y=0$ and using integration by parts in (8) we can write

$$A^{s,r}_{n,m,1}=\frac{B^{s,r}_{n,m,1}}{z_m^{1+\beta(m,\bar{\alpha})}\left[\frac{\bar{\alpha}}{z_m}\sinh\left(\frac{z_m}{H}\right)+\cosh\left(\frac{z_m}{H}\right)\right]},$$

where $B^{s,r}_{n,m,1}$ is uniformly bounded for all $m\in\mathbb{N}$. The same relationship holds between $A^{s,r}_{n,m,i}$ and uniformly bounded quantities $B^{s,r}_{n,m,i}$ for $i\in\{2,3,4\}$. Plugging these equalities in (3)-(6) and applying the nonhomogeneous boundary conditions, we obtain the expression of the coefficients at iteration $(n+1)$ in terms of those at iteration n. For example, with $\bar{\gamma}=\gamma/H$, we have for a specific index k,

$$\begin{aligned}
B^{s,r}_{n+1,k,1} &= \frac{\left(z_k+\frac{\bar{\alpha}^2}{z_k}\right)\sinh(2\bar{\gamma}z_k)+2\bar{\alpha}\cosh(2\bar{\gamma}z_k)}{\left(z_k+\frac{\bar{\alpha}^2}{z_k}\right)\sinh(z_k)+2\bar{\alpha}\cosh(z_k)}B^{s,r-1}_{n,k,1}\\
&+\sum_{m=1}^{\infty}\left\{\frac{4z_k^{1+\beta(k,\bar{\alpha})}\left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m+\frac{\bar{\alpha}^2}{z_m}\right)\sin((1-2\bar{\gamma})z_m)}{\left[\left(z_k+\frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k)+2\bar{\alpha}\right]z_m^{1+\beta(m,\bar{\alpha})}\left(z_m z_k^3+z_k z_m^3\right)}\right.\\
&\left.\frac{\left\{\tanh(z_m)\left[\bar{\alpha}(z_k^2+z_m^2)\sin(z_k)-z_k(\bar{\alpha}^2-z_m^2)\cos(z_k)\right]+z_m(\bar{\alpha}^2+z_k^2)\sin(z_k)\right\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2)-\bar{\alpha}^2\sin(2z_k)+2z_k(\bar{\alpha}^2+z_k^2+\bar{\alpha})-2\bar{\alpha}z_k\cos(2z_k)\right]}B^{s,r-1}_{n,m,2}\right\}\\
&+\frac{\left(-z_k+\frac{\bar{\alpha}^2}{z_k}\right)\sinh((1-2\bar{\gamma})z_k)}{\left(z_k+\frac{\bar{\alpha}^2}{z_k}\right)\sinh(z_k)+2\bar{\alpha}\cosh(z_k)}B^{s,r-1}_{n,k,3}\\
&+\sum_{m=1}^{\infty}\left\{\frac{4z_k^{1+\beta(k,\bar{\alpha})}\left[\frac{\bar{\alpha}}{z_k}\tanh(z_k)+1\right]\left(z_m+\frac{\bar{\alpha}^2}{z_m}\right)\sin((1-2\bar{\gamma})z_m)}{\left[\left(z_k+\frac{\bar{\alpha}^2}{z_k}\right)\tanh(z_k)+2\bar{\alpha}\right]z_m^{1+\beta(m,\bar{\alpha})}\left(z_m z_k^3+z_k z_m^3\right)}\right.\\
&\left.\frac{\left\{\tanh(z_m)z_k(\bar{\alpha}^2+z_m^2)-z_m\left[-2\bar{\alpha}z_k+\frac{(\bar{\alpha}^2-z_k^2)\sin(z_k)+2\bar{\alpha}z_k\cos(z_k)}{\cosh(z_m)}\right]\right\}}{\left[\frac{\bar{\alpha}}{z_m}\tanh(z_m)+1\right]\left[(z_k^2)-\bar{\alpha}^2\sin(2z_k)+2z_k(\bar{\alpha}^2+z_k^2+\bar{\alpha})-2\bar{\alpha}z_k\cos(2z_k)\right]}B^{s,r-1}_{n,m,4}\right\}.
\end{aligned} \tag{9}$$

