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in Rectangular Domains

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SYNCHRONOUS AND ASYNCHRONOUS OPTIMIZED SCHWARZ METHODS FOR POISSON’S EQUATION IN RECTANGULAR DOMAINS

JOSÉ C. GARAY†, FRÉDÉRIC MAGOULÈS‡, AND DANIEL B. SZYLD‡

Abstract. Convergence results on Optimized Schwarz methods (OSM) applied as solvers for Poisson’s Equation in a bounded rectangular domain with Dirichlet (physical) boundary conditions and artificial transmission conditions of the family $OOO$ between subdomains are presented. The analysis presented applies to an arbitrary number of subdomains. Both synchronous and asynchronous versions of OSM are discussed. Numerical experiments illustrate the theoretical results.

Key words. Asynchronous iterations, Optimized Schwarz methods, Infinite-dimensional Operator.

AMS subject classifications. 65F10, 65N22, 65N55

1. Introduction. Our overarching goal is to solve very large linear systems arising from the discretization of PDEs using parallel iterative methods in extreme-scale supercomputers.

Synchronous iterative algorithms are parallel iterative algorithms in which iterations and communications are synchronized among processors. In this synchronous paradigm, any load imbalance or non-uniformity in hardware performance causes processing units to idle at the synchronization point, waiting for the slowest unit, and impacts performance. Thus, given the heterogeneous and distributed architecture of the anticipated exascale computers, idle times in processing units will be an issue in terms of efficiency.

State-of-the-art solvers based on appropriately preconditioned Krylov Subspace methods are very fast (in terms of iteration count), but they are inherently synchronous methods. Consequently, the communication among processors is expected to be the bottleneck in future supercomputers (when these methods are used as outer solvers), implying low efficiency and large execution times.

Asynchronous iterative algorithms are parallel iterative algorithms in which communications and iterations are not synchronized among processors [7]. Thus, as soon as a processing unit finishes its own calculations, it starts the next cycle with the latest data received during a previous cycle, without waiting for any other processing unit. These algorithms increase the number of updates in some processors (with respect to the synchronous case) but suppress all the idle times. This usually results in a reduction of the (execution) time to achieve convergence.

Classical Schwarz methods are Domain Decomposition (DD) methods in which the transmission conditions between subdomains are Dirichlet boundary conditions [6, 18]. Optimized Schwarz methods are DD methods in which the transmission conditions are chosen in such a way to minimize convergence bounds, and thus improve upon the classical method [6, 8, 13]. These transmission conditions are optimized approximations of the optimal transmission conditions which are obtained by approx-
imating the global Poincaré-Steklov operator by local differential operators. There is more than one family of transmission conditions that can be used for a given PDE, e.g. the zeroth order optimized interface condition \( \text{OO0} \) and the second order condition \( \text{OO2} \) for the Poisson’s equation; see, e.g. [8]. Each of these families consists of a particular approximation of the optimal transmission conditions.

We are exploring the use of Optimized Schwarz methods as outer solvers for the solution of PDEs. These type of outer solvers are fast and can be implemented asynchronously; see [9, 16]. In this paper, we analyze the convergence properties of an Asynchronous Optimized Schwarz method applied as a solver for Poisson’s Equation in a bounded rectangular domain with Dirichlet (physical) boundary conditions.

In [5], a convergence analysis of Optimized Schwarz is presented for a bounded domain with multiple subdomains, for the case in which the subdomains form a one-dimensional array and in which each overlapped region is shared by two subdomains. In [10] we have analyzed for the synchronous case the convergence of Optimized Schwarz for a problem defined in a bounded domain and for an arbitrary number of subdomains, when the subdomains form a two-dimensional array containing cross points. In this paper, a more detailed version of that analysis is presented and the analysis is extended to include the asynchronous case. The standard results one uses for the analysis of asynchronous iterations are those in [3, 4, 17] (see also [7]), where the iteration operator is finite dimensional. In this work, we deal with an infinite dimensional operator \( \hat{T} \). Thus, none of the theorems from those references apply to our current case. Our convergence proof of the asynchronous method, although inspired in part by [4], is new.

Our convergence proofs in sections 4 and 6 are based on the hypothesis that the spectral radii of certain finite dimensional operators \( \hat{T}_3 \) and \( |\hat{T}_3| \) are each less than one (see the definition of these operators in the mentioned sections). We have computed the spectral radii of these operators numerically and we show the results in section 7 as an evidence that these assumptions hold.

The paper is organized as follows. In section 2 we present the description of the problem to solve. In section 3 we show how to recast the original problem into an equivalent problem that is easier to analyze, and which contains an infinite dimensional operator \( \hat{T} \) (infinite matrix) acting on infinite vectors. In section 4, details on some properties of \( \hat{T} \) are given and we show that it is possible to study the convergence of the iterative methods by studying the spectral properties of a truncated version of the operator \( \hat{T} \). In section 5, the model of asynchronous iterations is presented. Our main convergence results are presented in sections 6 and 7. Numerical Experiments are in section 8 followed by the conclusions in section 9.

2. OS for Poisson’s in rectangular domain for the OO0 case. We want to solve Poisson’s equation in a rectangular domain subject to nonhomogeneous Dirichlet boundary conditions, i.e,

\[
\begin{aligned}
-\Delta u &= f & \text{in } \Omega, \\
u &= g & \text{on } \partial \Omega.
\end{aligned}
\]

where \( \Omega = [0, L_1] \times [0, L_2] \).

We divide the physical domain into \( p \times q \) overlapping rectangular subdomains. To simplify the presentation, we consider square subdomains where each side is of length \( H \) and the same overlap on each side, but the analysis presented here is also valid for arbitrary rectangles and arbitrary overlaps. Each of these subdomains is represented by a pair of indexes, \( (s, r) \), with \( s \in \{1, \ldots, p\} \) and \( r \in \{1, \ldots, q\} \). Let \( h \) be the length
of the side of each subdomain as if it were a partition with no overlap. Let us now displace (outward) each of the boundaries of the nonoverlapping subdomains by an amount $\gamma$. We have then overlapping square subdomains with side $H = h + 2\gamma$ and can use $\gamma$ as a parameter to quantify the amount of overlap between subdomains.

The Optimized Schwarz (OS) iteration process associated with problem (1) and with $OO0$ transmission conditions is defined, for an interior subdomain (i.e., for $1 < s < p$, $1 < r < q$), by

$$
\begin{align*}
\frac{-\Delta u_{n+1}^{s,r}}{\partial x} &= f & \text{in } \Omega^{s,r} \\
\frac{\partial^2 u_{n+1}^{s,r}}{\partial x^2} + \alpha u_{n+1}^{s,r} &= -\frac{\partial u_{n+1}^{s-1,r}}{\partial x} + \alpha u_{n+1}^{s-1,r} & \text{for } x = (s-1)h - \gamma \\
\frac{\partial^2 u_{n+1}^{s,r}}{\partial x^2} + \alpha u_{n+1}^{s,r} &= -\frac{\partial u_{n+1}^{s+1,r}}{\partial x} + \alpha u_{n+1}^{s+1,r} & \text{for } x = sh + \gamma \\
\frac{\partial^2 u_{n+1}^{s,r}}{\partial y^2} + \alpha u_{n+1}^{s,r} &= -\frac{\partial u_{n+1}^{s,r-1}}{\partial y} + \alpha u_{n+1}^{s,r-1} & \text{for } y = (r-1)h - \gamma \\
\frac{\partial^2 u_{n+1}^{s,r}}{\partial y^2} + \alpha u_{n+1}^{s,r} &= -\frac{\partial u_{n+1}^{s+1,r}}{\partial y} + \alpha u_{n+1}^{s+1} & \text{for } x = rh + \gamma,
\end{align*}
$$

where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are, in this instance, normal derivatives\(^1\). The parameter $\alpha$ is the one which we want to optimize, so as to minimize the convergence bounds. The exterior subdomains have one or two boundaries that are actually physical (not artificial) boundaries. The equations for the exterior subdomains are similar to (2) with the exception that one or two of the boundary conditions are Dirichlet, namely, the ones associated to the physical boundaries.

![Partition of Domain](image)

**Fig. 1. Partition of Domain**

We want to study the convergence of the OS iteration (2). To that end, in the next sections, we write an equation similar to (2) for the error $\eta_n = u_n - u^*$, where

---

\(^1\)The usual formulation of the $OO0$ condition is with the normal derivative across the artificial interfaces
$u^*$ satisfies (1). We write the restriction of this error to each subdomain in terms of a generalized Fourier series and then recast the iteration (2) (with $f = 0$) as an iteration on the vector containing the coefficients of the generalized Fourier Series.

3. Recasting equations as a fix point iteration on the error coefficients. We begin by analyzing the local error of an interior subdomain, but the same analysis holds for exterior subdomains. Let $\eta_{n}^{s,r}$ be the local error after $n$ iterations corresponding to the subdomain $(s, r)$. By linearity, we can see that the local error (of interior subdomains) of the iteration process is described by (2) with $f = 0$. Furthermore, by superposition principle, we can write $\eta_{n}^{s,r} = \eta_{n,1}^{s,r} + \eta_{n,2}^{s,r} + \eta_{n,3}^{s,r} + \eta_{n,4}^{s,r}$, where $\eta_{n,i}^{s,r}, i = 1, ..., 4$, is the solution of (2) with $f = 0$ and with one non-homogeneous boundary condition and the rest homogeneous. Thus, using separation of variables, superposition principle and Sturm-Liouville theory, we can write each part of the local error $\eta_{n}^{s,r}$ as

$$\eta_{n,1}^{s,r}(x_l, y_l) = \sum_{m=1}^{\infty} \{ A_{n,m,1}^{s,r} \left[ \frac{\alpha}{z_m} \sin \left( \frac{2m x_l}{H} \right) + \cos \left( \frac{2m x_l}{H} \right) \right] \times$$

$$\left[ -\frac{\alpha}{z_m} \sinh \left( \frac{z_m(y_l - H)}{H} \right) + \cosh \left( \frac{z_m(y_l - H)}{H} \right) \right] \}$$

(3)

$$\eta_{n,2}^{s,r}(x_l, y_l) = \sum_{m=1}^{\infty} \{ A_{n,m,2}^{s,r} \left[ \frac{\alpha}{z_m} \sinh \left( \frac{z_m x_l}{H} \right) + \cosh \left( \frac{z_m x_l}{H} \right) \right] \left[ \frac{\alpha}{z_m} \sin \left( \frac{z_m y_l}{H} \right) + \cos \left( \frac{z_m y_l}{H} \right) \right] \}$$

(4)

$$\eta_{n,3}^{s,r}(x_l, y_l) = \sum_{m=1}^{\infty} \{ A_{n,m,3}^{s,r} \left[ \frac{\alpha}{z_m} \sinh \left( \frac{z_m y_l}{H} \right) + \cosh \left( \frac{z_m y_l}{H} \right) \right] \left[ \frac{\alpha}{z_m} \sin \left( \frac{z_m x_l}{H} \right) + \cos \left( \frac{z_m x_l}{H} \right) \right] \}$$

(5)

$$\eta_{n,4}^{s,r}(x_l, y_l) = \sum_{m=1}^{\infty} \{ A_{n,m,4}^{s,r} \left[ \frac{\alpha}{z_m} \sin \left( \frac{z_m y_l}{H} \right) + \cos \left( \frac{z_m y_l}{H} \right) \right] \times$$

$$\left[ -\frac{\alpha}{z_m} \sinh \left( \frac{z_m(x_l - H)}{H} \right) + \cosh \left( \frac{z_m(x_l - H)}{H} \right) \right] \} \},$$

(6)

where $z_m$ satisfies the transcendental equation

$$\tan(z) = \frac{2z\bar{\alpha}}{\bar{\alpha}^2 - z^2}$$

with $\bar{\alpha} = \alpha H$ (thus, $\bar{\alpha}$ is a parameter scaled with the width of the subdomain $H$).

