Inexact Newton with Krylov projection and recycling for Riccati equations

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Abstract
Our purpose is to solve Riccati equations which arise in many applications in Control Theory of the form

\[ AX + XA^T - XBB^T X + C^T C = 0, \]

where \( A, X, C \in \mathbb{R}^{m \times m}, B \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times n} \). It is standard to seek low-rank solutions of the form \( X = ZZ^T \), \( Z \in \mathbb{R}^{m \times m} \) with \( z \neq 0 \) (for storage issues). Since this is a nonlinear matrix equation, Newton’s method is a successful strategy; it requires the solution of a Lyapunov equation at each iteration \( j \), with the form

\[ M_j X_j S_j = -D_j D_j^T, \]

where \( M_j = A - X_j B B^T \) and \( D_j = (C^T X_j B) \). In this work in progress, we propose to use an Inexact Newton method to solve (1) and a Krylov projection method to solve (2) with a recycling process.

1. Project the Inexact Newton Methods
We are proposing the following to solve (1) projection methods. In particular:

- There are several good approaches to find \( X(0) \).
- For \( j = 1, 2, \ldots \)
  - (i) \( M_j = A - X_j B B^T \)
  - (ii) \( D_j^T = \{C^T X_j B\} \)
  - (iii) Find \( X_{j+1} \), solution of \( M_{j+1} X_{j+1} = D_j D_j^T \) at tol \( j \).


Krylov Projection Method

\[ M_j = S_j M_j^T = -D_j D_j^T \]

Project onto a Krylov subspace \( K_j(M_j) \) (state \( s \)).

Find \( Y_j \), solution of a small Lyapunov

\[ T = VV^T = \delta \]

The effectiveness of these methods depend on the subspace \( K_j(M_j) \). There are two attractive approaches to find \( K_j(M_j) \):

- Global Arnoldi
  1. Use Frobenius norm and its inner product.
  2. Produce \( V_j \) and \( V_j \) such that \( M_j V_j = V_j B_{\infty} + O(\epsilon) \)
  3. The small Lyapunov is \( \| M_j V_j \| = O(\epsilon) \).
  4. The dimension of \( V_j \) is \( O(\epsilon) \).

- Extended Arnoldi
  1. Use Euclidean norm and its inner product.
  2. Produce \( V_j \) and \( V_j \) such that \( M_j V_j = V_j B_{\infty} + O(\epsilon) \).
  3. The small Lyapunov is \( \| M_j V_j \| = O(\epsilon) \).
  4. The dimension of \( V_j \) is \( O(\epsilon) \).

Recycling Ideas: Take advantage of the Krylov subspace obtained in the j-th step to enrich the Krylov subspace at the next step.

The tolerance is given by

\[ \delta = \| X_j - X_j \| \]

1. Accelerating the convergence in the j-th step.
2. Saving storage and computational effort.

The work already done to create the basis at the j step is not thrown away; it enriches the space at the next step.

3. Problem # 1: This problem described as

\[ \begin{cases}
  X(0) = 0 \quad \text{when} \quad F(X) = 0 \quad \text{is a Riccati equation}
  \\
  X = X^T \quad \text{and} \quad |X| < \infty
\end{cases} \]

The solution was carried out on a 36x36 grid and central differences are used for the approximation. The tolerance is given by \( |X| < 8 \times 10^{-12} \).

Inexact Newton method with
- Extended and Global Krylov projections
- ADI

Experimental results problem # 2:
- Linear quadratic problem of one-dimensional heat flow.
- The tolerance is given by

\[ \| X - X \| = O(\epsilon) \]

Dimensions \( \| X \| = O(\epsilon) \).

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Outlook
- Encouraging numerical results
- Extended Krylov projection most promising
- Further theoretical and implementation details to be worked out.

References

Krylov projection method for Lyapunov equations: Roughly speaking, this approach to solve (2) consists in

1. Project the Lyapunov equation onto a Krylov subspace \( K_j(M_j) \) of dimension it
2. The approximate solution of (2) will be \( X = VV^T \), where \( V \) is an orthogonal basis of \( K_j(M_j) \) and \( V \) is the solution of a small Lyapunov equation (dimension \( j \)).