Let B_n be the infinite vector containing all the error series coefficients at iteration n, i.e., $B_n = (b_{n_1}, b_{n_2}, ...)$ with $b_{n_j} \in \Big\{B^{s,r}_{n,k,i} : s \in \{1,...,p\}, r \in \{1,...,q\}, k \in \mathbb{N}$, $i \in \{1,...,4\}\Big\}$. Then the relation between coefficients can be written as $B_{n+1} = \hat{T} B_n$, where $\hat{T} : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is an infinite matrix. Note that $\hat{T} = (\hat{T}^{1,1}, ..., \hat{T}^{p,q})$, where $\hat{T}^{s,r}$ is a local operator such that $B^{s,r}_{n+1} = \hat{T}^{s,r} B_n$ with $B^{s,r}_{n+1}$ being a vector containing all the error coefficients of the local problem (s,r) at iteration $(n+1)$.

4 Approximation of the infinite operator $\hat{T}$ by a matrix of finite dimensions

Note that the following statements hold

- In the r.h.s. of (9), the factors in front of each coefficient $B^{s,r-1}_{n,k,i}$, $i = 1,3$, decrease with k.
- In the r.h.s. of (9), the factors in front of each coefficient $B^{s,r-1}_{n,m,i}$, where m is the index of the series and $i = 2,4$, decrease with k and m.
- For a given $n \in \mathbb{N}_0$, $B^{s,r-1}_{n,m,i}$ is uniformly bounded in $m \in \mathbb{N}$ and $i = 1,...,4$.
- For any number $\delta > 0$ there exists a number k_δ, such that for $k > k_\delta$, the sum of the absolute values of the factors of all the coefficients $B^{s,r-1}_{n,m,i}$, $i = 2,4$, and $B^{s,r-1}_{n,k,i}$, $i = 1,3$, in the r.h.s. of (9) is less than δ.

Let $(B_n)_{|k \le k_\delta}$ denote the vector resulting after discarding all the entries of B_n corresponding to $k > k_\delta$. Then, based on the above three facts, we can write

$$(B_{n+1})_{|k \le k_\delta} = \left(\hat{T}(B_n)\right)_{|k \le k_\delta} = \tilde{T}\left((B_n)_{|k \le k_\delta}\right) + \xi_{n+1,k_\delta}((B_n)_{|k > k_\delta}), \tag{10}$$

where $\tilde{T}$ is a finite matrix obtained by discarding the rows and columns of $\hat{T}$ related to the coefficients pertaining to $k > k_\delta$, and $\xi_{n+1,k_\delta}((B_n)_{|k > k_\delta})$ is the error obtained by approximating $(B_{n+1})_{|k \le k_\delta}$ by $\tilde{T}\left((B_n)_{|k \le k_\delta}\right)$.

We will discuss in the next section situations in which $\rho(\tilde{T}) < 1$, i.e., the spectral radius of $\tilde{T}$ is less than one. In the rest of this section we show that in addition the error $\xi_{n+1,k_\delta}((B_n)_{|k > k_\delta})$ tends to zero as $n \to \infty$, and consequently $B_n \to 0$ as $n \to \infty$.

A necessary condition for convergence of Optimized Schwarz is that $B_n \to 0$ as $n \to \infty$. Note that each entry of $\xi_{n+1,k_\delta}((B_n)_{|k > k_\delta})$ is the truncation error that results after truncating the series in the formulas of the coefficients $B^{s,r}_{n+1,k,i}$, $k \le k_\delta$, by keeping only the terms corresponding to $k \le k_\delta$. Thus, as it can be seen in (9), $\xi_{n+1,k_\delta}((B_n)_{|k > k_\delta})$ is just a linear combination of the entries of $(B_n)_{|k > k_\delta}$. Note also that the entries of $(B_n)_{|k > k_\delta}$ are linear combinations of the entries of B_{n-1}. Therefore, by choosing a small enough δ (of course $\delta < 1$) and a large enough k_δ, one can obtain the following estimates

$$||(B_n)_{|k>k_\delta}||_\infty \leq \delta||B_{n-1}||_\infty, \tag{11}$$

$$||\xi_{n+1,k_\delta}((B_n)_{|k>k_\delta})||_\infty \leq \delta^2||\tilde{T}^{n-1}\left((B_0)_{|k\leq k_\delta}\right)||_\infty + O(\delta^3), \tag{12}$$