Note that $0 < z_1 < z_2 < \ldots$. The variables $x_l$ and $y_l$ are local coordinates related to the global coordinates $x$ and $y$ by

$$x_l = x - (s - 1)h + \gamma$$

$$y_l = y - (r - 1)h + \gamma,$$

and $\left\{ \frac{\alpha}{z_m} \sin \left( \frac{2m y_l}{H} \right) + \cos \left( \frac{2m y_l}{H} \right) \right\}_{m \in \mathbb{N}}$ and $\left\{ \frac{\alpha}{z_m} \sin \left( \frac{2m x_l}{H} \right) + \cos \left( \frac{2m x_l}{H} \right) \right\}_{m \in \mathbb{N}}$ are complete orthogonal sets in $[0, H]$. Therefore, equations (3) and (5) can be seen as Generalized Fourier series in $x_l$ and equations (4) and (6) as Generalized Fourier series in $y_l$. Then, we have that

(9)$A_{n,m,1}^{s,r} =

$$\int_0^H \eta_{n,1}^{s,r}(x_l, y_l) \left[ \frac{\alpha}{z_m} \sin \left( \frac{z_m x_l}{H} \right) + \cos \left( \frac{z_m x_l}{H} \right) \right] dx_l$$

$$\left[ -\frac{\alpha}{z_m} \sinh \left( \frac{z_m(y_l - H)}{H} \right) + \cosh \left( \frac{z_m(y_l - H)}{H} \right) \right] \int_0^H \left[ \frac{\alpha}{z_m} \sin \left( \frac{z_m x_l}{H} \right) + \cos \left( \frac{z_m x_l}{H} \right) \right]^2 dx_l.$$
Our goal is to show that the series in (3)-(6) converge uniformly to zero. To that end, we want to express the coefficients of the series as a quotient with a denominator having \( z_m^{1+\beta} \) for some \( \beta > 0 \). Let \( S_m = \{ \nu \in [0, \infty) : |z_m^\nu \sin(z_m)| \leq 1 \} \). Let \( \beta : \mathbb{N} \times \mathbb{R} \to \{-1\} \cup [0,1] \) such that

\[
\beta(m, \hat{\alpha}) = \begin{cases} 
-1, & \text{if } z_m < 1 \\
\max(S_m), & \text{if } z_m \geq 1.
\end{cases}
\]

Let \( \bar{\sigma} : \mathbb{N} \times [0, \infty) \to [0, 1] \) such that \( \bar{\sigma}(m, \hat{\alpha}) = z_m^{\beta(m, \hat{\alpha})} \sin(z_m) \). Note that, using integration by parts, we have

\[
(10) \int_0^H \frac{\bar{\alpha}H}{z_m}\sin\left(\frac{z_m x_t}{H}\right) \eta_{m, 1}^{s, r}(x_t, y_t) dx_t = \\
= \frac{\bar{\alpha}H}{z_m}\left[ -\eta_{m, 1}^{s, r}(H, y_t) \cos(z_m) + \eta_{m, 1}^{s, r}(0, y_t) + \int_0^H \frac{\bar{\alpha}H}{z_m}\cos\left(\frac{z_m x_t}{H}\right) \frac{\partial \eta_{m, 1}^{s, r}(x_t, y_t)}{\partial x} dx_t \right]
\]

and

\[
(11) \int_0^H \cos\left(\frac{z_m x_t}{H}\right) \eta_{m, 1}^{s, r}(x_t, y_t) dx_t = \\
= \frac{H}{z_m}\left[ \eta_{m, 1}^{s, r}(H, y_t) \sin(z_m) - \int_0^H \sin\left(\frac{z_m x_t}{H}\right) \frac{\partial \eta_{m, 1}^{s, r}(x_t, y_t)}{\partial x} dx_t \right] + \\
+ \int_0^H \frac{\bar{\alpha}H}{z_m}\cos\left(\frac{z_m x_t}{H}\right) \frac{\partial^2 \eta_{m, 1}^{s, r}(x_t, y_t)}{\partial x^2} dx_t = \\
+ \frac{H}{z_m}\eta_{m, 1}^{s, r}(H, y_t) \frac{\bar{\sigma}(m, \hat{\alpha})}{z_m^{\beta(m, \hat{\alpha})}} - \left(\frac{H}{z_m}\right)^2 \left[ -\frac{\partial \eta_{m, 1}^{s, r}(H, y_t) \cos(z_m)}{\partial x} + \frac{\partial \eta_{m, 1}^{s, r}(0, y_t)}{\partial x} \right] + \\
+ \int_0^H \frac{\bar{\alpha}H}{z_m}\cos\left(\frac{z_m x_t}{H}\right) \frac{\partial^2 \eta_{m, 1}^{s, r}(x_t, y_t)}{\partial x^2} dx_t \right].
\]

Using (10) and (11) we have

\[
(12) \int_0^H \eta_{m, 1}^{s, r}(x_t, y_t) \left[ \frac{\bar{\alpha}H}{z_m}\sin\left(\frac{z_m x_t}{H}\right) + \cos\left(\frac{z_m x_t}{H}\right) \right] dx_t = \\
= \frac{\bar{\alpha}H}{z_m}\left[ -\eta_{m, 1}^{s, r}(H, y_t) \cos(z_m) + \eta_{m, 1}^{s, r}(0, y_t) + \int_0^H \frac{\bar{\alpha}H}{z_m}\cos\left(\frac{z_m x_t}{H}\right) \frac{\partial \eta_{m, 1}^{s, r}(x_t, y_t)}{\partial x} dx_t \right] + \\
+ \eta_{m, 1}^{s, r}(H, y_t) H \frac{\bar{\sigma}(m, \hat{\alpha})}{z_m^{1+\beta(m, \hat{\alpha})}} - \left(\frac{H}{z_m}\right)^2 \left[ -\frac{\partial \eta_{m, 1}^{s, r}(H, y_t) \cos(z_m)}{\partial x} + \frac{\partial \eta_{m, 1}^{s, r}(0, y_t)}{\partial x} \right] + \\
+ \int_0^H \frac{\bar{\alpha}H}{z_m}\cos\left(\frac{z_m x_t}{H}\right) \frac{\partial^2 \eta_{m, 1}^{s, r}(x_t, y_t)}{\partial x^2} dx_t \right].
\]
\[
\frac{1}{z_m} \left\{ \frac{\hat{\alpha}}{1-\hat{\beta}(m,\alpha)} \left[-\eta_{n,1}^{s,r}(H, y_\ell) \cos(z_m) + \eta_{n,1}(0, y_\ell) + \int_0^H \frac{\hat{\alpha}}{z_m} \cos \left( \frac{z_m x_\ell}{H} \right) \frac{\partial \eta_{n,1}^{s,r}(x_\ell, y_\ell)}{\partial x} dx_\ell \right] \right. \\
\left. + \eta_{n,1}^{s,r}(H, y_\ell) H \sigma(m, \bar{\alpha}) - \left( \frac{H^2}{z_m} \right) \left[ - \frac{\partial \eta_{n,1}^{s,r}(H, y_\ell) \cos(z_m) + \partial \eta_{n,1}(0, y_\ell)}{\partial x} \right. \right.
\]
\[
+ \left. \int_0^H \frac{\hat{\alpha}}{z_m} \cos \left( \frac{z_m x_\ell}{H} \right) \frac{\partial^2 \eta_{n,1}^{s,r}(x_\ell, y_\ell)}{\partial x^2} dx_\ell \right\}.
\]

Then, with \( y_\ell = 0 \) and using (12) we can write
\[
A_{n,m,1}^{s,r} = \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\bar{\alpha})}} \left[ \frac{\hat{\alpha}}{z_m} \sin(z_m) + \cosh(z_m) \right],
\]with \( B_{n,m,1}^{s,r} = N_{n,m,1}^{s,r} / D_{n,m,1}^{s,r} \), where
\[
N_{n,m,1}^{s,r} = \frac{\hat{\alpha}}{z_m^{1-\beta(m,\bar{\alpha})}} \left[ -\eta_{n-1}^{s,r-1}(H,0) \cos(z_m) + \eta_{n-1}^{s,r-1}(0,0) + \right.
\]
\[
\left. \int_0^H \frac{\partial \eta_{n-1}^{s,r-1}(x,0) \cos(z_m)}{\partial x} \right] \left[ H + \eta_{n-1}^{s,r-1}(H,0) \bar{\sigma}(m, \bar{\alpha}) H - \right.
\]
\[
\left. \frac{H^2}{z_m^{1-\beta(m,\bar{\alpha})}} \left[ - \frac{\partial \eta_{n-1}^{s,r-1}(L,0)}{\partial x} \cos(z_m) + \frac{\partial \eta_{n-1}^{s,r-1}(0,0)}{\partial x} \right. \right.
\]
\[
\left. \int_0^H \frac{\partial^2 \eta_{n-1}^{s,r-1}(x,0) \cos(z_m)}{\partial x^2} \right] \right.
\]
\[
\text{and}
\]
\[
D_{n,m,1}^{s,r} = \int_0^H \left[ \frac{\hat{\alpha}}{z_m} \sin \left( \frac{z_m x_\ell}{H} \right) + \cos \left( \frac{z_m x_\ell}{H} \right) \right]^2 dx_\ell.
\]

We have that
\[
D_{n,m,1}^{s,r} = \frac{H \left( -\alpha^2 \sin(2z_m) + 2\alpha z_m (\alpha - \cos(2z_m) + 1) + \frac{z_m^2 \sin(2z_m)}{4z_m^3} \right)}{2} H + \frac{H^2}{2}
\]

Note that \( D_{n,m,1}^{s,r} \geq 0 \) for all \( m \in \mathbb{N} \) and that the first term in (14) goes to zero as \( m \) goes to infinity. Then, there exist an \( \hat{m} \) such that for all \( m \geq \hat{m} \) we have \( D_{n,m,1}^{s,r} \geq H/4 \). Let
\[
\bar{\omega} = \min_{m \in \{1,...,\hat{m}\}} D_{n,m,1}^{s,r}.
\]

Then, \( \bar{\omega} > 0 \) (since it is the minimum of a finite set of positive numbers), and for all \( m \in \mathbb{N} \) we have \( D_{n,m,1}^{s,r} \geq \bar{\omega} \) and thus,
\[
\frac{1}{D_{n,m,1}^{s,r}} \leq \frac{1}{\bar{\omega}} < \infty.
\]

