SIAM J. NUMER. ANAL.
Vol. 40, No. 6, pp. 2318–2336

CONVERGENCE THEORY OF RESTRICTED MULTIPLICATIVE SCHWARZ METHODS*

REINHARD NABBEN† AND DANIEL B. SZYLD‡

Abstract. Convergence results for the restricted multiplicative Schwarz (RMS) method, the multiplicative version of the restricted additive Schwarz (RAS) method for the solution of linear systems of the form $Ax = b$, are provided. An algebraic approach is used to prove convergence results for nonsymmetric M-matrices. Several comparison theorems are also established. These theorems compare the asymptotic rate of convergence with respect to the amount of overlap, the exactness of the subdomain solver, and the number of domains. Moreover, comparison theorems are given between the RMS and RAS methods as well as between the RMS and the classical multiplicative Schwarz method.

Key words. restricted Schwarz methods, domain decomposition, multisplittings, parallel algorithms

AMS subject classifications. 65F10, 65F35, 65M55

PII. S003614290138944X

1. Introduction.

We consider restricted Schwarz methods for the solution of linear systems of the form

$$Ax = b, \tag{1}$$

where A is $n \times n$ and nonsingular. These methods were introduced by Tai [26] and by Cai and Sarkis [10] for the parallel solution of (1); see also [8, 9]. In [10], it is shown by numerical examples that the restricted additive Schwarz (RAS) method is an efficient alternative to the classical additive Schwarz preconditioner. RAS preconditioners are widely used and are the default preconditioner in the PETSc software package [1]. In [26] and [10], the multiplicative variant of the RAS method, the restricted multiplicative Schwarz (RMS) method, is also mentioned; see also [9]. Although restricted Schwarz methods work very well in practice, until recently no theoretical results were available. In [16], convergence and comparison results for the RAS method were established when the matrix A in (1) is a (possibly nonsymmetric) M-matrix (or more generally an H-matrix). Those results use a new algebraic formulation of Schwarz methods and a connection with the well-known concept of multisplittings [7, 22]; see [2, 14, 15].

In this paper, we consider the RMS method. Again using the algebraic approach we are able to establish convergence results for the RMS method applied to M-matrices. Thus, this paper is the counterpart to [16] for the multiplicative case, although we prove some new results on RAS iterations as well. Furthermore, we are able to present a comparison result between the RMS and RAS methods. We show that, as measured in a certain norm, the convergence of the RMS method is never

*Received by the editors May 17, 2001; accepted for publication (in revised form) May 7, 2002; published electronically January 14, 2003.
http://www.siam.org/journals/sinum/40-6/38944.html
†Fakultät für Mathematik, Universität Bielefeld, Postfach 10 01 31, 33501 Bielefeld, Germany (nabben@mathematik.uni-bielefeld.de).
‡Department of Mathematics, Temple University (038-16), 1805 N. Broad Street, Philadelphia, PA 19122-6094 (szyld@math.temple.edu). The research of this author was supported by National Science Foundation grant DMS-0207525.

worse than that of the RAS method. More precisely, if $M_{RAS}^{-1}A$ and $M_{RMS}^{-1}A$ are the preconditioned matrices using the RAS and RMS methods, respectively, we show that in some norm

$$\|I - M_{RMS}^{-1}A\| \leq \|I - M_{RAS}^{-1}A\|;$$

see section 3. In some cases, this comparison is valid for the spectral radii $\rho_1 := \rho(I - M_{RMS}^{-1}A) \leq \rho_2 := \rho(I - M_{RAS}^{-1}A)$. This implies that the spectrum of the preconditioned matrix with RAS $\sigma(M_{RAS}^{-1}A) \subseteq B(1, \rho_2)$, the ball centered at 1 with radius ρ_2, while the spectrum of the preconditioned matrix with RMS $\sigma(M_{RMS}^{-1}A)$ is contained in the smaller ball $B(1, \rho_1)$. These results remain true if we allow an inexact (or approximate) solution of the subdomain problems; see section 4. We point out that such a theoretical comparison has only recently become available between the classical additive and multiplicative Schwarz methods [20].

In section 3, we prove that the asymptotic rate of convergence of the RMS method is no faster than that of the classical multiplicative Schwarz method. The reason why the restricted Schwarz methods are attractive is that the communication time between processors is reduced, usually converging in less overall computational time [10].

We prove several other comparison theorems. We compare the speed of convergence with respect to the amount of overlap of the domains (section 5), the exactness of the subdomain solver (section 4), and the number of domains (section 6). Some variants of the RMS method are analyzed in section 7. We finish the paper with some comments on coarse grid corrections.

2. The algebraic representation and notations.

As in [10, 16] we consider p nonoverlapping subspaces $W_{i,0}$, $i = 1, \ldots, p$, which are spanned by columns of the identity I over $\mathbb{R}^n$ and which are then augmented to produce overlap. For a precise definition, let $S = \{1, \ldots, n\}$, and let

$$S = \bigcup_{i=1}^{p} S_{i,0}$$

be a partition of S into p disjoint, nonempty subsets. For each of these sets $S_{i,0}$ we consider a nested sequence of larger sets $S_{i,\delta}$ with

$$S_{i,0} \subseteq S_{i,1} \subseteq S_{i,2} \subseteq \cdots \subseteq S = \{1, \ldots, n\}, \tag{2}$$

so that we again have $S = \cup_{i=1}^{p} S_{i,\delta}$ for all values of δ, but for $\delta > 0$ the sets $S_{i,\delta}$ are not pairwise disjoint; i.e., there is *overlap.* A common way to obtain the sets $S_{i,\delta}$ is to add those indices to $S_{i,0}$ which correspond to nodes lying at distance δ or less from those nodes corresponding to $S_{i,0}$ in the (undirected) graph of A. This approach is particularly adequate in discretizations of partial differential equations where the indices correspond to the nodes of the discretization mesh; see [6, 8, 9, 10, 13, 25].

Let $n_{i,\delta} = |S_{i,\delta}|$ denote the cardinality of the set $S_{i,\delta}$. For each nested sequence of the form (2) we can find a permutation π_i on $\{1, \ldots, n\}$ with the property that for all $\delta \geq 0$ we have $\pi_i(S_{i,\delta}) = \{1, \ldots, n_{i,\delta}\}$.

We now build matrices $R_{i,\delta} \in \mathbb{R}^{n_{i,\delta} \times n}$ whose rows are precisely those rows j of the identity for which $j \in S_{i,\delta}$. Formally, such a matrix $R_{i,\delta}$ can be expressed as

$$R_{i,\delta} = [I_{i,\delta} | O]\, \pi_i \tag{3}$$

with $I_{i,\delta}$ the identity on $\mathbb{R}^{n_{i,\delta}}$. Finally, we define the weighting (or masking) matrices

$$E_{i,\delta} = R_{i,\delta}^T R_{i,\delta} \quad = \pi_i^T \begin{bmatrix} I_{i,\delta} & O \\ O & O \end{bmatrix} \pi_i \quad \in \mathbb{R}^{n \times n} \tag{4}$$

and the subspaces

$$W_{i,\delta} = \text{range}(E_{i,\delta}), \qquad i = 1, \ldots, p.$$

Note the inclusion $W_{i,\delta} \supseteq W_{i,\delta'}$ for $\delta \geq \delta'$ and, in particular, $W_{i,\delta} \supseteq W_{i,0}$ for all $\delta \geq 0$.

