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# Local convergence of the (exact and inexact) iterative aggregation method for linear systems and Markov operators

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**Summary.** The iterative aggregation method for the solution of linear systems is extended in several directions: to operators on Banach spaces; to the method with inexact correction, i.e., to methods where the (inner) linear system is in turn solved iteratively; and to the problem of finding stationary distributions of Markov operators. Local convergence is shown in all cases. Convergence results apply to the particular case of stochastic matrices. Moreover, an argument is given which suggests why the iterative aggregation method works so well for nearly uncoupled Markov chains, as well as for Markov chains with other zero-nonzero structures.

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## 1. Introduction and notation

Iterative aggregation refers to a class of methods, sometimes called aggregation/disaggregation, which has been an effective tool for the solution of linear systems and certain eigenvalue problems; see Cao and Stewart [2], Chatelin and Miranker [4], [5], Haviv [15], Mandel and Sekerka [22], Miranker and Pan [30], and the references given therein.

Briefly, the general idea is that at each step of these methods, the linear system is replaced with a system in another (smaller) space $\mathscr{F}$ (aggregation step), this (smaller) linear system is solved and its solution is used to improve the current iterate in the original space (disaggregation step), often the improved iterate is used to begin one or more relaxation steps; see Sect. 2 for a detailed description of the method we study. Of course, the iterative aggregation method is related to the principle of the multigrid method, but no underlying mesh or even a discretization of a differential equation is assumed; see e.g. Mandel [21].

The idea of aggregation appeared naturally in input-output economic models, where goods and services are aggregated (grouped together) according to certain

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economic criteria; see the survey by Vakhutinsky, Dudkin and Ryvkin [41] and the extensive bibliography there. The iterative aggregation method was extended to other linear systems and to the problem of finding the stationary distribution of Markov chains; see the mentioned references and also Mandel [20], and Marek [24]. As mentioned, there are several versions of these methods; see the surveys by Schweitzer [33], [34], and the recent paper by Kafeety, Meyer and Stewart [16]. The method we choose to study is the extension of the method found in Mandel and Sekerka [22]. This method is not expensive, since no solution of linear systems at the disaggregated level are performed; only solution of systems in the (smaller) aggregated spaces are computed.

Global convergence of the iterative aggregation method for Markov chains was shown only for the nearly uncoupled case for the special method studied by Cao and Stewart [2]; see also Stewart, Stewart and McAllister [35]. Local convergence for linear systems was first proved by Mandel and Sekerka [22].

In this paper the local convergence proofs in [22] are extended in several directions. First, in Sect. 2, the iterative aggregation method is described for linear systems on Banach spaces, examples given, and local convergence shown. Further, in Sect. 3, local convergence is shown for the method applied to Markov operators. In our analysis of the method for Markov operators, we do not restrict ourselves to the nearly decomposable case, as is done e.g. in [2]. Our convergence proof applies to the general case. In Sect. 5 we relate our proof to the nearly decomposable case, as well as to other cases.

In practice, often the linear system in the (smaller) space $\mathscr{F}$ is not solved exactly. Instead, it is in turn solved iteratively, for example, using any of the methods in the package MARCA [36], or the one suggested by Freund and Hochbruck [9]. This situation is similar to inner/outer iterations or two-stage methods, see e.g. [10], [11], [14], [19] or [38]. We call the resulting method *Iterative Aggregation with Inexact Correction*. In Sect. 4, we show local convergence for the inexact method and extend the result to Markov operators. Some preliminary results related to this paper can be found in Marek [25] and in Marek and Szyld [27]; see also Szyld [37]. We believe that this is the first time local convergence proofs for (exact or inexact) iterative aggregation methods applied to general Markov chains – not necessarily nearly decomposable – and Markov operators is given.

In Sect. 5 a new mathematical tool is presented whereby we can apply our theory to stochastic matrices and finite Markov chains. In particular we obtain a better understanding on why the iterative aggregation method works so well when applied to Markov chains. Numerical examples are also given.