Note that \( \eta_{n-1}^{s,r-1} \) is harmonic and \([0,L]^2\) is compact. Consequently, \( \eta_{n-1}^{s,r-1} \) and all its derivatives are bounded. Note also that \( \bar{\sigma} \leq 1 \) and \( 0 < z_1 < z_2 < ... \). Then, we have

\[
\text{...}
\]
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\[
(16) \quad |N_{n,m}^{s,r}| \leq \frac{\bar{\alpha}}{z_1(1-\beta(1,\alpha))} \left[ 2||\eta_n^{s,r-1}||_\infty + ||\frac{\partial \eta_n^{s,r-1}}{\partial x}||_\infty H \right] H \\
+ ||\eta_n^{s,r-1}||_\infty H + \frac{H^2}{z_1(1-\beta(1,\alpha))} \left[ 2||\frac{\partial \eta_n^{s,r-1}}{\partial x}||_\infty + H||\frac{\partial^2 \eta_n^{s,r-1}}{\partial x^2}||_\infty \right].
\]

Thus, from (15) and (16), we have that for all \( m \in \mathbb{N} \)

\[
|B_{n,m,1}^{s,r}| = \left| \frac{N_{n,m,1}^{s,r}}{D_{n,m,1}^{s,r}} \right| \leq \frac{1}{\omega} \left\{ \frac{\alpha}{z_1(1-\beta(1,\alpha))} \left[ 2||\eta_n^{s,r-1}||_\infty + ||\frac{\partial \eta_n^{s,r-1}}{\partial x}||_\infty H \right] H \\
+ ||\eta_n^{s,r-1}||_\infty H + \frac{H^2}{z_1(1-\beta(1,\alpha))} \left[ 2||\frac{\partial \eta_n^{s,r-1}}{\partial x}||_\infty + H||\frac{\partial^2 \eta_n^{s,r-1}}{\partial x^2}||_\infty \right] \right\}.
\]

i.e., \( B_{n,m,1}^{s,r} \) is uniformly bounded for all \( m \in \mathbb{N} \). Similar type of bounds can be obtained for \( B_{n,m}^{s,r} \) with \( i \in \{ 2, 3, 4 \} \). Plugging (13) into (3)-(6), applying the nonhomogeneous boundary conditions and the orthogonality property of the set \( \left\{ \frac{\alpha}{z_m} \sin \left( \frac{2\pi z_k}{H} \right) + \cos \left( \frac{2\pi z_k}{H} \right) \right\}_{m \in \mathbb{N}} \), we obtain the expression of the coefficients at iteration \((n+1)\) in terms of those at iteration \(n\). For example, with \( \gamma = \gamma/H \), we have

\[
(17) \quad B_{n+1,1,1}^{s,r} = \left( z_k + \frac{\bar{\alpha}}{z_k} \right) \sinh \left( 2\gamma z_k \right) + 2\bar{\alpha} \cosh \left( 2\gamma z_k \right) B_{n,1,1}^{s,r-1} \\
+ \infty \sum_{m=1} \left\{ \frac{4z_k^{1+\beta(m,\bar{\alpha})}}{z_k} \tanh(z_k) + 1 \right\} \left[ \frac{z_m + \frac{\bar{\alpha}}{z_m}}{z_k + \frac{\bar{\alpha}}{z_k}} \tanh(z_k) + 2\bar{\alpha} \right] N_{n,m,1}^{s,r-1} \\
+ \infty \sum_{m=1} \left\{ \frac{\alpha}{z_m} \tanh(z_m) + 1 \right\} \left[ \frac{\bar{\alpha}}{z_m} \sinh(2z_k) + 2z_k(\bar{\alpha}^2 + z_k^2 + \bar{\alpha}) - 2\bar{\alpha}z_k \cos(2z_k) \right] B_{n,m,1}^{s,r-1} \\
+ \infty \sum_{m=1} \left\{ \frac{\bar{\alpha}}{z_m} \sinh(2z_k) + 2z_k(\bar{\alpha}^2 + z_k^2 + \bar{\alpha}) - 2\bar{\alpha}z_k \cos(2z_k) \right\} B_{n,m,1}^{s,r-1}.
\]

In the derivation of the expressions of these coefficients we have interchanged the order of integrals, derivatives and infinite summation. See Appendix B for the justification of these procedures.

Let \( B_n \) be the infinite vector containing all the error series coefficients at iteration \( n \), i.e., \( B_n = (b_{n_1}, b_{n_2}, \ldots) \) with \( b_{n_j} \in \{ B_{n,k,i}^{s,r} \} \) for \( s \in \{ 1, \ldots, p \}, r \in \{ 1, \ldots, q \}, k \in \mathbb{N} \).
Then the relation between coefficients can be written as $B_{n+1} = \hat{T} B_n$, where $\hat{T}: \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ is an infinite matrix. Note that $\hat{T} = (\hat{T}_{p,q}^{s,r})$, where $\hat{T}_{s,r}$ is a local operator such that $B_{n+1}^{s,r} = \hat{T}_{s,r} B_n$ with $B_{n+1}^{s,r}$ being a vector containing all the error coefficients of the local problem $(s, r)$ at iteration $(n+1)$.

4. Approximation of the infinite operator $\hat{T}$ by a matrix of finite dimensions. We begin with several observations.

**Remark 4.1.**

1. In the r.h.s. of (17), the factors in front of each coefficient $B_{n,k,i}^{s,r-1}$, $i = 1, 3$, decrease with $k$.

2. In the r.h.s. of (17), the factors in front of each coefficient $B_{n,m,i}^{s,r-1}$, where $m$ is the index of the series and $i = 2, 4$, decrease with $k$ and $m$.

3. For a given $n \in \mathbb{N}_0$, $B_{n,m,i}^{s,r-1}$ is uniformly bounded in $m \in \mathbb{N}$ for $i = 1, ..., 4$.

4. For any number $\delta > 0$ there exists a number $k_\delta$, such that for $k > k_\delta$, the sum of the absolute values of the factors of all the coefficients $B_{n,m,i}$, $i = 2, 4$, and $B_{n,k,i}^{r-1}$, $i = 1, 3$, in the r.h.s. of (17) is less than $\delta$.

5. Similar facts hold for the coefficients of the exterior subdomains. Let $(B_n)_{k \leq k_\delta}$ denote the vector resulting after discarding all the entries of $B_n$ corresponding to $k > k_\delta$.

Then, based on the facts of Remark 4.1, we can write

\begin{equation}
(B_{n+1})_{k \leq k_\delta} = \left(\hat{T} (B_n)\right)_{k \leq k_\delta} = \hat{T}_s \left( (B_n)_{k \leq k_\delta} \right) + \xi_{n+1,k_\delta} ((B_n)_{k > k_\delta}),
\end{equation}

where $\hat{T}_s$ is a finite matrix obtained by discarding the rows and columns of $\hat{T}$ related to the coefficients pertaining to $k > k_\delta$, and $\xi_{n+1,k_\delta} ((B_n)_{k > k_\delta})$ is the error vector obtained by approximating $(B_{n+1})_{k \leq k_\delta}$ by $\hat{T}_s \left( (B_n)_{k \leq k_\delta} \right)$.

We discuss in section 7 conditions for the spectral radius $\rho(\hat{T}_s) < 1$, i.e., that the (truncated) operator is contracting, so that the vector of the local error series $B_n$ would go to zero (as $n \rightarrow \infty$) is the absence of the term $\xi_{n+1,k_\delta} ((B_n)_{k > k_\delta})$. In the rest of this section we show that in addition the error $\xi_{n+1,k_\delta} ((B_n)_{k > k_\delta})$ tends to zero as $n \rightarrow \infty$, and consequently $B_n \rightarrow 0$ as $n \rightarrow \infty$.

Note that each entry of $\xi_{n+1,k_\delta} ((B_n)_{k > k_\delta})$ is the truncation error that results after truncating the series in the formulas of the coefficients $B_{n+1,k,i}^{s,r}$, $k \leq k_\delta$, by keeping only the terms corresponding to $k \leq k_\delta$. Thus, as it can be seen in (17), $\xi_{n+1,k_\delta} ((B_n)_{k > k_\delta})$ is just a linear combination of the entries of $(B_n)_{k > k_\delta}$. Note also that the entries of $(B_n)_{k > k_\delta}$ are linear combinations of the entries of $B_{n-1}$. Therefore, by choosing a small enough $\delta$ (of course $\delta < 1$) and a large enough $k_\delta$, one can obtain the following estimates

\begin{equation}
|| (B_n)_{k > k_\delta} ||_{\infty} \leq \delta \| B_{n-1} \|_{\infty},
\end{equation}

\begin{equation}
|| \xi_{n+1,k_\delta} ((B_n)_{k > k_\delta}) ||_{\infty} \leq \delta^2 || \hat{T}^{n-1} (B_0)_{k \leq k_\delta} ||_{\infty} + O(\delta^3).
\end{equation}

Using equation (18) recursively, we obtain the following equation

\begin{equation}
(B_{n+1})_{k \leq k_\delta} = \hat{T}^{n+1} ((B_0)_{k \leq k_\delta}) + \sum_{j=1}^{n+1} \hat{T}^{n+1-j} (\xi_{j,k_\delta} ((B_{j-1})_{k > k_\delta})).
\end{equation}
Using (19), (20) and (21), and assuming that the spectral radius of $T_\delta$ is less than one and that remains practically constant for large values of $k_\delta$ (and thus small values of $\delta$), it can be shown that given a $0 < \delta < 1$ there exists a $n_3$ such that $\|B_n\|_\infty \leq \delta\|B_0\|_\infty$ for all $n \geq n_3$. Repeating this argument, we can then show that $\lim_{n \to \infty} B_n = 0$. Hence, to prove that $B_n \to 0$ as $n \to \infty$ all that remains is to show that $\rho(T_\delta)$ is less than one and approaches a constant as $k_\delta$ increases (i.e., as $\delta$ decreases). We do so in section 7. The fact that $\lim_{n \to \infty} B_n = 0$ together with the fact that the local error series converge uniformly imply that the synchronous implementation of OS converges.

5. Asynchronous Optimized Schwarz methods. As mentioned in the introduction, we want to analyze the convergence of an asynchronous implementation of the OS iteration (2). We begin by reviewing the mathematical model of asynchronous iterations.