Using equation (10) recursively, we obtain the following equation

$$(B_{n+1})_{|k\leq k_\delta} = \tilde{T}^{n+1}((B_0)_{|k\leq k_\delta}) + \sum_{j=1}^{n+1} \tilde{T}^{n+1-j}(\xi_{j,k_\delta}((B_{j-1})_{|k>k_\delta})). \tag{13}$$

Using (11), (12) and (13), and assuming that the spectral radius of $\tilde{T}$ is less than one, it can be shown that there exists a n_δ such that $||B_n||_\infty \leq \delta||B_0||_\infty$ for all $n \geq n_\delta$. Repeating this argument, we can then show that $\lim_{n\to\infty} B_n = 0$. Hence in order to prove that $B_n \to 0$ as $n \to \infty$, it suffices to show that $\rho(\tilde{T}) < 1$.

5 Spectral Radius of $\tilde{T}$

The spectral radius of $\tilde{T}$ describes the convergence rate of the Optimized Schwarz method. Thus, we define the optimal $\bar{\alpha} = \alpha H$ as the one which minimizes the spectral radius of $\tilde{T}$ and thus gives the optimal convergence rate.

The values of the entries of the matrix $\tilde{T}$ depend on $\bar{\gamma}$, $\bar{\alpha}$ and k_δ. The structure of the matrix depends on k_δ, p, q and the way we order the entries of B_n, i.e., the way we order each coefficient $B^{s,r}_{n,k,i}$ based on its values of s, r, k and i. For the ordering we have chosen, we computed the spectral radius of the resulting matrix $\tilde{T}$, for $\bar{\gamma} \in \{0.01, 0.04, 0.08\}$, a set of values of $\bar{\alpha}$ in the range $[0.1, 500]$, $k_\delta \in \{20, 50, 100, 200\}$, and $p, q \in \{5, 10, 20, 30\}$. In these computations we have observed the following.

1. There exist values of $\bar{\alpha}$ for which the spectral radius of $\tilde{T}$ is less than one.
2. For a given $\bar{\gamma}$ and the range of $\bar{\alpha}$ considered in the experiments, $\rho(\tilde{T})$ has two local minima and it approaches a constant less than one for large values of $\bar{\alpha}$.
3. Given $\bar{\gamma}$, $\bar{\alpha}$, p and q, the value of $\rho(\tilde{T})$ remains practically constant for all $k_\delta \in \{20, 50, 100, 200\}$ (see Figure 3)
4. For a given $\bar{\gamma}$, the optimal spectral radius of $\tilde{T}$ remains practically constant as p and q increase.

In Figures 1 and 2, the results for the cases $\bar{\gamma} = 0.01$ and $\bar{\gamma} = 1/30$, with $p, q = 10$, $k_\delta = 20$, $\bar{\alpha} \in [1, 10]$, are shown.

6 Convergence of Optimized Schwarz

It can be shown that the series describing the local errors converge uniformly in $\Omega^{s,r}$. This implies that if each term of the error series goes to zero as n goes to infinity, so

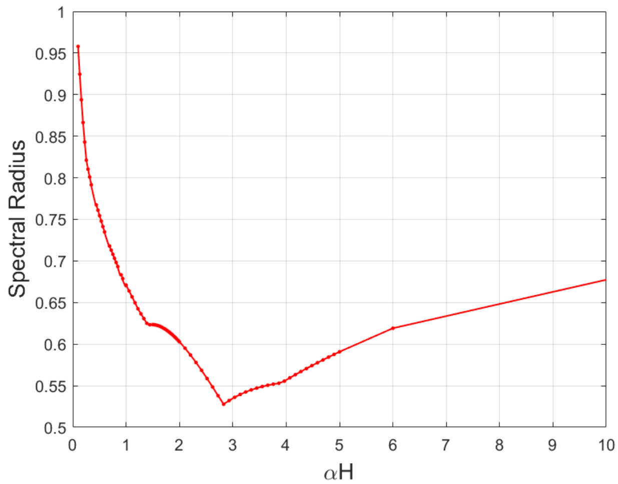

Fig. 1 Spectral radius of $\tilde{T}$ for $p,q = 10$, $k_\delta = 20$, $\bar{\gamma} = 0.01$ and $\bar{\alpha} \in [1,10]$