We view the matrices $R_{i,\delta}$ as restriction operators and $R_{i,\delta}^T$ as prolongations. We can identify the image of $R_{i,\delta}^T$ with the subspace $W_{i,\delta}$. For each subspace $W_{i,\delta}$ we define a restriction of the operator A on $W_{i,\delta}$ as

$$A_{i,\delta} = R_{i,\delta} A R_{i,\delta}^T.$$

To describe and analyze the classical Schwarz methods, the theory of orthogonal projections plays an important role; see, e.g., [17, Chap. 11], [25], and especially [5]. Therefore let

$$P_{i,\delta} = R_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} A, \tag{5}$$

provided that $A_{i,\delta}$ is nonsingular. It is not hard to see that this is a projection onto the subspace $W_{i,\delta}$. (In the case of symmetric A, this projection is orthogonal.) The additive Schwarz preconditioner is

$$M_{AS,\delta}^{-1} = \sum_{i=1}^{p} R_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} \tag{6}$$

and the preconditioned matrix is

$$M_{AS,\delta}^{-1} A = \sum_{i=1}^{p} P_{i,\delta}.$$

Similarly, the multiplicative Schwarz preconditioner $M_{MS,\delta}^{-1}$ is such that

$$T_{MS,\delta} = I - M_{MS,\delta}^{-1} A = (I - P_{p,\delta})(I - P_{p-1,\delta}) \cdots (I - P_{1,\delta}) = \prod_{i=p}^{1} (I - P_{i,\delta}). \tag{7}$$

Next we describe the *restricted* additive and multiplicative Schwarz preconditioners. We introduce "restricted" operators

$$\tilde{R}_{i,\delta} = R_{i,\delta} E_{i,0} \in \mathbb{R}^{n_{i,\delta} \times n}.$$

The image of $\tilde{R}_{i,\delta}^T = E_{i,0} R_{i,\delta}^T$ can be identified with $W_{i,0}$, so $\tilde{R}_{i,\delta}^T$ "restricts" $R_{i,\delta}^T$ in the sense that the image of the latter, $W_{i,\delta}$, is restricted to its subspace $W_{i,0}$, the space from the nonoverlapping decomposition. In the restricted (additive and multiplicative) Schwarz methods from [8, 10] the prolongation operator $R_{i,\delta}^T$ is replaced by $\tilde{R}_{i,\delta}^T$ and the (oblique) projection

$$Q_{i,\delta} = \tilde{R}_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} A = E_{i,0} P_{i,\delta}$$

is used; see [16]. Thus, the restricted counterparts to the operators (6) and (7) are

$$M_{RAS,\delta}^{-1} = \sum_{i=1}^{p} \tilde{R}_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta}$$

and

$$T_{RMS,\delta} = (I - Q_{p,\delta})(I - Q_{p-1,\delta}) \cdots (I - Q_{1,\delta}) = \prod_{i=p}^{1} (I - Q_{i,\delta}), \tag{8}$$

respectively. The iteration matrix of the RAS method is then

$$T_{RAS,\delta} = I - M_{RAS,\delta}^{-1} A = I - \sum_{i=1}^{p} \tilde{R}_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} A = I - \sum_{i=1}^{p} Q_{i,\delta}. \tag{9}$$

For practical parallel implementations, replacing $R_{i,\delta}^T$ by $\tilde{R}_{i,\delta}^T$ means that the corresponding part of the computation does not require any communication, since the images of the $\tilde{R}_{i,\delta}^T$ do not overlap. In addition, the numerical results in [10] indicate that the RAS method is faster (in terms of number of iterations and/or CPU time) than the classical one.

For the analysis of preconditioned Krylov subspace methods, the relevant matrices are $M_{AS,\delta}^{-1} A$ and $M_{RAS,\delta}^{-1} A$ for additive Schwarz and $I - T_{MS,\delta}$ and $I - T_{RMS,\delta}$ for multiplicative Schwarz. Alternatively, we can consider and compare the iteration matrices $T_{AS,\delta} = I - M_{AS,\delta}^{-1} A$, $T_{RAS,\delta}$ and $T_{MS,\delta}$, $T_{RMS,\delta}$. These correspond to stationary iterative methods, e.g., of the form

$$x^{k+1} = T_{RAS,\delta} x^k + M_{RAS,\delta}^{-1} b, \qquad k = 0, 1, \ldots,$$

for the RAS case; see, e.g., [18] for another example of such Schwarz iterations.

As in [2, 15, 16], the key to our analysis is a new (algebraic) representation of the restricted Schwarz methods. We construct a set of matrices $M_{i,\delta}$ associated with $R_{i,\delta}$ as follows:

$$M_{i,\delta} = \pi_i^T \begin{bmatrix} A_{i,\delta} & O \\ O & D_{\neg i,\delta} \end{bmatrix} \pi_i \tag{10}$$

and $D_{\neg i,\delta}$ is the diagonal part of the principal submatrix of A "complementary" to $A_{i,\delta}$; i.e.,

$$D_{\neg i,\delta} = \operatorname{diag}\left([O | I_{\neg i,\delta}] \cdot \pi_i \cdot A \cdot \pi_i^T \cdot [O | I_{\neg i,\delta}]^T \right)$$

with $I_{\neg i,\delta}$ the identity on $\mathbb{R}^{n - n_{i,\delta}}$. Here, we assume that $A_{i,\delta}$ and $D_{\neg i,\delta}$ are nonsingular. It can be shown (see [16]) that

$$\tilde{R}_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} = E_{i,0} M_{i,\delta}^{-1}, \qquad i = 1, \ldots, p,$$

and therefore

$$Q_{i,\delta} = \tilde{R}_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} A = E_{i,0} M_{i,\delta}^{-1} A, \qquad i = 1, \ldots, p.$$

With these fundamental identities the RAS and RMS methods can be described by the iteration matrices

$$T_{RAS,\delta} = I - \sum_{i=1}^{p} E_{i,0} M_{i,\delta}^{-1} A,$$

$$T_{RMS,\delta} = \prod_{i=p}^{1} (I - E_{i,0} M_{i,\delta}^{-1} A). \tag{11}$$

Moreover, we have

$$M_{RAS,\delta}^{-1} = \sum_{i=1}^{p} E_{i,0} M_{i,\delta}^{-1}.$$

In the rest of this section, we list some basic terminology and some well-known results which we use in the rest of the paper.

The natural partial ordering $\leq$ between matrices $A = (a_{ij})$, $B = (b_{ij})$ of the same size is defined componentwise; i.e., $A \leq B$ iff $a_{ij} \leq b_{ij}$ for all i, j. If $A \geq O$, we call A nonnegative. If all entries of A are positive, we say that A is positive and write $A > O$. This notation and terminology carries over to vectors as well.

A nonsingular matrix $A \in \mathbb{R}^{n \times n}$ is called monotone if $A^{-1} \geq O$. A monotone matrix $A \in \mathbb{R}^{n \times n}$ is called a (nonsingular) M-matrix if it has nonpositive off-diagonal elements. The following lemma states some useful properties of M-matrices; see, e.g., [4, 28].

Lemma 2.1. *Let $A, B \in \mathbb{R}^{n \times n}$ be two nonsingular M-matrices with $A \leq B$. Then we have the following:*

(i) *Every principal submatrix of A or B is again an M-matrix.*
(ii) *Every matrix D such that $A \leq D \leq B$ is an M-matrix. In particular, if $A \leq D \leq \operatorname{diag}(A)$, then D is an M-matrix.*
(iii) *$B^{-1} \leq A^{-1}$.*

Our convergence results are formulated in terms of nonnegative splittings according to the following definition.

Definition 2.2. *Consider the splitting $A = M - N \in \mathbb{R}^{n \times n}$ with M nonsingular. This splitting is said to be*

(i) regular *if $M^{-1} \geq O$ and $N \geq O$,*
(ii) weak nonnegative of the first type *(also called* weak regular*) if $M^{-1} \geq O$ and $M^{-1}N \geq O$,*
(iii) weak nonnegative of the second type *if $M^{-1} \geq O$ and $NM^{-1} \geq O$, and*
(iv) nonnegative *if $M^{-1} \geq O$, $M^{-1}N \geq O$, and $NM^{-1} \geq O$.*

Note that all the above splittings $A = M - N$ are *convergent* splittings for M-matrices A; i.e., the spectral radius $\rho(M^{-1}N)$ of the iteration matrix $M^{-1}N$ is less than one. Given an iteration matrix, there is a unique splitting for it, which is stated by the following result; see [3].

Lemma 2.3. *Let A and T be square matrices such that A and $I - T$ are nonsingular. Then there exists a unique pair of matrices B, C such that B is nonsingular, $T = B^{-1}C$, and $A = B - C$. The matrices are $B = A(I - T)^{-1}$ and $C = B - A = A((I - T)^{-1} - I)$.*

For a positive vector w, we denote $\|x\|_w$ the weighted max norm in $\mathbb{R}^n$ given by

$$\|x\|_w = \max_{i=1,\ldots,n} |x_i|/w_i.$$

The resulting operator norm in $\mathbb{R}^{n\times n}$ is denoted similarly, and for $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ we have (see, e.g., [24])

$$\|B\|_w = \max_{i=1,\ldots,n} \left(\sum_{j=1}^{n} |b_{ij}| w_j \right) / w_i. \tag{12}$$

The following lemma follows directly from (12).