Let $X^{(1)}, \ldots, X^{(p)}$ be given sets and $X$ be their Cartesian product, i.e., $X = X^{(1)} \times \cdots \times X^{(p)}$. Thus $x \in X$ implies $x = (x^{(1)}, \ldots, x^{(p)})$ with $x^{(s)} \in X^{(s)}$ for $s \in \{1, \ldots, p\}$. Let $T^{(s)} : X \to X^{(s)}$ where $s \in \{1, \ldots, p\}$, and let $T : X \to X$ be a vector-valued map (the iteration map) given by $T = (T^{(1)}, \ldots, T^{(p)})$ with a fixed point $x^*$, i.e., $x^* = T(x^*)$. Let us define a time stamp as the instant of time at which at least one processor finishes its computation and produces a new update. Thus, let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of time stamps at which at least one processor updates its associated component. Let $\{\sigma(n)\}_{n \in \mathbb{N}}$ be a sequence with $\sigma(n) \subset \{1, \ldots, p\} \forall n \in \mathbb{N}$. The set $\sigma(n)$ consists of labels (numbers) of the processors that update their associated component at the $n$-th time stamp. Define for $s, q \in \{1, \ldots, p\}$, $\{\tau_q^s(n)\}_{n \in \mathbb{N}}$ a sequence of integers, representing the time-stamp index of the update of the data coming from processor $q$ and available in processor $s$ at the $n$-th time stamp. Let $x(0) = (x^{(1)}(0), \ldots, x^{(p)}(0))$ be the initial approximation (of the fixed point $x^*$). Then, the new computed value updated by processor $s$ at the $(n+1)$st time stamp is

$$x^{(s)}(n+1) = \begin{cases} T^{(s)}(\tau_1^s(n)), & s \in \sigma(n+1) \\ x^{(s)}(n), & s \notin \sigma(n+1). \end{cases}$$

in other words, at the time stamp $t_{n+1}$ either $x^{(s)}$ is updated (if $s \in \sigma(n+1)$) or it is not (if $s \notin \sigma(n+1)$). It is assumed that the three following conditions (necessary for convergence) are satisfied

\[
\forall s, q \in \{1, \ldots, p\}, \forall n \in \mathbb{N}, \tau_q^s(n) \leq n, \tag{22}
\]

\[
\forall s \in \{1, \ldots, p\}, \text{card } \{n \in \mathbb{N} | s \in \sigma(n)\} = +\infty, \tag{23}
\]

\[
\forall s, q \in \{1, \ldots, p\}, \lim_{n \to +\infty} \tau_q^s(n) = +\infty. \tag{24}
\]

Condition (22) indicates that data used at the time $t_n$ must have been produced before time $t_n$, i.e., time does not flow backward. Condition (23) means that no process will ever stop updating its components. Condition (24) corresponds to the fact that new data will always be provided to the process. In other words, no process will have a piece of data that is never updated.

We are ready to finally define the Asynchronous Optimized Schwarz iterations. Let $l_1 = \tau_{s-1,r}^s(n)$, $l_2 = \tau_{s+1,r}^s(n)$, $l_3 = \tau_{s-1}^s(n)$ and $l_4 = \tau_{s+1}^s(n)$, i.e., the timestamp indexes of the updates of the data coming from the neighboring processors and
available in processor \((s, r)\) at the \(n - \text{th}\) time stamp. Let

\[
\begin{cases}
-\Delta u_{n+1}^{s,r} = f^{s,r} & \text{in } \Omega^{s,r} \\
-\frac{\partial u_{n+1}^{s,r}}{\partial x} + \alpha u_{n+1}^{s,r} = -\frac{\partial u_{t_{l_1}}^{s-1,r}}{\partial x} + \alpha u_{n+1}^{s-1,r} & \text{for } x = (s-1)h - \gamma \\
-\frac{\partial u_{n+1}^{s,r}}{\partial y} + \alpha u_{n+1}^{s,r} = -\frac{\partial u_{t_{l_1}}^{s-1,r}}{\partial y} + \alpha u_{n+1}^{s-1,r} + \alpha u_{t_{l_1}}^{s+1,r} & \text{for } x = s h + \gamma \\
-\frac{\partial u_{n+1}^{s,r}}{\partial y} + \alpha u_{n+1}^{s,r} = -\frac{\partial u_{t_{l_1}}^{s-1,r}}{\partial y} + \alpha u_{n+1}^{s-1,r} & \text{for } y = (r-1)h - \gamma \\
-\frac{\partial u_{n+1}^{s,r}}{\partial y} + \alpha u_{n+1}^{s,r} = -\frac{\partial u_{t_{l_1}}^{s-1,r}}{\partial y} + \alpha u_{n+1}^{s-1,r} & \text{for } x = r h + \gamma.
\end{cases}
\]

(25)

Then, the local approximation of the solution at the time stamp \(t_{n+1}\) corresponding to the \((s, r)\) interior subdomain is

\[
\begin{cases}
\text{Solution of (25)}, & \text{if } (s, r) \in \sigma(n + 1) \\
u_{t_{l_1}}^{s,r}, & \text{if } (s, r) \notin \sigma(n + 1).
\end{cases}
\]

(26)

Note that following the same process as in the synchronous case we can obtain local operators that relate the error coefficients at different time stamps. These local operators are the same as in the synchronous case. In the asynchronous case, the local operations are performed without synchronization, therefore the expression of the global operator is more complex than in the synchronous case. However, as it is shown in the next section, we can study the convergence of the asynchronous method by studying the spectral properties of the operator \(|\hat{T}|\), where \(\hat{T}\) is the global operator of the synchronous case, and the absolute value is understood componentwise.

6. Convergence proof of Asynchronous OS. As we shall see in section 7, there are values of \(\bar{\alpha}\) and \(\bar{\gamma}\) (OOO parameter and overlap normalized with the subdomain size \(H\)) for which \(\rho(|\hat{T}|) < 1\). Also, the value of \(\rho(|\hat{T}|) < 1\) remains practically constant for large enough \(k_3\), which implies that if \(\rho(|\hat{T}|) < 1\) for large enough \(k_3\), then there exists \(\delta > 0\) such that \(\rho(|\hat{T}|) + \delta < 1\). We use these results to prove the convergence of Asynchronous OS (AOS) for the given Poisson’s problem. We divide the convergence proof into two parts. In the first part we show that, for an arbitrary sequence of time stamps \(\{t_j\}_{j \in \mathbb{N}}\), \(B_{t_j} \to 0\) as \(j \to \infty\) for values of \(\bar{\alpha}\) and \(\bar{\gamma}\) such that \(\rho(|\hat{T}|) < 1\). In the second part we show that the series expansions of the local errors converge to zero as the number of time stamps go to infinity.

The standard results one uses for the analysis of asynchronous iterations are those in \([3, 4, 17]\) (see also \([7]\)), where the the iteration operator is finite dimensional. Note that in our case \(\hat{T}\) is an infinite dimensional operator. Thus, none of the theorems from those references apply to our current case. Our following convergence proof of the asynchronous method, although inspired in part by \([4]\), is new.

**Theorem 6.1.** Given any \(\delta > 0\), there is a \(k_3\) as defined in Remark 4.1, point 4. Let \(\hat{T}_3\) be the operator (matrix of finite dimension) obtained by dropping the entries of \(\hat{T}\), defined by (17), corresponding to \(k > k_3\). Let us then assume that there exists \(\delta > 0\) and a corresponding \(k_3\) for which \(\rho(|\hat{T}_3|) + \delta < 1\). Then, the asynchronous implementation of the optimized Schwarz iteration (26) converges.

We prove this theorem in two parts.

**Part 1 of the Proof of Theorem 6.1.** We want to show that, for an arbitrary sequence of time stamps, the infinite vector of the error series coefficients \(B_{t_j}\) goes to zero as \(j \to \infty\). We begin by quoting a result in \([4]\).
Lemma 6.2. If $A$ is a non-negative matrix with spectral radius $\rho(A) < 1$, then for any $\epsilon > 0$ there exist a positive vector $v_\epsilon$ such that $Av_\epsilon \leq (\rho + \epsilon)v_\epsilon$.

We have, by hypothesis, that $\rho(\|\hat{T}_\delta\|) < 1$. Given that $\|\hat{T}_\delta\|$ is a non-negative matrix, then, for an arbitrarily small $\epsilon > 0$ there exist a positive vector $v$ such that $\|\hat{T}_\delta\|v \leq (\rho(\|\hat{T}_\delta\|) + \epsilon)v$. By hypothesis, $\rho(\|\hat{T}_\delta\|) + \delta < 1$. Then, since we can choose $\epsilon > 0$ as small as we want, we choose $\epsilon$ small enough so that $\rho + \epsilon + \delta < 1$. Let $B_{t_0}$ be the initial vector of local error coefficients. Let $w \in \mathbb{R}_\infty$ be an infinite vector of the same size as $B_{t_0}$ such that

$$ (w)|_{k \leq k_\delta} = \frac{\|B_{t_0}\|_\infty}{v_{\min}} v, $$

where $v_{\min}$ is the smallest entry of $v$, and

$$ (w)|_{k > k_\delta} = \|B_{t_0}\|_\infty \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, $$

Note that $\|B_{t_0}\| \leq w$ (this inequality is component wise). Let us denote by $v^{s,r}$ and $\xi_{t,j,k_\delta}$ the restriction of $v$ and $\xi_{t,j,k_\delta}$ to the subdomain $s, r$, respectively, where $\xi_{t,j,k_\delta}$ is defined by the following equation (analogous to equation (18))

$$ (27) \quad (B_{t_{n+1}}^{s,r})|_{k \leq k_\delta} = (\hat{T}_\delta^{s,r} (B_s))|_{k \leq k_\delta} = \hat{T}_\delta^{s,r} (\xi_{t,j,k_\delta})(B_s) + \xi_{t_{n+1},k_\delta}^{s,r} ((B_s)|_{k > k_\delta}). $$

where $B_s$ is the coefficients vector whose values are the ones available at processor $(s, r)$ when the computation of $B_{t_{n+1}}^{s,r}$ started.

Note that

$$ (28) \quad \hat{T}_\delta^{s,r} (\|B_{t_0}\|_{k \leq k_\delta}) \leq \|\hat{T}_\delta^{s,r}\| \|B_{t_0}\|_{k \leq k_\delta} \leq \|\hat{T}_\delta^{s,r}\| \frac{\|B_{t_0}\|_\infty v}{v_{\min}} \leq (\rho(\|\hat{T}_\delta\|) + \delta) \frac{\|B_{t_0}\|_\infty v^{s,r}}{v_{\min}}. $$

Let $t_j$ be the time stamp at which the processor $(s, r)$ produces its first new update. Then, by (27) and (28) we have

$$ (B_{t_j}^{s,r})|_{k \leq k_\delta} = \hat{T}_\delta^{s,r} ((B_{t_0})|_{k \leq k_\delta}) + \xi_{t_j,k_\delta}^{s,r} ((B_{t_0})|_{k > k_\delta}) $$

$$ \leq (\rho(\|\hat{T}_\delta\|) + \delta) \frac{\|B_{t_0}\|_\infty v^{s,r}}{v_{\min}} + \xi_{t_j,k_\delta}^{s,r} ((B_{t_0})|_{k > k_\delta}). $$

Note that

$$ \xi_{t_j,k_\delta}^{s,r} ((B_{t_0})|_{k > k_\delta}) \leq \rho(\|\hat{T}_\delta\|) \frac{\|B_{t_0}\|_\infty}{v_{\min}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \leq \delta \frac{\|B_{t_0}\|_\infty}{v_{\min}} v^{s,r}. $$

Hence, the error series coefficients corresponding to $k \leq k_\delta$ of the error at subdomain $s, r$ are bounded as follows

$$ (29) \quad (B_{t_j}^{s,r})|_{k \leq k_\delta} \leq (\rho(\|\hat{T}_\delta\|) + \epsilon + \delta) \frac{\|B_{t_0}\|_\infty}{v_{\min}} v^{s,r}. $$
As for the coefficients corresponding to \( k > k_{\delta} \), we have the following bound

\[
(B^{s,r}_{t_j})_{k > k_{\delta}} \leq \delta \|B_{t_0}\|_{\infty} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \leq (\rho(|\hat{T}_\delta|) + \epsilon + \delta) \|B_{t_0}\|_{\infty} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.
\]

Thus, from (29) and (30) we obtain

\[
(B^{s,r}_{t_j})_{k > k_{\delta}} \leq (\rho(|\hat{T}_\delta|) + \epsilon + \delta)w.
\]