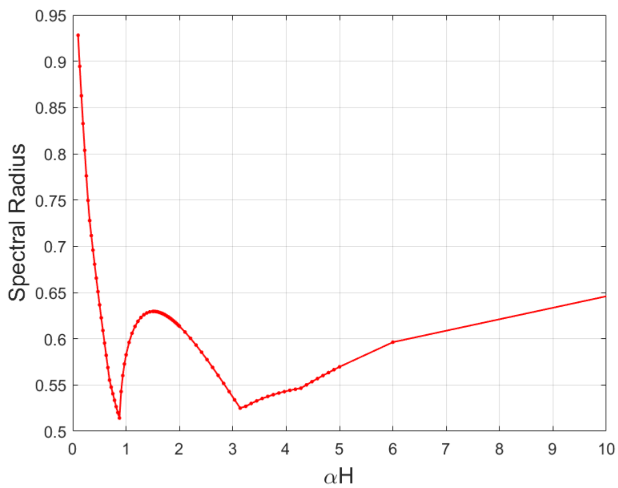

Fig. 2 Spectral radius of $\tilde{T}$ for $p,q = 10$, $k_\delta = 20$, $\bar{\gamma} = 1/30$ and $\bar{\alpha} \in [1,10]$

will do the series. From the previous section, we know that there exist values of $\bar{\alpha}$ for which the spectral radius of $\tilde{T}$ is less than one. This implies, as stated in section 4, that $B_n \to 0$ as $n \to \infty$. Consequently, $\lim_{n\to\infty} B^{s,r}_{n,m,i} = 0$ and $\lim_{n\to\infty} A^{s,r}_{n,m,i} = 0$, i.e., the coefficients of the error series go to zero as n goes to infinity. Therefore, the error of the iterative process converges to zero as n goes to infinity, which means that Optimized Schwarz converges for the given Poisson's problem for any initial error.

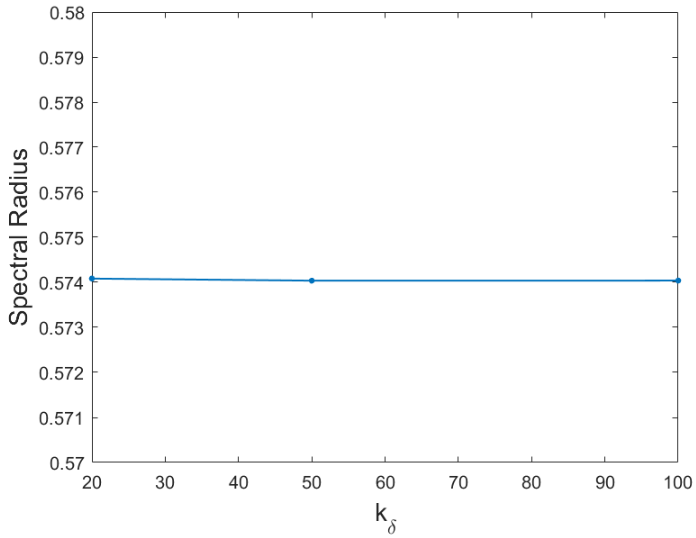

Fig. 3 Spectral radius of $\tilde{T}$ vs. k_δ for $p,q = 10$, $k_\delta = 20$, $\bar{\gamma} = 0.01$ and $\bar{\alpha} = 2.3684$

7 Conclusion

We analyzed the convergence of the Optimized Schwarz method applied to Poisson's equation in a bounded rectangular domain subject to nonhomogeneous Dirichlet boundary conditions and transmission conditions of the family *OO*0. The spectral radius of $\tilde{T}$ can be less than one if there is enough overlap. One can obtain the optimal Robin parameter that minimizes this spectral radius. We outlined a proof showing that this bound on the spectral radius, together with other results, can guarantee convergence of OSM for the problem studied.

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