LEMMA 2.4. *Let $T, \tilde{T}$ be nonnegative matrices. Assume that $Tw \le \tilde{T}w$ for some vector $w > 0$. Then $\|T\|_w \le \|\tilde{T}\|_w$.*

3. Convergence and comparisons of RMS. In this section, we show that for a monotone matrix A the restricted multiplicative Schwarz iteration is convergent. Moreover, we establish that the spectral radius of the RMS iteration matrix is less than or equal to the spectral radius of the RAS iteration matrix, and it is no smaller than the spectral radius of the classical multiplicative Schwarz method (Theorems 3.5 and 3.8).

We begin by stating a lemma proved in [2].

LEMMA 3.1. *Let A be monotone, and let a collection of p triples (E_i, M_i, N_i) be given such that $O \le E_i \le I$, $\sum_{i=1}^{p} E_i \ge I$, and $A = M_i - N_i$ is a weak regular splitting for $i = 1, \ldots, p$. Let*

$$T = (I - E_p M_p^{-1} A)(I - E_{p-1} M_{p-1}^{-1} A) \cdots (I - E_1 M_1^{-1} A).$$

Then T is nonnegative and, for any vector $w = A^{-1}e > 0$ with $e > 0$, $\rho(T) \le \|T\|_w < 1$.

Now we formulate one of the main results of this section. It is the counterpart to Theorem 4.4 [16], where it was shown that the RAS method is convergent, and the iteration matrix (9) induces a weak regular splitting.

THEOREM 3.2. *Let A be a nonsingular M-matrix. Then for each value of $\delta \ge 0$ and for any $w = A^{-1}e > 0$ with $e > 0$, we have $\rho(T_{RMS,\delta}) \le \|T_{RMS,\delta}\|_w < 1$. Furthermore, there exists a unique splitting $A = M_{RMS,\delta} - N_{RMS,\delta}$ such that $T_{RMS,\delta} = M_{RMS,\delta}^{-1} N_{RMS,\delta}$, and this splitting is weak regular (i.e., weak nonnegative of the first type). The matrix $M_{RMS,\delta}$ is given by $M_{RMS,\delta} = A(I - T_{RMS,\delta})^{-1}$.*

Proof. The proof we present is almost the same as the proof of the convergence of the classical multiplicative Schwarz method given in [2]. Let $E_{i,0}$ as in (4) and $M_{i,\delta}$ as in (10). Observe that $O \le E_{i,0} \le I$, $i = 1, \ldots, p$. We have already seen that

$$I - Q_{i,\delta} = I - E_{i,0} M_{i,\delta}^{-1} A, \qquad i = 1, \ldots, p.$$

Moreover, it is not hard to see that the splittings $A = M_{i,\delta} - N_{i,\delta}$ (with $N_{i,\delta} = M_{i,\delta} - A$) are regular. Hence, by Lemma 3.1, $T_{RMS,\delta} \ge O$ and $\rho(T_{RMS,\delta}) \le \|T_{RMS,\delta}\|_w < 1$ for any $w = A^{-1}e > 0$ with $e > 0$. Furthermore, by Lemma 2.3, there exists a unique splitting $A = M_{RMS,\delta} - N_{RMS,\delta}$ such that $T_{RMS,\delta} = M_{RMS,\delta}^{-1} N_{RMS,\delta}$. To prove that the splitting is weak regular it suffices to show that

$$M_{RAS,\delta}^{-1} = (I - T_{RMS,\delta}) A^{-1} \ge O \tag{13}$$

or, equivalently, that $M_{RAS,\delta}^{-1} z \ge 0$ for all $z \ge 0$. Letting $v = A^{-1} z \ge 0$, all we need to show is that $(I - T_{RMS,\delta}) v \ge 0$ or $T_{RMS,\delta} v \le v$. This is proved in the same way as Lemma 3.1; see [2]. Hence, the unique splitting $A = M_{RMS,\delta} - N_{RMS,\delta}$ is weak regular. $\square$

In Example 3.3, we show that the splittings induced by the RAS method and the RMS method are, in general, not nonnegative, i.e., are not weak nonnegative of the second type. This is in contrast with the classical Schwarz methods; see [2].

Example 3.3. For the RAS method, we have to consider

$$\bar{T} = I - AM_{RAS}^{-1} = I - A\sum_{i=1}^{p} E_{i,0}M_{i,\delta}^{-1},$$

while for the RMS method

$$\tilde{T} = N_{RMS,\delta}M_{RMS,\delta}^{-1} = I - AM_{RMS,\delta}^{-1} = I - A(I - T_{RMS,\delta})A^{-1}.$$

It is not hard to see that $\tilde{T} = \prod_{i=p}^{1}(I - \tilde{Q}_{i,\delta})$ with $\tilde{Q}_{i,\delta} = AE_{i,\delta}M_{i,\delta}^{-1}$. Now let

$$A = \begin{bmatrix} 6 & -1 & -2 \\ -2 & 8 & -3 \\ -1 & -1 & 4 \end{bmatrix}, \quad M_{1,\delta} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad E_{1,0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M_{2,\delta} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad E_{2,0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$M_{3,\delta} = \begin{bmatrix} 6 & -1 & 0 \\ -2 & 8 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad E_{3,0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We obtain

$$\bar{T} = I - A\sum_{i=1}^{3} E_{i,0}M_{i,\delta}^{-1} = \begin{bmatrix} 0.0435 & -0.0054 & 0.5000 \\ 0.3478 & 0.0435 & 0.7500 \\ 0.1739 & 0.1467 & 0 \end{bmatrix} \ngeq 0,$$

$$\tilde{T} = \prod_{i=3}^{1}(I - AE_{i,0}M_{i,\delta}^{-1}) = \begin{bmatrix} -0.0435 & -0.0054 & -0.0258 \\ 0.3478 & 0.0435 & 0.2065 \\ 0.1739 & 0.1467 & 0.1970 \end{bmatrix} \ngeq 0.$$

In the case of no overlap, the RMS method as well as the classical multiplicative Schwarz method reduce to a block Gauss–Seidel method. Similarly, with no overlap, the RAS and the classical additive Schwarz methods reduce to the block Jacobi method. The classical Stein–Rosenberg theorem (see, e.g., [28]) says that for M-matrices the Gauss–Seidel method converges faster than the Jacobi method. The next theorem extends this statement to the case of overlap. We are able to compare the RMS method with the RAS method. We point out that only recently a similar result was obtained for the classical Schwarz methods [20].

We need the following lemma; see [15, 21].

LEMMA 3.4. *Let $A^{-1} \geq O$. Let $A = \bar{M} - \bar{N} = M - N$ be two weak regular splittings such that*

$$\bar{M}^{-1} \geq M^{-1}.$$

Let $w > 0$ be such that $w = A^{-1}e$ for some $e > 0$. Then $\|\bar{M}^{-1}\bar{N}\|_w \leq \|M^{-1}N\|_w$.

THEOREM 3.5. *Let A be a nonsingular M-matrix, and let $w > 0$ be any positive vector such that $Aw > 0$, e.g., $w = A^{-1}v$ with $v > 0$. Then*

$$\|T_{RMS,\delta}\|_w \leq \|T_{RAS,\delta}\|_w.$$

Moreover, if the Perron vector w_δ of $T_{RAS,\delta}$ satisfies $w_\delta > 0$ and $Aw_\delta \geq 0$, then we also have

$$\rho(T_{RMS,\delta}) \leq \rho(T_{RAS,\delta}).$$

Proof. We will use Lemma 3.4. The splittings corresponding to the RAS and RMS methods are weak regular splittings, and, in particular, the matrices $M_{RAS,\delta}^{-1}$ and $M_{RMS,\delta}^{-1}$ are nonnegative; see (13). Next we show that

$$M_{RMS,\delta}^{-1} \geq M_{RAS,\delta}^{-1}.$$

To that end, we write explicitly $M_{RMS,\delta}^{-1}$ using (13) and (11) as follows:

$$\begin{aligned} M_{RMS,\delta}^{-1} &= \left(I - \prod_{i=p}^{1} (I - E_{i,0} M_{i,\delta}^{-1} A) \right) A^{-1} \\ &= \Big(I - (I - E_{p,0} M_{p,\delta}^{-1} A)(I - E_{p-1,0} M_{p-1,\delta}^{-1} A) \cdots (I - E_{1,0} M_{1,\delta}^{-1} A) \Big) A^{-1}. \end{aligned}$$

Thus, by computing the product $M_{RMS,\delta}^{-1}$ can be written as

$$\begin{aligned} M_{RMS,\delta}^{-1} &= \sum_{i=1}^{p} E_{i,0} M_{i,\delta}^{-1} - \sum_{i<j} E_{j,0} M_{j,\delta}^{-1} A E_{i,0} M_{i,\delta}^{-1} \\ &\quad + \sum_{m=3}^{p} \sum_{\substack{(j_1,\ldots,j_m) \\ j_i \in \{1,\ldots,p\}, \\ j_i > j_k \text{ if } i<k}} (-1)^{m+1} \left(\prod_{k=1}^{m} E_{j_k,0} M_{j_k,\delta}^{-1} A \right) A^{-1}. \end{aligned} \tag{14}$$

Note that in each product above all $E_{j_k,0}$ are different, i.e., $E_{j_k,0} \neq E_{j_i,0}$ for $k \neq i$.