Since \( \rho(|\hat{T}_\delta|) + \epsilon + \delta < 1 \), we have that \( (B^{s,r}_{t_j})_{k > k_{\delta}} \leq (\rho(|\hat{T}_\delta|) + \epsilon + \delta)w < w \). Thus, for this processor it holds that \( B^{s,r}_{t_j} \leq (\rho(|\hat{T}_\delta|) + \epsilon + \delta)w \leq w \), for all \( t_m \geq t_j \). Then, after every processor has produced its first update, say at \( t_{j_1} \), we have that (31) holds for all \( s,r \), and consequently \( B_{t_j} \leq (\rho(|\hat{T}_\delta|) + \epsilon + \delta)w \) for all \( t_j \geq t_{j_1} \). By a similar reasoning, we can see that once every processor has produced a new update after \( t_{j_1} \), say at \( t_{j_2} \), we have \( B_{t_j} \leq (\rho(|\hat{T}_\delta|) + \epsilon + \delta)^2w \) for all \( t_j \geq t_{j_2} \). Thus, we have

\[
B_{t_j} \leq (\rho(|\hat{T}_\delta|) + \epsilon + \delta)^{\phi(j)}w,
\]

where, denoting by \( t_j \) the first time stamp at which all processors have updated their values at least \( i \) times, we have \( \phi(j) = \max\{i \in \mathbb{N} : t_{j_n} \leq t_j \} \), i.e., \( \phi(j) \) is the update number (at time \( t_j \)) of the processor that produced the least number of updates among all processors until the instant of time \( t_j \). Conditions 2 and 3 of the asynchronous model imply that new updates will always be produced and used by the processors (see section 5), thus we have that \( \phi(j) \to \infty \) as \( j \to \infty \). Consequently, from (32) we have that \( B_{t_j} \to 0 \) as \( j \to \infty \). This concludes the first part of the proof.

The weighted max norm of an operator \( A \) with weight vector \( w \) is defined as

\[
|||A|||_w^w = \left( \frac{|||A|||w||_{\infty}}{||w||_{\infty}} \right).
\]

Hence, note that we just proved that the infinite dimensional operator \( \hat{T} \) is a contraction in the weighted max norm corresponding to the (infinite) weight vector \( w > 0 \). In fact, we have

\[
||\hat{T}||w \leq (\rho(|\hat{T}_\delta|) + \delta + \epsilon)w.
\]

Then,

\[
|||\hat{T}||w||_{\infty} \leq (\rho(|\hat{T}_\delta|) + \delta + \epsilon)||w||_{\infty}
\]

which implies

\[
||\hat{T}||_w^w = \left( \frac{||\hat{T}||w||_{\infty}}{||w||_{\infty}} \right) \leq (\rho(|\hat{T}_\delta|) + \delta + \epsilon < 1.
\]

In other words, we have shown the following.

**Theorem 6.3.** Let \( \hat{T}_\delta \) be the operator (matrix of finite dimension) obtained by dropping the entries of \( \hat{T} \) corresponding to \( k > k_{\delta} \). Then, if the spectral radius of \( |\hat{T}_\delta| \) is less than one, i.e., \( \rho(|\hat{T}_\delta|) < 1 \), and \( \rho(|\hat{T}_\delta|) + \delta < 1 \), we have that \( \hat{T} \) (infinite dimensional operator) is a contraction in a weighted max norm.
Part 2 of the Proof of Theorem 6.1. The local error, for an interior subdomain, is given by

$$
\eta_{n,i}^{s,r}(x_t, y_t) = \sum_{i=1}^{4} \eta_{n,i}^{s,r}(x_t, y_t)
$$

$$
= \sum_{i=1}^{4} \sum_{m=1}^{\infty} \left\{ \frac{A_{t_n,m,i}^{s,r}}{z_m} \left[ \frac{\alpha}{z_m} \sin \left( \frac{z_m x_t}{H} \right) + \cos \left( \frac{z_m y_t}{H} \right) \right] \times
\right.
\left[ -\frac{\alpha}{z_m} \sin \left( \frac{z_m (y_t - H)}{H} \right) + \cosh \left( \frac{z_m (y_t - H)}{H} \right) \right]\right\}
$$

$$
= \sum_{i=1}^{4} \sum_{m=1}^{\infty} \left\{ \frac{B_{t_n,m,i}^{s,r}}{z_m^{1+\beta(m,\alpha)}} \left[ \frac{\alpha}{z_m} \sin \left( \frac{z_m x_t}{H} \right) + \cos \left( \frac{z_m y_t}{H} \right) \right] \times
\right.
\left[ -\frac{\alpha}{z_m} \sin \left( \frac{z_m (y_t - H)}{H} \right) + \cosh \left( \frac{z_m (y_t - H)}{H} \right) \right]\right\}.
$$

So far, we have shown that each of the coefficients $B_{t_n,m,i}^{s,r}$ goes to zero as $n \to \infty$. This implies that each term of the infinite sum in (33) go to zero. But this fact alone does not guarantee that the infinite sum will go to zero as $n \to \infty$. In order to insure that this series goes to zero (and consequently $\eta_{n,i}^{s,r}$ goes to zero), it suffices to show that it converges uniformly in $(x_t, y_t) \in [0, H] \times [0, H]$, since this implies that the order of the limit and infinite sum operations can be interchanged, and thus

$$
\lim_{n \to \infty} \eta_{n,i}^{s,r}(x_t, y_t) = \lim_{n \to \infty} \sum_{m=1}^{\infty} \left\{ \frac{B_{t_n,m,i}^{s,r}}{z_m^{1+\beta(m,\alpha)}} \left[ \frac{\alpha}{z_m} \sin \left( \frac{z_m x_t}{H} \right) + \cos \left( \frac{z_m y_t}{H} \right) \right] \times
\right.
\left[ -\frac{\alpha}{z_m} \sin \left( \frac{z_m (y_t - H)}{H} \right) + \cosh \left( \frac{z_m (y_t - H)}{H} \right) \right]\right\} = 0.
$$

Hence, in order to complete the convergence proof, we shall show that the series

$$
\eta_{n,i}^{s,r}(x_t, y_t) = \sum_{m=1}^{\infty} \left\{ \frac{B_{t_n,m,i}^{s,r}}{z_m^{1+\beta(m,\alpha)}} \left[ \frac{\alpha}{z_m} \sin \left( \frac{z_m x_t}{H} \right) + \cos \left( \frac{z_m y_t}{H} \right) \right] \times
\right.
\left[ -\frac{\alpha}{z_m} \sin \left( \frac{z_m (y_t - H)}{H} \right) + \cosh \left( \frac{z_m (y_t - H)}{H} \right) \right]\right\}
$$

converges uniformly. We show this in Appendix A, and this would conclude part 2 of the proof of Theorem 6.1.

7. Spectral Radius of $\hat{T}_\delta$ and $|\hat{T}_\delta|$. Recall that $k_\delta$ is the number of terms that are kept in each of the series contained in the local errors series expansions (see equations (3)-(6)) so that the tail of each of the series is less than $\delta$. The subdomains
form a two-dimensional array, \(p\) is the number of subdomains per row and \(q\) the number of subdomains per column. The values of the entries of the matrix \(\hat{T}_\delta\) depend on \(\hat{\gamma}, \hat{\alpha}\) (the normalized overlap and normalized OO0 parameter) and \(k_\delta\). The structure of the matrix depends on \(k_\delta, p, q\) and the way we order the entries of \(B_n\), i.e., the way we order each coefficient \(B_{n,r,s,k,i}\) based on the values of \(s, r, k\) and \(i\). However, the eigenvalues (and thus the spectral radius) do not depend on the ordering of the entries, since a change in the order is a just a similarity transformation obtained through permutation matrices. For the ordering we have chosen, we computed the spectral radius of the resulting matrix \(\hat{T}_\delta\), for \(\hat{\gamma} \in \{0, 0.01, 1/30, 0.04, 0.1, 0.13, 0.18, 0.2, 0.25\}\), a set of values of \(\hat{\alpha}\) in the range \([0.01, 500]\), \(k_\delta \in \{1, 2, 3, 5, 10, 20, 50, 100, 200\}\), and \(p, q \in \{5, 10, 20, 30, 40\}\). In these computations we have observed the following.

1. There exist values of \(\hat{\alpha}\) for which the spectral radius of \(\hat{T}_\delta\) is less than one.
2. For a given \(\hat{\gamma}\) and the range of \(\hat{\alpha}\) considered in the experiments, \(\rho(\hat{T}_\delta)\) has two local minima and it approaches a constant less than one for large values of \(\hat{\alpha}\).
3. Given \(\hat{\gamma}, \hat{\alpha}, p\) and \(q\), the value of \(\rho(\hat{T}_\delta)\) remains practically constant for all \(k_\delta \in \{20, 50, 100, 200\}\) (see Figure 4)
4. For a given \(\hat{\gamma}\), the optimal spectral radius of \(\hat{T}_\delta\) remains practically constant as \(p\) and \(q\) increase.

In Figures 2 and 3, the results for the cases \(\hat{\gamma} = 0.01\) and \(\hat{\gamma} = 1/30\), with \(p, q = 10, k_\delta = 20, \hat{\alpha} \in [1, 10]\), are shown.

![Fig. 2. Spectral radius of \(\hat{T}_\delta\) for \(p, q = 10, k_\delta = 20, \hat{\gamma} = 0.01\) and \(\hat{\alpha} \in [1, 10]\)](image-url)

As for \(\rho(|\hat{T}|)\), i.e., the he asynchronous case, we observed the following.

1. There is only one minimum of \(\rho(|\hat{T}|)\) and there is a relatively large interval around the minimizer for which the convergence of the asynchronous method is guaranteed.
2. The spectral radius of \(|\hat{T}_\delta|\) is pretty insensitive to the values of \(k_\delta\).
3. Figure 8 indicates that the contraction factor (i.e., the spectral radius of \(|\hat{T}_\delta|\)) of the error coefficients bound varies for different amounts of overlap, which suggests that the convergence speed of the method varies with the amount of overlap. In particular, as the overlap increases from \(\gamma = 0\) to \(\gamma = 0.1\), this
Fig. 3. Spectral radius of $\hat{T}_3$ for $p, q = 10$, $k_3 = 20$, $\bar{\gamma} = 1/30$ and $\bar{\alpha} \in [1, 10]$

Fig. 4. Spectral radius of $\hat{T}_3$ vs. $k_3$ for $p, q = 10$, $k_3 = 20$, $\bar{\gamma} = 0.01$ and $\bar{\alpha} = 2.8276$

the contraction factor decreases monotonically.

4. We see that the optimal $\alpha$ remains practically constant for different amounts of overlap, except for the cases with minimum overlap $\bar{\gamma} = 0$ and maximum overlap $\bar{\gamma} = 0.25$ (Figure 7).

5. Figure 9 indicates that the optimal $\bar{\alpha}$ is practically insensitive for large number of subdomains.

6. Figure 10 indicates that, for a fixed overlap $\bar{\gamma}$ the contraction factor (the spectral radius of $|\hat{T}_3|$) of the error coefficients bound does not vary much as we increase the number of subdomains.

From Figures 2 to 6, we see that we see that the values of $\rho(\hat{T}_3)$ and $\rho(|\hat{T}_3|)$ remain practically constant as we increase $k_3$ (or equivalently, decrease $\delta$) and that
there are values of \( \bar{\alpha} \) and \( \bar{\gamma} \) for which these spectral radii are less than one. Thus, the assumptions in our convergence proofs in sections 6 and 7 are true. Consequently, Synchronous and Asynchronous OS converge for appropriate values of \( \bar{\alpha} \) and \( \bar{\gamma} \).

