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Equivalence of Conditions for Convergence of Iterative Methods for Singular Equations

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Equivalence is shown between different conditions for convergence of iterative methods for consistent singular systems of linear equations on Banach spaces. These systems appear in many applications, such as Markov chains and Markov processes. The conditions considered relate the range and null spaces of different operators.

KEY WORDS Linear systems Singular systems Iterative methods Convergent methods Index of the iteration operator

Let $\mathscr{E}$ be a Banach space and let A be a bounded linear operator on $\mathscr{E}$. For the solution of equations of the form

$$Au = f \tag{1.1}$$

it is customary to consider a splitting $A = M - N$, where M is bounded and invertible, and iterative methods of the form

$$u_{k+1} = Tu_k + M^{-1}f, \quad k = 0, 1, \ldots \tag{1.2}$$

where $T = M^{-1}N$. In this article we consider solutions of (1.1) where A is a singular operator and f is such that a solution exists, i.e., consistent singular systems. Singular systems of linear equations appear in many applications, such as when computing the stationary probability distributions of Markov chains [9] or Markov processes [2]. These particular applications appear in the analysis of telecommunications or computer networks [9].

Let $\mathcal{N}(A)$ be the null space of A. H. B. Keller [3] showed that for the finite-dimensional case $\mathscr{E} = \mathbb{R}^n$ the following two conditions are equivalent: (a) for every $u_0 \in \mathscr{E}$, the sequence $\{u_k\}$ of (1.2) converges to a solution of (1.1); (b) there exists a subspace C, complementary to $\mathcal{N}(A)$, such that C is invariant under T, T is the identity on $\mathcal{N}(A)$, and

$$\lim_{k\to\infty} T^k u \quad \text{exists for every} \quad u \in C \tag{1.3}$$

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Let $\sigma(T)$ be the spectrum of T, and let $r(T)$ be its spectral radius [10]. Let $\lambda \in \sigma(T)$ be a pole of the resolvent operator $R(\mu, T) = (\mu I - T)^{-1}$. The multiplicity of λ as a pole of $R(\mu, T)$ is called the index of T with respect to λ and denoted $ind_\lambda T$. When the condition (1.3) is satisfied it is said that the operator T is convergent in the subspace C; see [6] or [8]. Let $r(T) = 1$. It is well known that (1.3) holds if and only if $\lambda \in \sigma(T)$, $|\lambda| = 1$, implies $\lambda = 1$ and, in addition, $ind_1 T = 1$.

O'Carroll [7] showed that for the finite-dimensional case $\mathscr{E} = \mathbb{R}^n$ and for T the S.O.R. operator (see, for example, [11], [12]), the subspace C in (b) can be taken as the range of $I - T$, denoted $\mathcal{R}(I - T)$ and that

$$ind_1 T = 1 \quad \text{if and only if } \mathcal{N}(A) \cap \mathcal{R}(I - T) = \{0\} \tag{1.4}$$

The proof by O'Carroll [7], as well as that of Keller [3], considers the size of the Jordan blocks of the matrix, and cannot be directly generalized to infinite dimensions.

In this article we show that (1.4) is valid for general Banach spaces and for all iteration operators of processes of the form (1.2). At the same time, we show that this condition is equivalent to others available in the literature. Our main result is the following

Theorem *Let $A = M - N$ be a splitting, and $T = M^{-1}N$, $r(T) = 1$. Let 0 be an isolated pole of the resolvent of A. Let $A = I - B$, where $B = P + Z$, $P^2 = P \neq 0$, $PZ = ZP = 0$, $1 \notin \sigma(Z)$, i.e., P is the first term of the Laurent expansion of the resolvent of A. Then, the following are equivalent*

1. $\mathcal{N}(A) \bigcap \mathcal{R}(I - T) = \{0\}$.

2. $\mathcal{N}(AM^{-1}) \bigcap \mathcal{R}(A) = \{0\}$.

3. $M\mathcal{R}(P) \bigcap \mathcal{R}(I - P) = \{0\}$.

4. $ind_1 T = 1$.

Condition (2) is similar to one by Berman and Neumann [1]. They work with rectangular matrices, and a pseudoinverse replaces M^{-1} in their condition. Condition (3) was introduced by Marek and Szyld [4], and applied in [5]. We note that this condition is independent of the splitting chosen. The equivalence of conditions (3) and (4) was shown in Lemma 6.2 and Theorem 6.4 in [4] together with other results. A different proof follows from the theorem.

Before proceeding with the proof of the theorem we prove the following

Lemma *Let the hypothesis of the Theorem hold. Then, the following subspace equalities hold:*

1. $M\mathcal{N}(A) = \mathcal{N}(AM^{-1})$,
2. $\mathcal{R}(A) = \mathcal{R}(I - P)$,
3. $\mathcal{N}(A) = \mathcal{R}(P)$,
4. $M\mathcal{R}(I - T) = \mathcal{R}(A)$.

Proof

1. $x \in M\mathcal{N}(A)$ if and only if $y = M^{-1}x \in \mathcal{N}(A)$, if and only if $x \in \mathcal{N}(AM^{-1})$.

2. Since $A = I - P - Z = (I - P)(I - Z)$, $\mathcal{R}(A) \subseteq \mathcal{R}(I - P)$, and if $y \in \mathcal{R}(I - P)$, which implies $y = (I - P)y$, we can consider $x = (I - Z)^{-1}y$, and $Ax = (I - P)(I - Z)x = (I - P)y = y$.
3. Since $AP = 0$, it follows that $\mathcal{R}(P) \subseteq \mathcal{N}(A)$. Let $x \in \mathcal{N}(A)$, thus $x - Zx = Px$, and since $1 \notin \sigma(Z)$, $x = (I - Z)^{-1}Px = Px$, and therefore $\mathcal{N}(A) = \mathcal{R}(P)$.
4. It follows directly from the equality $M(I - T) = A$. ■

Proof of the Theorem

From the lemma it follows that

$$\begin{aligned} M(\mathcal{N}(A) \cap \mathcal{R}(I - T)) &= \mathcal{N}(AM^{-1}) \cap \mathcal{R}(A) \\ &= M\mathcal{R}(P) \cap \mathcal{R}(I - P) \end{aligned}$$

and thus the equivalence of the conditions (1)–(3) follows. For the equivalence of the conditions (4) and (1) we have

$$\begin{aligned} ind_1 T = 1 &\Leftrightarrow \{(I - T)^2 z = 0 \Rightarrow (I - T)z = 0\} \\ &\Leftrightarrow \{\mathcal{R}(I - T) \cap \mathcal{N}(I - T) = \{0\}\} \\ &\Leftrightarrow \{\mathcal{R}(I - T) \cap \mathcal{N}(A) = \{0\}\} \end{aligned}$$

as $A = M(I - T)$. ■

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