The first sum in (14) is just $M_{RAS,\delta}^{-1}$. Thus, all that remains to be shown is that the remaining part of (14) is a nonnegative matrix. To do so, we first consider matrices of the form

$$E_{j,0} M_{j,\delta}^{-1} A E_{i,0} M_{i,\delta}^{-1}.$$

Since $E_{j,0} M_{j,\delta}^{-1} = E_{j,0} M_{j,\delta}^{-1} E_{j,\delta}$ and $E_{i,0} M_{i,\delta}^{-1} = E_{i,0} M_{i,\delta}^{-1} E_{i,\delta}$ we have that

$$E_{j,0} M_{j,\delta}^{-1} A E_{i,0} M_{i,\delta}^{-1} = E_{j,0} M_{j,\delta}^{-1} \left(E_{j,\delta} A E_{i,0} \right) M_{i,\delta}^{-1} E_{i,\delta}. \tag{15}$$

We consider two cases.

Case (a). $S_{j,\delta} \cap S_{i,0} = \emptyset$. Since A is an M-matrix,

$$(E_{j,\delta} A E_{i,0})_{s,t} \begin{cases} \leq 0 & \text{if } \ s \in S_{j,\delta},\ t \in S_{i,0} \\ = 0 & \text{otherwise.} \end{cases}$$

Thus, since $E_{j,0} M_{j,\delta}^{-1}$ and $M_{i,\delta}^{-1}$ are nonnegative, we obtain

$$E_{j,0} M_{j,\delta}^{-1} A E_{i,0} M_{i,\delta}^{-1} \leq O.$$

Case (b). $S_{j,\delta} \cap S_{i,0} \neq \emptyset$. With the construction on $M_{j,\delta}^{-1}$ in (10), we obtain here

$$\left(E_{j,0} M_{j,\delta}^{-1} A\right)_{s,t} \begin{cases} = 0 & \text{if } \ s \notin S_{j,0} \\ = 1 & \text{if } \ s \in S_{j,0}, \ s = t \\ = 0 & \text{if } \ s \in S_{j,0}, \ t \in S_{j,\delta}, \ s \neq t \\ \leq 0 & \text{otherwise.} \end{cases} \tag{16}$$

Since $S_{j,0} \cap S_{i,0} = \emptyset$ it follows that

$$E_{j,0} M_{j,\delta}^{-1} A E_{i,0} M_{i,\delta}^{-1} \leq O.$$

Therefore in both cases we obtain

$$-\sum_{i<j} E_{j,0} M_{j,\delta}^{-1} A E_{i,0} M_{i,\delta}^{-1} \geq O.$$

Moreover, in both cases we have with (15) that

$$\left(E_{j,0} M_{j,\delta}^{-1} A E_{i,0} M_{i,\delta}^{-1}\right)_{s,t} \begin{cases} \leq 0 & \text{if } \ s \in S_{j,0}, \ t \in S_{i,\delta} \\ = 0 & \text{otherwise.} \end{cases} \tag{17}$$

Finally, consider the terms of the third sum in (14), i.e., consider

$$\begin{aligned} &(-1)^{m+1} \left(\prod_{k=1}^{m} E_{j_k,0} M_{j_k,\delta}^{-1} A \right) A^{-1} \\ &= (-1)^{m+1} \left(\prod_{k=1}^{m-2} E_{j_k,0} M_{j_k,\delta}^{-1} A \right) E_{j_2,0} M_{j_2,\delta}^{-1} A E_{j_1,0} M_{j_1,\delta}^{-1} \end{aligned}$$

with $m \geq 3$. Each of these contains one factor of the form (17), while the other $m-2$ factors are of the form (16). Since the matrices $E_{j_k,0}$ are different for different values of k, the entries with value 1 in (16) get multiplied by zeros when performing the product. This implies that in this case every entry in (16) is nonpositive.

We proceed now by induction on the number of factors. If we have an even number of factors, we have $(-1)^{m+1} = -1$, but since the factor of the form (17) is nonpositive and each of the other $m-2$ factors of the form (16) is also nonpositive, the product is a nonnegative matrix. Similarly, if m is odd $(-1)^{m+1} = 1$, but we have an odd number of nonpositive factors of the form (16) and the nonpositive factor of the form (17). Thus, the product is a nonnegative matrix. Hence in both cases we have

$$(-1)^{m+1} \left(\prod_{k=1}^{m} E_{j_k,0} M_{j_k,\delta}^{-1} A \right) A^{-1} \geq O.$$

Therefore $M_{RMS,\delta}^{-1} \geq M_{RAS,\delta}^{-1}$. Hence with Lemma 3.4 we obtain

$$\|T_{RMS,\delta}\|_w \leq \|T_{RAS,\delta}\|_w.$$

Now, if the Perron vector w_δ of $T_{RAS,\delta}$ satisfies $w_\delta > 0$ and $Aw_\delta \geq 0$, we also have $\|T_{RAS,\delta}\|_{w_\delta} = \rho(T_{RAS,\delta})$. Thus $\rho(T_{RMS,\delta}) \leq \rho(T_{RAS,\delta})$. ∎

To end this section, we compare the RMS method with its classical version. We need first the following two lemmas. The first one is well known and can be found, e.g., in [4], while the second is from [2].

LEMMA 3.6. *Assume that a square matrix T is nonnegative and that for some $\alpha \geq 0$ and for some nonzero vector $x \geq 0$ we have $Tx \geq \alpha x$. Then $\rho(T) \geq \alpha$. The inequality is strict if $Tx > \alpha x$.*

LEMMA 3.7. *Let $A^{-1} \geq O$. Let $A = M - N$ be a splitting such that $M^{-1} \geq O$ and $NM^{-1} \geq O$. Then $\rho(M^{-1}N) < 1$ and there exists a nonzero vector $x \geq 0$ such that $M^{-1}Nx = \rho(M^{-1}N)x$ and $Ax \geq 0$.*

THEOREM 3.8. *Let A be a nonsingular M-matrix, and let $w > 0$ be any positive vector such that $Aw > 0$. Let $T_{RMS,\delta}$ and $T_{MS,\delta}$ be as in (8) and (7); then, for any $\delta \geq 0$,*

$$\|T_{MS,\delta}\|_w \leq \|T_{RMS,\delta}\|_w < 1.$$

Moreover, $\rho(T_{MS,\delta}) \leq \rho(T_{RMS,\delta})$.

Proof. The proof is similar to that of Theorem 4.7 of [2]. We have already seen that $T_{RMS,\delta}$ and $T_{MS,\delta}$ are nonnegative matrices. By Theorem 3.5 of [2] the iteration matrix $T_{MS,\delta}$ induces a nonnegative splitting of A. Let $x \geq 0$, $x \neq 0$ be an eigenvector of $T_{MS,\delta}$ with eigenvalue $\rho(T_{MS,\delta})$. We will show that

$$T_{RMS,\delta}\, x \;\geq\; T_{MS,\delta}\, x \;=\; \rho(T_{MS,\delta})\, x, \tag{18}$$

so that by Lemma 3.6 we get the desired result $\rho(T_{RMS,\delta}) \geq \rho(T_{MS,\delta})$. Let $x^0 = \bar{x}^0 = x$ and define $x^i := (I - E_{i,\delta}M_{i,\delta}^{-1}A)x^{i-1}$ and $\bar{x}^i := (I - E_{i,0}M_{i,\delta}^{-1}A)\bar{x}^{i-1}$, $i = 1, \ldots, p$. Thus, $x^p = T_{MS,\delta}x$ and $\bar{x}^p = T_{RMS,\delta}x$. To establish (18) we proceed by induction and show that

$$Ax^i \geq 0, \qquad i = 1, \ldots, p-1, \tag{19}$$

and

$$0 \leq x^i \leq \bar{x}^i, \qquad i = 1, \ldots, p. \tag{20}$$

We then have (18) since $x^p = T_{MS,\delta}x$ and $\bar{x}^p = T_{RMS,\delta}x$; see (7) and (8).