For large enough \( k_\delta \), such that \( \rho(\hat{T}_\delta) < 1 \), the spectral radius of \( \hat{T}_\delta \) describes the asymptotic convergence rate of the Optimized Schwarz method for the synchronous case. For cases in which \( \rho(|\hat{T}_\delta|) < 1 \), the (worst-case-scenario) bounds of the error series coefficients contract at a rate \( \rho(|\hat{T}_\delta|) + \epsilon + \delta \) (see equation (32)). Thus, in the synchronous case we define the optimal \( \bar{\alpha} \), for a given overlap amount \( \gamma \), as the
one which minimizes the spectral radius of $\hat{T}_\delta$ and thus gives the optimal asymptotic convergence rate. For the asynchronous case, we define the optimal $\bar{\alpha}$ as the one that minimizes $\rho(|\hat{T}_\delta|)$ and thus minimizes the worst-case-scenario convergence bounds.

![Fig. 7. Optimal spectral radius of $|\hat{T}_\delta|$ vs. $\bar{\gamma}$ for $p,q=10$ and $k_\delta=20$](image)

![Fig. 8. Optimal $\bar{\alpha}$ vs. $\bar{\gamma}$ for $p,q=10$ and $k_\delta=20$](image)

Comparing figures 2 and 5 we see that the optimal $\bar{\alpha}$ for the asynchronous case is not far from the one for the synchronous case.

Note that $T_\delta$ is a banded square matrix of dimension $N = 2k_\delta(2pq - p - q)$. Let $\rho_{T_\infty} = \lim_{k_\delta \to \infty} \rho(T_\delta)$ and $\rho_{|T|_\infty} = \lim_{k_\delta \to \infty} \rho(|T_\delta|)$. Usually, for $k_\delta = 3$ we have that $\rho(T_\delta)$ and $\rho(|T|_\delta)$ are good estimations of $\rho_{T_\infty}$ and $\rho_{|T|_\infty}$, respectively. Thus, computing the spectral radius of $T_\delta$ and $|T_\delta|$ is not an expensive operation.
Consequently, finding the optimal $\bar{\alpha}$ is not an expensive operation in comparison with the cost of solving the discretized version of the given problem.

![Fig. 9. Optimal $\bar{\alpha}$ vs. $p$ for the asynchronous case with $p = q$, $k_\delta = 5$ and $\bar{\gamma} = 0.01$](image)

![Fig. 10. Optimal spectral radius of $|\tilde{T}_h|$ vs. $p$ (asynchronous case) with $p = q$, $k_\delta = 5$ and $\bar{\gamma} = 0.01$](image)

8. Numerical experiments. We present numerical experiments that illustrate the performance of the proposed Asynchronous Optimized Schwarz method on a two-dimensional bounded domain (as well as the synchronous counterpart). The experiments show the convergence of the methods, illustrating the results of our theorems. In addition it can be observed that the asynchronous version is faster in terms of execution time.

The test cases are related to the study of the heat analysis in a car compartment,
modelled as follows:
\[ \rho \frac{\partial u}{\partial t} - \nabla (k \nabla u) = f \]
where \( u \) denotes the thermal field, \( k \) is the thermal conductivity and \( f \) the heat-flux density of the source. Here the steady-state heat equation is considered, i.e.,
\[ \frac{\partial u}{\partial t} = 0 \]
which is by definition not time-dependent, and which corresponds to the case where enough time has passed such that the thermal field no longer evolves in time. This leads to the following reduced equation
\[ -k \nabla^2 u = f. \]

The numerical solution of the steady-state heat equation required for this study was performed on a two-dimensional domain of dimensions 1760 mm \( \times \) 745 mm. This area is meshed with unstructured quadrangle finite elements. One example of the finite element mesh with 465 degrees of freedom (DOF) is shown Figure 11. We considered for the simulation a refinement of the mesh which contain 46945 DOF. Lagrange \( Q_1 \) finite elements are used for the discretization. The domain is split into subdomains and one example of the partitioning into 16 subdomains with the METIS software \[11\] is shown Figure 12.

We consider three different situations, namely partitioning of the domain into 16, 25, and 36 subdomains. In all cases, the overlap used is the minimum overlap, i.e., one set of common nodes in the boundary between subdomains. The optimized coefficients of the Schwarz algorithms are obtained using the values of \( \rho(T_\beta) \) and \( \rho(|T_\beta|) \) in terms of \( \alpha = \alpha H \) (see, e.g., Figures 2, 3, and 5), with the minimum found using the CMA-ES algorithm, first introduced in \[13\]. Here \( H \) is the diameter of the subdomains. The resulting parameters where divided by \( H \), leading to (non-normalized) values
of the optimized parameter $\alpha$ equal to $0.00719 \times 10^3$, $0.00899 \times 10^2$, $0.01001 \times 10^3$ (synchronous), and to $0.00408 \times 10^3$, $0.00510 \times 10^3$, $0.00612 \times 10^3$ (asynchronous), for a partitioning into 16, 25, and 36 subdomains, respectively.

To solve the resulting linear system, the synchronous and asynchronous optimized Schwarz methods with zeroth order optimized interface conditions were implemented in the recently developed C++ library Alinea [12]. The parallel implementation of the asynchronous optimized Schwarz methods is quite similar to the synchronous implementation we have described in [15], except that the asynchronous iterations and asynchronous communications are managed by a new additional layer. This new additional layer, JACK [14], a recently developed C++ library, is defined on top of the MPI library; the version of the MPI library used in the experiments is MPICH2 [1]. This layer allows us to use asynchronous communications between the processors and to deal with continuous requests. This new layer also contains new functionality such as the detection of the asynchronous convergence of the algorithm for a given stopping criteria. Here we use the stopping criterion developed in [2]; it is based on a leader election protocol over a tree topology, where cancellation messages are introduced in order to avoid erroneous detections.

The experiments are performed on a heterogeneous cluster composed of four nodes (Intel(R) Xeon(R) E5410, 2.33GHz, 8 cores, RAM: 8 GB), four nodes (Intel(R) Core(TM) i7 2.80GHz, 8 cores, RAM: 8 GB) each with graphics processing units accelerator (Tesla K20c 4799MB, GTX 570 1279MB) and four nodes (Intel(R) Xeon(R) E5-2609, 2.10GHz, 24 cores, RAM: 16 GB) each with graphics processing units accelerator (three of them with Quadro K4000 3071MB, and one with Quadro K600 1023MB), for a total of 160 cores. The interconnected network is a switched, star shaped 10Mb/s Ethernet network. We report our computational results in Table 1, where we vary the number of subdomains using the same discretization.

<table>
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<th>time</th>
<th># updates</th>
<th>avg</th>
<th>max</th>
<th>time</th>
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<td>201</td>
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<td>338</td>
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<td>374</td>
<td>677</td>
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</tbody>
</table>

Table 1

Number of iterations or average and maximum updates, and computational time (in seconds)

Recall that in asynchronous iterations we cannot talk about (global) iteration steps, since each processor may update its approximation to the (local) solution at different moments, i.e., on different time stamps. In fact, each processor (corresponding to each subdomain) would usually perform a different number of updates, i.e., of local solutions. Thus, in Table 1, for the synchronous case, the number of iterations are reported, while for the asynchronous case, the average and the maximum number of updates among all processors are reported. In both cases, the total computational time (in seconds) are shown. It can be appreciated that the asynchronous optimized Schwarz performs better in terms of execution time than its synchronous counterpart. In fact, for each case, the time to converge for the asynchronous runs is about half the time needed in the synchronous case.

9. Conclusion. We have analyzed the convergence of the Optimized Schwarz method when it is applied as an outer solver for the solution of Poisson’s problem in a rectangular domain with Dirichlet (physical) boundary conditions and zeroth
order artificial boundary conditions (OOO). We presented convergence proofs for the synchronous and asynchronous implementations of OS. As a key preliminary step to prove convergence we recasted the problem into a fixed point iteration with an infinite matrix as the iteration operator \( \hat{T} \). Then we showed that to prove convergence of the method in the synchronous case it suffices to study the spectral properties of a truncated version of this operator, \( \hat{T}_s \). For the convergence proof of the asynchronous case, it suffices to study the spectral properties of \( \hat{T}_s \). We defined as optimal values of the OOO parameter as those whose normalized values minimize the spectral radius of \( \hat{T}_s \), for the synchronous case, and that of \( \hat{T}_s \) for the asynchronous case. Finally, we presented some numerical experiments from practical applications illustrating that on the one hand the method indeed converges, as the theory indicates, and on the other hand the asynchronous implementation is faster than its synchronous counterpart, in terms of execution time.

**Appendix A. Part 2 of the proof of Theorem 6.1.**

From part 1 of the proof of Theorem 6.1 we know that \(|B_{t_n,m,i}^s| \leq (\rho(|\hat{T}_s|) + \epsilon + \delta)^{\phi(n)} ||w||_\infty\), where \( \rho(|\hat{T}_s|) + \epsilon + \delta < 1 \). Note that

\[
\frac{\tilde{\alpha}}{z_m} \sin \left( \frac{z_m x_T}{H} \right) + \cos \left( \frac{z_m x_T}{H} \right) \leq \frac{1}{z_m} + 1.
\]

Also, \[\frac{-\tilde{\alpha}}{z_m} \sinh \left( \frac{z_m (y_T - H)}{H} \right) + \cosh \left( \frac{z_m (y_T - H)}{H} \right)\] is decreasing for \( y_T \in [0, H] \), with maximum at \( y_T = 0 \) given by \[\frac{-\tilde{\alpha}}{z_m} \sinh (z_m) + \cosh (z_m)\]. Therefore,

\[
\frac{-\tilde{\alpha}}{z_m} \sinh \left( \frac{z_m (y_T - H)}{H} \right) + \cosh \left( \frac{z_m (y_T - H)}{H} \right) \leq 1.
\]