For $i = 0$, (20) holds by assumption, while relation (19) is true by Lemma 3.7. Assume now that (19) and (20) are both true for some i. To obtain (19) for $i+1$, observe that $Ax^{i+1} = A(I - E_{i,\delta}M_{i,\delta}^{-1}A)x^i = (I - AE_{i,\delta}M_{i,\delta}^{-1})Ax^i$. We have $I - AE_{i,\delta}M_{i,\delta}^{-1} \geq O$, since

$$\begin{aligned} I - M_{i,\delta}^{-T}E_{i,\delta}^T A^T = I - E_{i,\delta}M_{i,\delta}^{-T}A^T &= I - E_{i,\delta} + E_{i,\delta}(I - M_{i,\delta}^{-T}A^T) \\ &= I - E_{i,\delta} + E_{i,\delta}M_{i,\delta}^{-T}N_{i,\delta}^T \geq O, \end{aligned}$$

with $N_{i,\delta} := M_{i,\delta} - A \geq 0$. Moreover, $Ax^i \geq 0$ by the induction hypothesis, and thus (19) holds for $i+1$. To prove that (20) holds for $i+1$, we use (19), the fact $E_{i,0} \leq E_{i,\delta}$, and the induction hypothesis to obtain

$$x^{i+1} = (I - E_{i,\delta}M_{i,\delta}^{-1}A)x^i \leq (I - E_{i,0}M_{i,\delta}^{-1}A)x^i \leq (I - E_{i,0}M_{i,\delta}^{-1}A)\bar{x}^i = \bar{x}^{i+1}.$$

To establish the inequalities for the weighted max norms, one proceeds in precisely the same manner as before (using w instead of x) to show $T_{RMS,\delta}w \geq T_{MS,\delta}w$. Since both matrices are nonnegative, by Lemma 2.4 we get $\|T_{MS,\delta}\|_w \leq \|T_{RMS,\delta}\|_w$. ◻

4. Inexact local solves. In the previous section, the subdomain problems were assumed to be solved exactly, and this is represented by the inverses of the matrices $A_{i,\delta}$. In this section, we consider the case were the subdomain problems are solved approximatively or, in other words, inexactly. We represent this fact by using an approximation $\tilde{A}_{i,\delta}$ of the matrix $A_{i,\delta}$. In practice, one uses, for example, an incomplete factorization of $A_{i,\delta}$; see, e.g., [19, 27].

As in [15], suppose that the inexact solves are such that the splittings

$$A_{i,\delta} = \tilde{A}_{i,\delta} - (\tilde{A}_{i,\delta} - A_{i,\delta}) \qquad \text{are weak regular splittings} \tag{21}$$

for $i = 1, \ldots, p$ or that

$$\tilde{A}_{i,\delta} \ \text{ is an } M\text{-matrix and } \ \tilde{A}_{i,\delta} \geq A_{i,\delta}, \qquad i = 1, \ldots, p. \tag{22}$$

Note that (22) implies (21). The incomplete factorizations satisfy (21) [19].

The restricted multiplicative Schwarz iteration with inexact solves on the subdomains is then given by

$$\tilde{T}_{RMS,\delta} = \prod_{i=p}^{1} (I - \tilde{R}_{i,\delta} \tilde{A}_{i,\delta}^{-1} R_{i,\delta} A).$$

In a way similar to (10), we construct matrices

$$\tilde{M}_{i,\delta} = \pi_i^T \begin{bmatrix} \tilde{A}_{i,\delta} & O \\ O & D_{\neg i,\delta} \end{bmatrix} \pi_i \tag{23}$$

such that

$$\tilde{R}_{i,\delta} \tilde{A}_{i,\delta}^{-1} R_{i,\delta} A = E_{i,0} \tilde{M}_{i,\delta}^{-1} A, \qquad i = 1, \ldots, p,$$

and thus

$$\tilde{T}_{RMS,\delta} = \prod_{i=p}^{1} (I - E_{i,0} \tilde{M}_{i,\delta}^{-1} A). \tag{24}$$

We can now establish our convergence result.

THEOREM 4.1. *Let A be a nonsingular M-matrix. Then the RMS iteration matrix* (24) *with inexact solves satisfying* (21) *satisfies $\rho(\tilde{T}_{RMS,\delta}) \leq \|\tilde{T}_{RMS,\delta}\|_w < 1$ for any $w = A^{-1}e > 0$ with $e > 0$. Furthermore, there exists a unique splitting $A = \tilde{B} - \tilde{C}$ such that $\tilde{T}_{RMS,\delta} = \tilde{B}^{-1}\tilde{C}$, and this splitting is weak regular.*

Proof. The proof proceeds in the same manner as that of Theorem 3.2. All we need to show is that each splitting $A = \tilde{M}_{i,\delta} - \tilde{N}_{i,\delta}$ with $\tilde{M}_{i,\delta}$ as in (23) is weak regular. Since $\tilde{A}_{i,\delta}$ is monotone, it follows from (23) that $\tilde{M}_{i,\delta}^{-1} \geq O$. With

$$\pi_i A \pi_i^T := \begin{bmatrix} A_{i,\delta} & K_i \\ L_i & A_{\neg i,\delta} \end{bmatrix}$$

and $\tilde{N}_{i,\delta} = \tilde{M}_{i,\delta} - A$ we have

$$\pi_i \tilde{M}_{i,\delta}^{-1} \tilde{N}_{i,\delta} \pi_i^T = \begin{bmatrix} \tilde{A}_{i,\delta}^{-1}(\tilde{A}_{i,\delta} - A_{i,\delta}) & -\tilde{A}_{i,\delta}^{-1} K_{i,\delta} \\ -D_{\neg i}^{-1} L_{i,\delta} & D_{\neg i}^{-1}(D_{\neg i} - A_{\neg i,\delta}) \end{bmatrix},$$

which, in view of (21) and the fact that A is an M-matrix, is nonnegative. □

Note that if (22) holds, then $\tilde{N}_{i,\delta} = \tilde{M}_{i,\delta} - A \geq O$ and $A = \tilde{M}_{i,\delta} - \tilde{N}_{i,\delta}$ is in fact a regular splitting.

It is shown in [16] that if (21) holds, the inexact RAS method given by

$$\tilde{T}_{RAS,\delta} = I - \sum_{i=1}^{p} \tilde{R}_{i,\delta}^T \tilde{A}_{i,\delta}^{-1} R_{i,\delta} A$$

is also convergent and that this matrix also induces a weak regular splitting. We use these properties to compare the inexact RMS method with the inexact RAS method.

THEOREM 4.2. *Let A be an M-matrix and consider the inexact RAS method and the inexact RMS method where the matrices $\tilde{A}_{i,\delta}$ corresponding to the inexact solves satisfy* (21)*. Then, for any positive vector w such that $Aw > 0$ and any $\delta \geq 0$, we have*

$$\|\tilde{T}_{RMS,\delta}\|_w \leq \|\tilde{T}_{RAS,\delta}\|_w < 1. \tag{25}$$

Moreover, if the Perron vector w_δ of $T_{RAS,\delta}$ satisfies $w_\delta > 0$ and $Aw_\delta \geq 0$, then we also have

$$\rho(\tilde{T}_{RMS,\delta}) \leq \rho(\tilde{T}_{RAS,\delta}). \tag{26}$$

Proof. The proof is similar to the proof of Theorem 3.5. With (21) and (23), all matrices $\tilde{M}_{i,\delta}^{-1}$ are nonnegative. We will show that

$$\tilde{M}_{RMS,\delta}^{-1} \geq \tilde{M}_{RAS,\delta}^{-1}.$$

Following the proof of Theorem 3.5 we have only to modify Case (b). However, with (21) we have $\tilde{A}_{i,\delta}^{-1} A_{i,\delta} \leq I$, and thus

$$\left(E_{j,0} \tilde{M}_{j,\delta}^{-1} A\right)_{s,t} \begin{cases} = 0 & \text{if } \; s \notin S_{j,0} \\ \leq 1 & \text{if } \; s \in S_{j,0}, \; s = t \\ \leq 0 & \text{otherwise.} \end{cases}$$

We then proceed as in the proof of Theorem 3.5.

If the Perron vector w_δ can be chosen as w, we have $\|T_{RAS,\delta}\|_{w_\delta} = \rho(T_{RAS,\delta})$, so that (25) yields $\|T_{RMS,\delta}\|_{w_\delta} \leq \rho(T_{RAS,\delta})$, and since the spectral radius is never larger than any induced operator norm we have (26). □

Next we relate the speed of convergence to the exactness of the subdomain solver.

THEOREM 4.3. *Let A be an M-matrix. Consider two inexact RMS methods where the matrices $\tilde{A}_{i,\delta}$ and $\hat{A}_{i,\delta}$ corresponding to the inexact solves satisfy* (22) *and*

$$O \leq \hat{A}_{i,\delta}^{-1} \leq \tilde{A}_{i,\delta}^{-1} \leq A_{i,\delta}^{-1}, \qquad i = 1, \ldots, p.$$

Let the corresponding iteration matrices be as in (24)*. Then, for any positive vector w such that $Aw > 0$ and any $\delta \geq 0$, we have*

$$\|T_{RMS,\delta}\|_w \leq \|\tilde{T}_{RMS,\delta}\|_w \leq \|\hat{T}_{RMS,\delta}\|_w < 1. \tag{27}$$

Proof. From the hypothesis, (10), and (23) it follows that

$$M_{i,\delta}^{-1} \geq \tilde{M}_{i,\delta}^{-1} \geq \hat{M}_{i,\delta}^{-1}.$$

Following the proof of Theorem 3.5, this establishes (27). □

5. The effect of overlap on RMS. We study in this section the effect of varying the overlap. More precisely, we prove comparison results on the spectral radii and/or on weighted max norms for the corresponding iteration matrices

$$T_{RMS,\delta} = I - M_{RMS,\delta}^{-1} A$$

for different values of $\delta \geq 0$.

We start with a result which compares one RMS iterative process, defined through the sets $S_{i,\delta'}$, with another one with more overlap defined through sets $S_{i,\delta}$, where $S_{i,\delta'} \subseteq S_{i,\delta}$, $i = 1, \ldots, p$. We show that the larger the overlap ($\delta \geq \delta'$) the faster the RMS method converges as measured in certain weighted max norms.

THEOREM 5.1. *Let A be a nonsingular M-matrix, and let $w > 0$ be any positive vector such that $Aw > 0$. Then, if $\delta \geq \delta'$,*

$$\|T_{RMS,\delta}\|_w \leq \|T_{RMS,\delta'}\|_w < 1. \tag{28}$$

Moreover, if the Perron vector $w_{\delta'}$ of $T_{RMS,\delta'}$ satisfies $w_{\delta'} > 0$ and $Aw_{\delta'} \geq 0$, then we also have

$$\rho(T_{RMS,\delta}) \leq \rho(T_{RMS,\delta'}). \tag{29}$$

Proof. Since $S_{i,\delta'} \subseteq S_{i,\delta}$, $i = 1, \ldots, p$, we have $A \leq M_{i,\delta} \leq M_{i,\delta'} \leq \text{diag}(A)$. Since A is an M-matrix, this yields

$$M_{i,\delta}^{-1} \geq M_{i,\delta'}^{-1} \quad \text{for} \quad i = 1, \ldots, p.$$

Next we compare the matrices $M_{RMS,\delta}^{-1}$ and $M_{RMS,\delta'}^{-1}$. To do so consider (14) in the proof of Theorem 3.5. Since all the parts in the sum are nonnegative, we get (28).

Now, if the Perron vector $w_{\delta'}$ can be chosen as w, we have $\|T_{RMS,\delta'}\|_{w_{\delta'}} = \rho(T_{RMS,\delta'})$ so that (28) yields (29). ◻

As a special case of Theorem 5.1 above we choose $\delta' = 0$, i.e., a block Gauss–Seidel method. In this case, we do not need any additional assumption for comparing the spectral radii. To that end, we use the following comparison theorem due to Woźnicki [29]; see also [12].

THEOREM 5.2. *Let $A^{-1} \geq O$ and two splittings $A = M - N = \tilde{M} - \tilde{M}$, where one of them is weak nonnegative of the first type (weak regular) and the other is weak nonnegative of the second type. If $M^{-1} \geq \tilde{M}^{-1}$, then*

$$\rho(M^{-1}N) \leq \rho(\tilde{M}^{-1}\tilde{N}).$$

THEOREM 5.3. *Let A be a nonsingular M-matrix. Then, for any value of $\delta \geq 0$, $\rho(T_{RMS,\delta}) \leq \rho(T_{RMS,0})$.*

Proof. The proof follows immediately from the above results and the fact that the block Gauss–Seidel splitting is a regular splitting. ◻

6. Varying the number of domains. In this section, we show how the partitioning of a subdomain into smaller subdomains affects the convergence of the restricted Schwarz method. In the M-matrix case, we show that, for both additive and multiplicative restricted Schwarz methods, the more subdomains the slower the convergence rate.

Formally, consider each block of variables $S_{i,\delta}$ partitioned into k_i subblocks; i.e., we have

$$S_{i_j,\delta} \subset S_{i,\delta}, \qquad j = 1, \ldots, k_i, \tag{30}$$

$\bigcup_{j=1}^{k_i} S_{i_j,\delta} = S_{i,\delta}$, and $S_{i_j,\delta} \cap S_{i_k,\delta} = \emptyset$ if $j \neq k$. Each set $S_{i_j,\delta}$ has associated matrices $R_{i_j,\delta}$ and $E_{i_j,\delta} = R^T_{i_j,\delta} R_{i_j,\delta}$. Since we have a partition,

$$E_{i_j,\delta} \leq E_{i,,\delta}, \qquad j = 1, \dots, k_i, \quad \text{and} \quad \sum_{j=1}^{k_i} E_{i_j,\delta} = E_{i,\delta}, \qquad i = 1, \dots, p. \tag{31}$$

We define the matrices $A_{i_j,\delta} = R_{i_j,\delta} A R^T_{i_j,\delta}$, and $M_{i_j,\delta}$ corresponding to the set $S_{i_j,\delta}$ as in (10) so that

$$E_{i_j,\delta} M^{-1}_{i_j,\delta} = R^T_{i_j,\delta} A^{-1}_{i_j} R_{i_j,\delta}, \qquad j = 1, \dots, k_i, \qquad i = 1, \dots, p.$$

The iteration matrix of the restricted additive Schwarz method with the refined partition is then

$$\bar{T}_{RAS,\delta} = I - \sum_{i=1}^{p} \sum_{j=1}^{k_i} E_{i_j,0} M^{-1}_{i_j,\delta} A, \tag{32}$$

and the unique induced splitting $A = \bar{M}_{RAS,\delta} - \bar{N}_{RAS,\delta}$ (which is a weak regular splitting) is given by

$$\bar{M}^{-1}_{RAS,\delta} = \sum_{i=1}^{p} \sum_{j=1}^{k_i} E_{i_j,0} M^{-1}_{i_j,\delta}.$$

THEOREM 6.1. *Let A be a nonsingular M-matrix. Consider two sets of subblocks of A defined by* (2) *and* (30), *respectively, and the two corresponding RAS iterations* (9) *and* (32). *Then, for every $\delta \geq 0$ and for any vector $w > 0$ for which $Aw > 0$, $\|T_{RAS,\delta}\|_w \leq \|\bar{T}_{RAS,\delta}\|_w$.*

Proof. The inclusion (30) implies that

$$M^{-1}_{i_j,\delta} \leq M^{-1}_{i,\delta}, \qquad j = 1, \dots, k_i, \qquad i = 1, \dots, p. \tag{33}$$

Thus, with (31) we have

$$\sum_{j=1}^{k_i} E_{i_j,0} M^{-1}_{i_j,\delta} \leq \sum_{j=1}^{k_i} E_{i_j,0} M^{-1}_{i,\delta} = E_{i,0} M^{-1}_{i,\delta}$$

and therefore $\bar{M}^{-1}_{RAS,\delta} \leq M^{-1}_{RAS,\delta}$, which implies the result, using Lemma 3.4. ∎

Next we consider the RMS method. The iteration matrix for the RMS method corresponding to the finer partition (more subdomains) is given by

$$\tilde{T}_{RMS,\delta} = \prod_{i=p}^{1} \prod_{j=k_i}^{1} (I - Q_{i_j,\delta}), \tag{34}$$

where $Q_{i_j,\delta} = E_{i_j,0} M^{-1}_{i_j,\delta} A = \tilde{R}^T_{i_j,\delta} A^{-1}_{i_j,\delta} R_{i_j,\delta} A$.