Thus, considering these results we obtain

\[
\eta_{\nu,n,m,i}^s(x_T, y_T) = \sum_{m=1}^{\infty} \left( \frac{\tilde{\alpha}}{z_m} \sin \left( \frac{z_m x_T}{H} \right) + \cos \left( \frac{z_m x_T}{H} \right) \right) \times \\
\left[ \frac{-\tilde{\alpha}}{z_m} \sinh \left( \frac{z_m (y_T - H)}{H} \right) + \cosh \left( \frac{z_m (y_T - H)}{H} \right) \right] \leq \sum_{m=1}^{\infty} (\rho(|\hat{T}_s|) + \epsilon + \delta)^{\phi(n)} ||w||_\infty \left( \frac{\tilde{\alpha}}{z_m} + 1 \right) \leq \sum_{m=1}^{\infty} (\rho(|\hat{T}_s|) + \epsilon + \delta)^{\phi(n)} ||w||_\infty \left( \frac{\tilde{\alpha}}{z_1} + 1 \right)
\]

\[\sum_{m=1}^{\infty} \frac{1}{z_m^{1+\beta(m,\alpha)}} < \infty, \]

where the second inequality follows from the fact that \( z_m \geq z_1 \) for \( m \in \mathbb{N} \). Thus, since the \( m - th \) term of the series in (33) is bounded by the \( m - th \) term of the last series in (34), if we prove that \( \sum_{m=1}^{\infty} \frac{1}{z_m^{1+\beta(m,\alpha)}} < \infty \), we have that the series expansion of \( \eta_{\nu,n,m,i}^s(x_T, y_T) \) converges uniformly. Recall that \( S_m = \{ \nu \in [0, \infty) : |z_m^\nu \sin(z_m)| \leq 1 \} \) and \( \beta(m,\tilde{\alpha}) = \max(S_m) \) for \( z_m > 1 \). Let \( 0 < \zeta < 1/2 \). We claim that for a given \( \tilde{\alpha} \)
there exist \( m_\alpha \) such that for \( m \geq m_\alpha \) we have \( \beta(m, \bar{\alpha}) > \zeta \). In fact, from (7) we have

\[
\sin(z_m) = \frac{2z_m \bar{\alpha} \cos(z_m)}{\bar{\alpha}^2 - z_m^2},
\]

then, multiplying both sides by \( z_m^\nu \) we have for \( \nu \in S_m \) that

\[
|z_m^\nu \sin(z_m)| = \frac{2z_m^{1+\nu} \bar{\alpha} |\cos(z_m)|}{|\bar{\alpha}^2 - z_m^2|} \leq 1.
\]

The above condition is satisfied, in particular, for values of \( \nu \) such that

\[
\frac{2z_m^{1+\nu} \bar{\alpha}}{|\bar{\alpha}^2 - z_m^2|} \leq 1.
\]

Solving this inequality for \( \nu \) yields

\[
(35) \quad \nu \leq \frac{\log \left( \frac{1}{2} \right) - \log \left( \frac{\bar{\alpha}}{|\bar{\alpha}^2 - z_m^2|} \right)}{\log(z_m)} - 1.
\]

Note that, \( z_m \to \infty \) as \( m \to \infty \). Thus, for a given \( \epsilon_1 > 0 \), for large enough \( m \), we have \( \bar{\alpha}/|\bar{\alpha}^2 - z_m^2| = \bar{\alpha}/(z_m^2 - \bar{\alpha}^2) < 1 \) and \( -\log(1/2)/\log(z_m) < \epsilon_1 \). Note that if \( \zeta > 0 \) and \( \epsilon_1 > 0 \) are small enough, for large \( m \) we have

\[
(36) \quad \frac{-\log \left( \frac{\bar{\alpha}}{|z_m^2 - \bar{\alpha}^2|} \right)}{\log(z_m)} > 1 + \zeta + \epsilon_1.
\]

In fact, operating on the above inequality we have that this relation holds if and only if the following inequality holds

\[
\log \left( \frac{z_m^2 - \bar{\alpha}^2}{\bar{\alpha} z_m^{1+\zeta+\epsilon_1}} \right) > 1,
\]

which is equivalent to

\[
\frac{z_m^2 - \bar{\alpha}^2}{\bar{\alpha} z_m^{1+\zeta+\epsilon_1}} > \epsilon_1,
\]

or

\[
z_m^{2-1-\zeta-\epsilon_1} - \frac{\bar{\alpha}^2}{z_m^{1+\zeta+\epsilon_1}} > \bar{\alpha} \epsilon_1,
\]

or

\[
z_m^{1-\zeta-\epsilon_1} > \bar{\alpha} \epsilon_1 + \frac{\bar{\alpha}^2}{z_m^{1+\zeta+\epsilon_1}} > \bar{\alpha} \epsilon_1.
\]

Thus (36) holds if and only if (37) holds. Since \( 0 < \zeta < 1/2 \) and we can choose \( \epsilon_1 \) as small as we want, we have that \( 1 - \zeta - \epsilon_1 > 0 \). Then, \( z_m^{1-\zeta-\epsilon_1} \to \infty \) as \( m \to \infty \). Therefore, there exist an \( m_\alpha \) such that for \( m \geq m_\alpha \) we have \( z_m^{1-\zeta-\epsilon_1} > \bar{\alpha} \epsilon_1 \) and thus (36) holds. Then, for \( m \geq m_\alpha \) we have

\[
(38) \quad 0 < \zeta < \left( \frac{\log(1/2)}{\log z_m} + \epsilon_1 \right) + \zeta < \frac{\log \left( \frac{1}{2} \right) - \log \left( \frac{\bar{\alpha}}{|\bar{\alpha}^2 - z_m^2|} \right)}{\log(z_m)} - 1.
\]
which implies that

$$\left[ \log \left( \frac{1}{2} \right) - \log \left( \frac{\tilde{m} + \alpha}{|z_m|} \right) - 1 \right] \subset S_m .$$

Recall that \( \beta(m, \alpha) = \max(S_m) \) for \( m \) such that \( z_m > 1 \). Then, for \( m \geq m_\alpha \) we have \( \beta(m, \alpha) > \zeta \). Thus, we have shown that given an \( \alpha \) and \( 0 < \zeta < 1/2 \), there exist an \( m_\alpha \) such that for \( m \geq m_\alpha \) we have \( \beta(m, \alpha) > \zeta \).

Note that \( \inf_{m \in \mathbb{N} \cap [m_\alpha, \infty)} \beta(m, \alpha) \geq \zeta > 0 \). The solutions of (7) are the zeros of the function \( \Phi : \mathbb{R} \rightarrow \mathbb{R} \) given by \( \Phi(z) = \tan(z) - \frac{2m}{\alpha + z^2} \). Thus, \( z_m \) is a positive zero of \( \Phi \) for all \( m \in \mathbb{N} \). Plotting the graph of \( \Phi \) we can see that \( z_m > (m - 1)\pi \) for all \( m \in \mathbb{N} \) (see Figure 13). Then we have

$$\sum_{m=1}^{\infty} \frac{1}{\beta(m, \alpha)} = \sum_{m=1}^{m_\alpha - 1} \frac{1}{\beta(m, \alpha)} + \sum_{m=m_\alpha}^{\infty} \frac{1}{\beta(m, \alpha)} \leq m_\alpha - 1 + \sum_{m=m_\alpha}^{\infty} \frac{1}{\zeta + 1}
\leq m_\alpha - 1 + \sum_{m=m_\alpha}^{\infty} \frac{1}{(m - 1)\pi + 1 + \zeta}
\leq m_\alpha - 1 + \sum_{j=1}^{\infty} \frac{1}{j + 1 + \zeta} .$$

Note that \( \sum_{j=1}^{\infty} \frac{1}{j} < \infty \) for any \( \alpha > 1 \). Then, since \( \zeta > 0 \), we have \( \sum_{m=1}^{\infty} \frac{1}{\beta(m, \alpha)} < \infty \). Consequently, \( \sum_{m=1}^{\infty} \frac{1}{\beta(m, \alpha)} < \infty \). Therefore, we have

$$\eta_{n,i}^{s,r}(x_t, y_t) \leq \sum_{m=1}^{\infty} \frac{||w||_{\infty}}{\zeta + 1 + \alpha} \left( \frac{\alpha}{z_m} + 1 \right) < \infty.$$ 

In conclusion, since the \( m - \text{th} \) term of the series expansion of \( \eta_{n,i}^{s,r} \) is bounded by \( \frac{||w||_{\infty}}{\zeta + 1 + \alpha} \left( \frac{\alpha}{z_m} + 1 \right) \) and \( \sum_{m=1}^{\infty} \frac{||w||_{\infty}}{\zeta + 1 + \alpha} \left( \frac{\alpha}{z_m} + 1 \right) < \infty \), we have that \( \eta_{n,i}^{s,r}(x_t, y_t) \) converges uniformly in \( (x_t, y_t) \in [0, H] \times [0, H] \). This completes the proof of part 2, and thus of the theorem.

Appendix B. Justification of order interchange between derivative as infinite summation. In the derivation of the expressions of the coefficients \( B_{n,m,i}^{s,r} \) we have used the following two identities

$$\left( -\frac{\partial}{\partial y_t} + \alpha \right) \left( \sum_{m=1}^{\infty} \frac{B_{n,m,i}^{s,r}}{z_m^{\beta(m, \alpha)} \zeta_m} \phi_m(x_t)\psi_m(y_t) \right) \phi_k(x_t)dx =$$

$$\left( -\frac{\partial}{\partial y_t} + \alpha \right) \sum_{m=1}^{\infty} \frac{B_{n,m,i}^{s,r}}{z_m^{\beta(m, \alpha)} \psi_m(y_t)\zeta_m} \int_{0}^{H} \phi_m(x_t)\phi_k(x_t)dx$$
and

\[
\int_0^H \left[ -\frac{\partial}{\partial y} + \alpha \right] \left( \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{\zeta_m} \phi_m(y) \psi_m(x) \right) \phi_k(x) dx = \\
\sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{\zeta_m} \int_0^H \psi_m(x) \phi_k(x) dx \left( -\frac{\partial}{\partial y} \phi_m(y) + \alpha \phi_m(y) \right).
\]

**Proof of identity (39).** Since \( \eta_{n,m}^{s,r} \) is harmonic, it is continuous and all its derivatives are continuous. We have that

\[
\eta_{n,1}^{s,r}(x,y) = \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{\zeta_m} \left[ \frac{\dot{\alpha}}{\dot{z}_m} \sin \left( \frac{z_m x}{H} \right) + \cos \left( \frac{z_m x}{H} \right) \right] \times \\
\left[ -\frac{\dot{\alpha}}{\dot{z}_m} \sin \left( \frac{z_m (y - H)}{H} \right) + \cosh \left( \frac{z_m (y - H)}{H} \right) \right]
\]

\[
= \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{\zeta_m} \phi_m(x) \psi_m(y),
\]

where

\[
\phi_m(x) = \frac{\dot{\alpha}}{\dot{z}_m} \sin \left( \frac{z_m x}{H} \right) + \cos \left( \frac{z_m x}{H} \right),
\]

\[
\psi_m(y) = -\frac{\dot{\alpha}}{\dot{z}_m} \sin \left( \frac{z_m (y - H)}{H} \right) + \cosh \left( \frac{z_m (y - H)}{H} \right),
\]

and

\[
\zeta_m = \frac{\dot{\alpha}}{\dot{z}_m} \sin (z_m) + \cosh (z_m).
\]

We claim that

\[
\int_0^H \left[ -\frac{\partial}{\partial y} + \alpha \right] \left( \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{\zeta_m} \phi_m(x) \psi_m(y) \right) \phi_k(x) dx = \\
\left( -\frac{\partial}{\partial y} + \alpha \right) \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{\zeta_m} \psi_m(y) \int_0^H \phi_m(x) \phi_k(x) dx.
\]
In fact, since \( n_{n,1}^{s,r}(x, y_t) \phi_m(x) \) and \( \frac{\partial}{\partial y_t} \left( n_{n,1}^{s,r}(x, y_t) \phi_m(x) \right) \) are continuous in \([0, H] \times [0, H]\) we have

\[
\int_0^H \frac{\partial n_{n,1}^{s,r}}{\partial y_t} \phi_m(x_t) dx_t = \int_0^H \frac{\partial}{\partial y_t} \left( n_{n,1}^{s,r} \phi_m(x_t) \right) dx_t = \frac{\partial}{\partial y_t} \int_0^H \left( n_{n,1}^{s,r} \phi_m(x_t) \right) dx_t.
\]