THEOREM 6.2. *Let A be a nonsingular M-matrix. Consider two sets of subblocks of A defined by* (2) *and* (30), *respectively, and the two corresponding RMS iterations* (8) *and* (34). *Then, for any $\delta \geq 0$ and for any vector $w > 0$ for which $Aw > 0$, $\|T_{RMS,\delta}\|_w \leq \|\tilde{T}_{RMS,\delta}\|_w$.*

Proof. Since each $Q_i = E_{i,0}M_{i,\delta}^{-1}A = R_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} A$ is a projection [16], we have

$$I - Q_i = (I - Q_i)^2 = \cdots = (I - Q_i)^{k_i}.$$

This allows us to represent $T_{RMS,\delta}$ and $\tilde{T}_{RMS,\delta}$ as a product with the same number of factors. We pair each factor $I - Q_{i_j} = I - E_{i_j,0}M_{i_j}^{-1}A$ of $\tilde{T}_{RMS,\delta}$ in (34) with the corresponding factor $I - Q_i = I - E_{i,0}M_{i,\delta}^{-1}A$ of $T_{RMS,\delta}$ in (8). The corresponding set of indices S_{i_j} and S_i satisfy $S_{i_j} \subseteq S_i$. By (31) and (33) we have that $E_{i_j,0}M_{i_j,\delta}^{-1} \leq E_{i,0}M_{i,\delta}^{-1}$. Therefore we can proceed in exactly the same manner as in the proof of Theorem 3.8 to establish the desired result. ∎

7. RMS variants: MSH, RMSH, WRMS, and WMSH. Cai and Sarkis [10] introduced restricted Schwarz methods with harmonic extension. In these variants, the projections $P_{i,\delta}$ in (5) of the classical Schwarz method are replaced by

$$H_{i,\delta} = R_{i,\delta}^T A_{i,\delta}^{-1} \tilde{R}_{i,\delta} A = R_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} E_{i,0} A,$$

in contrast to the restricted methods where

$$Q_{i,\delta} = \tilde{R}_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} A = E_{i,0} R_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} A$$

are used. The additive Schwarz method with harmonic extension (ASH method) can then be described in our notation by the iteration matrix $T_{ASH,\delta} = I - M_{ASH}^{-1}A$, where M_{ASH}^{-1} is given by

$$M_{ASH,\delta}^{-1} = \sum_{i=1}^{p} R_{i,\delta}^T A_{i,\delta}^{-1} \tilde{R}_{i,\delta} = \sum_{i=1}^{p} M_{i,\delta}^{-1} E_{i,0}.$$

Similarly, the multiplicative Schwarz method with harmonic extension (MSH method) is defined by

$$T_{MSH,\delta} = \prod_{i=p}^{1} (I - H_{i,\delta}) = \prod_{i=p}^{1} (I - M_{i,\delta}^{-1} E_{i,0} A).$$

It was observed in [10, Rem. 2.4] that the ASH method and the RAS method used as a preconditioner exhibit a similar convergence behavior. In fact, it was shown in [16] that in the case of a symmetric matrix A the two spectra coincide, i.e., $\sigma(M_{ASH,\delta}^{-1}A) = \sigma(M_{RAS,\delta}^{-1}A)$.

In the following, we establish similar results for the MSH method. We have, for a general nonsingular matrix A,

$$\begin{aligned} T_{MSH,\delta}^T &= \prod_{i=p}^{1} (I - R_{i,\delta}^T A_{i,\delta}^{-1} R_{i,\delta} E_{i,0} A)^T = \prod_{i=p}^{1} (I - A^T E_{i,0} R_{i,\delta}^T A_{i,\delta}^{-T} R_{i,\delta}) \\ &= A^T \left(\prod_{i=p}^{1} (I - E_{i,0} R_{i,\delta}^T A_{i,\delta}^{-T} R_{i,\delta} A^T) \right) A^{-T}. \end{aligned} \tag{35}$$

Hence the spectrum of the MSH method is the same as the spectrum of a RMS method for A^T. So, with the weighted column sum norm $\|\cdot\|_{1,w}$ defined for $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ as

$$\|B\|_{1,w} = \max_{j=1,\ldots,n} \left(\sum_{i=1}^{n} |b_{ij}| w_i \right) / w_j,$$

we immediately obtain the following result.

THEOREM 7.1. *Let A be a nonsingular M-matrix. Then the following hold.*

(i) *For any value of $\delta \geq 0$, the splitting $A = M_{MSH,\delta} - N_{MSH,\delta}$, corresponding to the MSH method, is weak nonnegative of the second type, hence*

$$\rho(T_{MSH,\delta}) < 1.$$

(ii) *If $A = A^T$, then for any value of $\delta \geq 0$*

$$\sigma(T_{MSH,\delta}) = \sigma(T_{RMS,\delta}) \qquad \text{and} \qquad \sigma(M_{MSH,\delta}^{-1}A) = \sigma(M_{RMS,\delta}^{-1}A).$$

(iii) *For any positive vector w such that $w^T A > 0$ and for $\delta \geq \delta'$, we have*

$$\|T_{MSH,\delta}\|_{1,w} \leq \|T_{MSH,\delta'}\|_{1,w}.$$

Moreover, if the Perron vector $w_{\delta'}$ of $T_{MSH,\delta'}^T$ satisfies $w_{\delta'} > 0$ and $A^T w_{\delta'} \geq 0$, then we also have

$$\rho(T_{MSH,\delta}) \leq \rho(T_{MSH,\delta'}).$$

(iv) *For any value of $\delta \geq 0$, $\rho(T_{MSH,\delta}) \leq \rho(T_{MSH,0})$.*

In the same way that we showed that the RMS method is faster than the RAS method (Theorem 3.5), we show that the MSH method is faster than the ASH method.

THEOREM 7.2. *Let A be a nonsingular M-matrix. Then, for any value $\delta \geq 0$, we have*

$$\|T_{MSH,\delta}\|_{1,w} \leq \|T_{ASH,\delta}\|_{1,w}.$$

Proof. By Lemma 2.3, we have that $M_{MSH,\delta} = A(I - T_{MSH,\delta})^{-1}$. Thus, using (35), we write

$$M_{MSH,\delta}^{-T} = A^{-T}(I - T_{MSH,\delta}^T) = \left(I - \prod_{i=p}^{1} (I - E_{i,0} M_{i,\delta}^{-T} A^T) \right) A^{-T}.$$

Since

$$\left(M_{ASH,\delta}^{-1} \right)^T = \sum_{i=1}^{p} \tilde{R}_{i,\delta}^T \left(A_{i,\delta}^{-1} \right)^T R_{i,\delta},$$

every ASH-splitting of A gives rise to a corresponding RAS-splitting of A^T [16]. We can then follow the proof of Theorem 3.5 verbatim considering the M-matrix A^T. ☐

We note that Theorems 7.1 and 7.2 hold if inexact solves on the subdomains are used; see section 4.

Combining the restricted and the harmonic versions we obtain the RASH and RMSH methods of [10] with

$$T_{RASH,\delta} = I - M_{RASH,\delta}^{-1} A = I - \sum_{i=1}^{p} \tilde{R}_{i,\delta}^T A_{i,\delta}^{-1} \tilde{R}_{i,\delta} A,$$

$$T_{RMSH,\delta} = I - M_{RMSH,\delta}^{-1} A = \prod_{i=p}^{1} (I - \tilde{R}_{i,\delta}^T A_{i,\delta}^{-1} \tilde{R}_{i,\delta} A).$$

However, the RASH method is, in general, not convergent as observed in [16]. The same holds for the RMSH method, as the following example illustrates.

Example 7.3. Consider the symmetric M-matrix

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

Let

$$R_{1,0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad R_{2,0} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and let $R_{1,1} = R_{2,1} = I$. We then have

$$T_{RMSH,1} = (I - E_{2,0}A^{-1}E_{2,0}A)(I - E_{1,0}A^{-1}E_{1,0}A) = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -3 & -3 & 2 & 2 \\ -3 & -3 & 2 & 2 \end{bmatrix}$$

with $\rho(T_{RMSH,1}) = 2$.

Other variants of the classical Schwarz methods are the weighted restricted Schwarz methods introduced by Cai and Sarkis [10]. For these modifications, one introduces weighted restriction operators $R^{\omega}_{i,\delta}$ which result from $R_{i,\delta}$ by replacing the entry 1 in column j by $1/k$, where k is the number of sets $S_{i,\delta}$ the component j belongs to, or more generally by some weights adding up to 1. With this notation, we define

$$\tilde{E}_{i,\delta} = R^T_{i,\delta} R^{\omega}_{i,\delta},$$

and we have

$$\sum \tilde{E}_{i,\delta} = \sum R^T_{i,\delta} R^{\omega}_{i,\delta} = I.$$

Then the weighted restricted additive Schwarz (WRAS) method and the weighted restricted multiplicative Schwarz (WRMS) method can be described in our notation by the iteration matrices

$$T_{WRAS,\delta} = I - M^{-1}_{WRAS}A = I - \sum_{i=1}^{p} (R^{\omega}_{i,\delta})^T A^{-1}_{i,\delta} R_{i,\delta} A = I - \sum_{i=1}^{p} \tilde{E}_{i,\delta} M^{-1}_{i,\delta} A \tag{36}$$

and

$$T_{WRMS,\delta} = \prod_{i=p}^{1} (I - (R^{\omega}_{i,\delta})^T A^{-1}_{i,\delta} R_{i,\delta} A) = \prod_{i=p}^{1} (I - \tilde{E}_{i,\delta} M^{-1}_{i,\delta} A), \tag{37}$$

respectively. Similarly weighted Schwarz methods with harmonic extensions can be defined. Observe that $(R^{\omega}_{i,\delta})^T A^{-1}_{i,\delta} R_{i,\delta} A$ is not a projection.

If we compare the WRMS method with the classical multiplicative Schwarz method we obtain the following result.

THEOREM 7.4. *Let A be nonsingular M-matrix. Then, for any $\delta \geq 0$, we have*

$$\rho(T_{MS,\delta}) \leq \rho(T_{WRMS,\delta}) \leq 1. \tag{38}$$

Proof. Following our analysis in the previous sections, we obtain that $T_{WRMS,\delta}$ induces a weak regular splitting of A. Since $\tilde{E}_{i,\delta} \leq E_{i,\delta}$ we get (38) using the same techniques as in the proof of Theorem 3.8. ◻

A similar result holds for the weighted MSH method.

8. Coarse grid corrections. It has been shown theoretically, and confirmed in practice, that a coarse grid correction improves the performance of the classical Schwarz methods. This coarse grid correction can be applied either additively or multiplicatively; see, e.g., [2, 11, 15, 23, 25]. This corresponds to a two-level scheme, the coarse correction being the second level. In [26], a coarse grid correction was used in connection with RAS iterations.

The analysis done for the RAS case in [16] applies almost without changes to the RMS methods of this paper, so we omit the details. All we will say is that in all cases where we have shown convergence the coarse grid correction can never degrade, and often improves, the convergence rate.

Acknowledgments. We thank Michele Benzi and Andreas Frommer for a careful reading of the manuscript and their suggestions which helped improve our presentation.

REFERENCES

[1] S. Balay, W.D. Gropp, L.C. McInnes, and B.F. Smith, *PETSc* 2.0 *User's Manual*, Technical Report ANL-95/11 - Revision 2.0.22, Argonne National Laboratory, Argonne, IL, 1998; also available online from http://www.mcs.anl.gov/petsc.

[2] M. Benzi, A. Frommer, R. Nabben, and D. B. Szyld, *Algebraic theory of multiplicative Schwarz methods*, Numer. Math., 89 (2001), pp. 605–639.

[3] M. Benzi and D. B. Szyld, *Existence and uniqueness of splittings for stationary iterative methods with applications to alternating methods*, Numer. Math., 76 (1997), pp. 309–321.

[4] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979; reprinted and revised, SIAM, Philadelphia, 1994.

[5] P. E. Bjørstad and J. Mandel, *On the spectra of sums of orthogonal projections with applications to parallel computing*, BIT, 31 (1991), pp. 76–88.

[6] P. Bjørstad and O. B. Widlund, *To overlap or not to overlap: A note on a domain decomposition method for elliptic problems*, SIAM J. Sci. Statist. Comput., 10 (1989), pp. 1053–1061.

[7] R. Bru, V. Migallón, J. Penadés, and D. B. Szyld, *Parallel, synchronous and asynchronous two-stage multisplitting methods*, Electron. Trans. Numer. Anal., 3 (1995), pp. 24–38.

[8] X.-C. Cai, C. Farhat, and M. Sarkis, *A Minimum Overlap Restricted Additive Schwarz Preconditioner and Applications to* 3*D Flow Simulations*, Contemp. Math. 218, AMS, Providence, RI, 1998, pp. 479–485.

[9] X.-C. Cai and Y. Saad, *Overlapping domain decomposition algorithms for general sparse matrices*, Numer. Linear Algebra Appl., 3 (1996), pp. 221–237.

[10] X.-C. Cai and M. Sarkis, *A restricted additive Schwarz preconditioner for general sparse linear systems*, SIAM J. Sci. Comput., 21 (1999), pp. 792–797.

[11] T. F. Chan and T. P. Mathew, *Domain decomposition methods*, Acta Numerica (1994), pp. 61–143.

[12] J.-J. Climent and C. Perea, *Some comparison theorems for weak nonnegative splittings of bounded operators*, Linear Algebra Appl., 275/276 (1998), pp. 77–106.

[13] M. Dryja and O. B. Widlund, *Towards a unified theory of domain decomposition algorithms for elliptic problems*, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, T.F. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., SIAM, Philadelphia, 1990, pp. 3–21.

[14] A. Frommer, R. Nabben, and D. B. Szyld, *An algebraic convergence theory for restricted additive and multiplicative Schwarz methods*, in Domain Decomposition Methods in Science and Engineering, N. Debit, M. Garbey, R. Hoppe, J. Périaux, D. Keyes, and Y. Kuznetzov, eds., CIMNE, UPS, Barcelona, 2002, pp. 371–377.

[15] A. Frommer and D. B. Szyld, *Weighted max norms, splittings, and overlapping additive Schwarz iterations*, Numer. Math., 83 (1999), pp. 259–278.

[16] A. Frommer and D. B. Szyld, *An algebraic convergence theory for restricted additive Schwarz methods using weighted max norms*, SIAM J. Numer. Anal., 39 (2001), pp. 463–479.

[17] W. Hackbusch, *Iterative Solution of Large Sparse Systems of Equations*, Springer-Verlag, New York, Berlin, Heidelberg, 1994.

[18] T. P. Mathew, *Uniform convergence of the Schwarz alternating method for solving singularly perturbed advection-diffusion equations*, SIAM J. Numer. Anal., 35 (1998), pp. 1663–1683.

[19] J. A. Meijerink and H. van der Vorst, *An iterative solution method for linear systems of which the coefficient matrix is a symmetric M-matrix*, Math. Comp., 31 (1977), pp. 148–162.

[20] R. Nabben, *Comparisons between additive and multiplicative Schwarz iterations in domain decomposition methods*, Numer. Math., to appear.

[21] M. Neumann and R. J. Plemmons, *Convergence of parallel multisplitting methods for M-matrices*, Linear Algebra Appl., 88/89 (1987), pp. 559–574.

[22] D. P. O'Leary and R. E. White, *Multi-splittings of matrices and parallel solution of linear systems*, SIAM J. Alg. Disc. Meth., 6 (1985), pp. 630–640.

[23] A. Quarteroni and A. Valli, *Domain Decomposition Methods for Partial Differential Equations*, Oxford Science Publications, Clarendon Press, Oxford University Press, New York, 1999.

[24] W. C. Rheinboldt and J. S. Vandergraft, *A simple approach to the Perron-Frobenius theory for positive operators on general partially-ordered finite-dimensional linear spaces*, Math. Comp., 27 (1973), pp. 139–145.

[25] B. F. Smith, P. E. Bjørstad, and W. D. Gropp, *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*, Cambridge University Press, Cambridge, New York, Melbourne, 1996.

[26] X.-C. Tai, *A space decomposition method for parabolic problems*, Numer. Methods Partial Differential Equations, 14 (1998), pp. 27–46.

[27] R. S. Varga, *Factorization and normalized iterative methods*, in Boundary Problems in Differential Equations, R. E. Langer, ed., The University of Wisconsin Press, Madison, WI, 1960, pp. 121–142.

[28] R. S. Varga, *Matrix Iterative Analysis*, Prentice–Hall, Englewood Cliffs, NJ, 1962; 2nd ed., Springer-Verlag, Berlin, 2000.

[29] Z. I. Woźnicki, *Nonnegative splitting theory*, Japan J. Indust. Appl. Math., 11 (1994), pp. 289–342.