Thus,

\[
\int_0^H \frac{\partial n_{n,1}^{s,r}}{\partial y_t} \phi_k(x_t) dx_t = \frac{\partial}{\partial y_t} \int_0^H n_{n,1}^{s,r} \phi_k(x_t) dx_t.
\]

Consequently,

\[
(41) \quad \int_0^H \left( \frac{\partial}{\partial y_t} \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)} \zeta_m} \phi_m(x_t) \psi_m(y_t) \right) \phi_k(x_t) dx_t =
\]

\[
- \frac{\partial}{\partial y_t} \int_0^H \left( \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)} \zeta_m} \phi_m(x_t) \psi_m(y_t) \right) \phi_k(x_t) dx_t =
\]

\[
\frac{\partial}{\partial y_t} \int_0^H \sum_{m=1}^{\infty} \left( \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)} \zeta_m} \phi_m(x_t) \psi_m(y_t) \phi_k(x_t) \right) dx_t
\]

Note that \( \phi_k(x_t) \) is bounded and \( \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)} \zeta_m} \phi_m(x_t) \psi_m(y_t) \) converges absolutely (and uniformly). Then, \( \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)} \zeta_m} \phi_m(x_t) \psi_m(y_t) \phi_k(x_t) \) converges absolutely. Therefore,

\[
(42) \quad \int_0^H \sum_{m=1}^{\infty} \left( \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)} \zeta_m} \phi_m(x_t) \psi_m(y_t) \phi_k(x_t) \right) dx_t =
\]

Thus, using (41) and (42) we have

\[
\int_0^H \left[ \left( \frac{\partial}{\partial y_t} + \alpha \right) \left( \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)} \zeta_m} \phi_m(x_t) \psi_m(y_t) \right) \right] \phi_k(x_t) dx_t =
\]

\[
- \int_0^H \left( \frac{\partial}{\partial y_t} \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)} \zeta_m} \phi_m(x_t) \psi_m(y_t) \right) \phi_k(x_t) dx_t +
\]

\[
\alpha \int_0^H \left( \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)} \zeta_m} \phi_m(x_t) \psi_m(y_t) \right) \phi_k(x_t) dx_t =
\]
Then, we can obtain the following equality

\[ -\frac{\partial}{\partial y_\ell} \int_0^H \sum_{m=1}^{\infty} \left( \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)}} \phi_m(x_\ell) \psi_m(y_\ell) \phi_k(x_\ell) \right) \, dx_\ell + \]

\[ \alpha \int_0^H \sum_{m=1}^{\infty} \left( \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)}} \phi_m(x_\ell) \psi_m(y_\ell) \phi_k(x_\ell) \right) \, dx_\ell = \]

\[ \left( -\frac{\partial}{\partial y_\ell} + \alpha \right) \int_0^H \sum_{m=1}^{\infty} \left( \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)}} \phi_m(x_\ell) \psi_m(y_\ell) \phi_k(x_\ell) \right) \, dx_\ell = \]

\[ \left( -\frac{\partial}{\partial y_\ell} + \alpha \right) \sum_{m=1}^{\infty} \int_0^H \phi_m(x_\ell) \phi_k(x_\ell) \, dx_\ell = \]

\[ \left( -\frac{\partial}{\partial y_\ell} + \alpha \right) \sum_{m=1}^{\infty} \int_0^H \phi_m(x_\ell) \phi_k(x_\ell) \, dx_\ell. \]

The proof is complete.

**Proof of Identity (40).** By a similar process as in the proof of the identity (39) we can obtain the following equality

\[ \int_0^H \left[ \left( -\frac{\partial}{\partial y_\ell} + \alpha \right) \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)}} \phi_m(y_\ell) \psi_m(x_\ell) \right] \phi_k(x_\ell) \, dx_\ell = \]

\[ \left( -\frac{\partial}{\partial y_\ell} + \alpha \right) \sum_{m=1}^{\infty} \int_0^H \phi_m(x_\ell) \phi_k(x_\ell) \, dx_\ell. \]

Thus, given (43), in order to prove (40) it suffices to show that

\[ \left( -\frac{\partial}{\partial y_\ell} + \alpha \right) \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)}} \phi_m(y_\ell) \int_0^H \psi_m(x_\ell) \phi_k(x_\ell) \, dx_\ell = \]

\[ \sum_{m=1}^{\infty} \left( -\frac{\partial}{\partial y_\ell} + \alpha \right) \frac{B_{n,m,1}^{s,r}}{z_m^{1+\beta(m,\alpha)}} \phi_m(y_\ell) \int_0^H \psi_m(x_\ell) \phi_k(x_\ell) \, dx_\ell. \]

Let

\[ \sigma_{m,k} = \int_0^H \psi_m(x_\ell) \phi_k(x_\ell) \, dx_\ell. \]

Then,

\[ \sigma_{m,k} = \left( \frac{H z_k \cos(z_k) \left( z_m^2 - \alpha^2 \right) \sinh(z_m) + H \sin(z_k) \left( z_m \left( \alpha^2 + z_k^2 \right) \cosh(z_m) + \alpha \left( z_k^2 + z_m^2 \right) \sinh(z_m) \right)}{z_m z_k \left( z_k^2 + z_m^2 \right)} \right) \]

\[ = \left( \frac{H z_k \cos(z_k) \left( z_m^2 - \alpha^2 \right) \sinh(z_m) + H \sin(z_k) \left( z_m \left( \alpha^2 + z_k^2 \right) \cosh(z_m) + \alpha \left( z_k^2 + z_m^2 \right) \sinh(z_m) \right)}{z_m z_k \left( z_k^2 + z_m^2 \right) \left[ z_k \sinh(z_m) + \cosh(z_m) \right]} \right) \]

\[ = \left( \frac{H z_k \cos(z_k) \left( \frac{1}{z_m} \alpha^2 + z_k^2 \right) \tanh(z_m) + H \sin(z_k) \left( \frac{1}{z_m} \alpha^2 + z_k^2 \right) \frac{z_k}{z_m} \left[ z_k \tanh(z_m) + 1 \right]}{z_m z_k \left( \frac{z_k^2}{z_m} + 1 \right) \left[ z_k \tanh(z_m) + 1 \right]} \right). \]
Note that
\[
|\sigma_{m,k}| \leq \left( \frac{Hz_k \left( 1 + \frac{\alpha^2}{z_k} \right) \tanh(z_1) + H \left( \frac{1}{z_1} \left( \alpha^2 + z_k^2 \right) + \alpha \left( \frac{z_k^2}{z_1} + 1 \right) \right)}{z_m z_k} \right) = \frac{M_k}{z_m} < \infty.
\]

Also, we have
\[
\frac{\partial}{\partial y_t} \phi_m(y_t) = \frac{\partial}{\partial y_t} \left[ \frac{\alpha}{z_m} \sin \left( \frac{z_m y_t}{H} \right) + \cos \left( \frac{z_m y_t}{H} \right) \right] = \frac{z_m}{H} \left[ \alpha \cos \left( \frac{z_m y_t}{H} \right) - \sin \left( \frac{z_m y_t}{H} \right) \right] \leq \frac{z_m}{H} (\bar{\alpha} + 1).
\]

Then, using the facts that \(|B_{n,m,1}^{s,r}| \leq ||w||_\infty (w \text{ is defined in section 6}), \zeta_m \geq \zeta_1\) for all \(m \in \mathbb{N}\), and \(\sum_{m=1}^{\infty} \frac{1}{z_m^{1+\beta(m,\alpha)}} < \infty\) (see Appendix A) we have
\[
\sum_{m=1}^{\infty} \frac{\partial}{\partial y_t} \left( \frac{B_{n,m,1}^{s,r} \phi_m(y_t)}{z_m^{1+\beta(m,\alpha)} \zeta_m} \right) \int_0^H \psi_m(x_t) \phi_k(x_t) dx_t \\
\sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r} \left( \frac{\partial}{\partial y_t} \phi_m(y_t) \right)}{z_m^{1+\beta(m,\alpha)} \zeta_m} \int_0^H \psi_m(x_t) \phi_k(x_t) dx_t \leq \sum_{m=1}^{\infty} \frac{||w||_\infty \bar{\alpha} + 1}{z_m^{1+\beta(m,\alpha)} \zeta_m} \frac{M_k}{z_m} \\
\frac{M_k ||w||_\infty (\bar{\alpha} + 1)}{H \zeta_1} \sum_{m=1}^{\infty} \frac{1}{z_m^{1+\beta(m,\alpha)}} < \infty,
\]
i.e., the series
\[
\sum_{m=1}^{\infty} \frac{\partial}{\partial y_t} \left( \frac{B_{n,m,1}^{s,r} \phi_m(y_t)}{z_m^{1+\beta(m,\alpha)} \zeta_m} \right) \int_0^H \psi_m(x_t) \phi_k(x_t) dx_t
\]
converges uniformly and consequently
\[
(44) \quad \frac{\partial}{\partial y_t} \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r} \phi_m(y_t)}{z_m^{1+\beta(m,\alpha)} \zeta_m} \int_0^H \psi_m(x_t) \phi_k(x_t) dx_t = \sum_{m=1}^{\infty} \frac{\partial}{\partial y_t} \left( \frac{B_{n,m,1}^{s,r} \phi_m(y_t)}{z_m^{1+\beta(m,\alpha)} \zeta_m} \right) \int_0^H \psi_m(x_t) \phi_k(x_t) dx_t.
\]

Then, we have
\[
\left( -\frac{\partial}{\partial y_t} + \alpha \right) \sum_{m=1}^{\infty} \frac{B_{n,m,1}^{s,r} \phi_m(y_t)}{z_m^{1+\beta(m,\alpha)} \zeta_m} \int_0^H \psi_m(x_t) \phi_k(x_t) dx_t =
\]
\[
\frac{\partial}{\partial y_\ell} \sum_{m=1}^{\infty} \frac{B^{s,r}_{n,m,1} \phi_m(y_\ell)}{z_m^{1+\beta(m,\alpha)}} \int_0^H \psi_m(x_\ell) \phi_k(x_\ell) dx_\ell + \alpha \sum_{m=1}^{\infty} \frac{B^{s,r}_{n,m,1} \phi_m(y_\ell)}{z_m^{1+\beta(m,\alpha)}} \int_0^H \psi_m(x_\ell) \phi_k(x_\ell) dx_\ell =
\]

\[
\sum_{m=1}^{\infty} - \frac{\partial}{\partial y_\ell} \left( \frac{B^{s,r}_{n,m,1} \phi_m(y_\ell)}{z_m^{1+\beta(m,\alpha)}} \int_0^H \psi_m(x_\ell) \phi_k(x_\ell) dx_\ell \right) + \sum_{m=1}^{\infty} \alpha \frac{B^{s,r}_{n,m,1} \phi_m(y_\ell)}{z_m^{1+\beta(m,\alpha)}} \int_0^H \psi_m(x_\ell) \phi_k(x_\ell) dx_\ell =
\]

\[
\sum_{m=1}^{\infty} \left( - \frac{\partial}{\partial y_\ell} + \alpha \right) \left( \frac{B^{s,r}_{n,m,1} \phi_m(y_\ell)}{z_m^{1+\beta(m,\alpha)}} \int_0^H \psi_m(x_\ell) \phi_k(x_\ell) dx_\ell \right).
\]

The proof is complete.

REFERENCES