Let $\mathscr{E}$ and $\mathscr{F}$ be Banach spaces over the field of real numbers (e.g. $\mathbb{R}^n$ and $\mathbb{R}^m$, $m < n$). Let $\mathscr{E}'$ and $\mathscr{F}'$ denote the dual space of $\mathscr{E}$ and $\mathscr{F}$, respectively, and $\mathscr{B}(\mathscr{E},\mathscr{F})$ the Banach space of bounded linear operators of $\mathscr{E}$ into $\mathscr{F}$; we let $\mathscr{B}(\mathscr{E}) = \mathscr{B}(\mathscr{E},\mathscr{E})$. When $\mathscr{E} = \mathbb{R}^n$, $\mathscr{B}(\mathscr{E})$ is the space of $n \times n$ matrices. If $T \in \mathscr{B}(\mathscr{E},\mathscr{F})$ then $T'$ denotes the dual of $T$, i.e., $T' \in \mathscr{B}(\mathscr{F}',\mathscr{E}')$. We further assume that $\mathscr{E}$ and $\mathscr{F}$ are generated by closed normal cones $\mathscr{K}$ and $\mathscr{H}$ respectively; see e.g. Krasnosel'skii [17] or Kreĭn and Rutman [18]. This assumption is satisfied by $\mathbb{R}^n$ by letting $\mathscr{K} = \mathbb{R}^n_+$, the set of nonnegative vectors. An operator $A \in \mathscr{B}(\mathscr{E})$ is called $\mathscr{K}$-nonnegative if $A\mathscr{K} \subset \mathscr{K}$ and denoted $A \geq 0$. When the identification of the cone $\mathscr{K}$ is clear from the context a $\mathscr{K}$-nonnegative operator is simply called nonnegative. For $\mathscr{K} = \mathbb{R}^n_+$, the nonnegative operators are the nonnegative matrices; see e.g. [1]. Let $\mathscr{K}' = \{x' \in \mathscr{E}' : \langle x, x'\rangle = x'(x) \geq 0 \text{ for all } x \in \mathscr{K}\}$. It can be shown that $\mathscr{K}'$ is also a closed normal cone generating $\mathscr{E}'$ [18]. An element

$x' \in \mathscr{K}'$ is called strictly positive if $\langle x, x'\rangle > 0$ whenever $x \in \mathscr{K}$, $x \neq 0$. When $\mathscr{E} = \mathbb{R}^n$, a strictly positive vector is just a vector with positive entries.

A suitable replacement to a topological interior of a cone (Int $\mathscr{K}$) is the concept of $d$-interior: $\mathscr{K}^{\mathrm{d}} = \{x \in \mathscr{K} : \langle x, x'\rangle > 0 \ \forall x' \in \mathscr{K}', x' \neq 0\}$. An example of a cone $\mathscr{K}$ having Int $\mathscr{K} = \emptyset$ while $\mathscr{K}^{\mathrm{d}} \neq \emptyset$ is the cone $\mathscr{L}^2(0,1)_+$ consisting of all elements $x \in \mathscr{E} = \mathscr{L}^2(0,1)$ having a representative $\overline{x} \geq 0$ almost everywhere in (0,1). If $\mathscr{E} = \mathbb{R}^n$, then $\mathscr{K}^{\mathrm{d}} = \mathrm{Int}\,\mathscr{K} = \{x : x_i > 0, i = 1, \cdots, n\}$. By $x \geq 0$ we denote $x \in \mathscr{K}$ and call it a nonnegative element, and by $x > 0$ we denote $x \in \mathscr{K}^{\mathrm{d}}$. We write $x \geq y$ if $x - y \geq 0$.

An element $x \in \mathscr{K}$ is called extremal if $x = \alpha y + \beta z$, $y \in \mathscr{K}$, $z \in \mathscr{K}$ $y \neq 0$, $z \neq 0$, implies that either $\alpha = 0$, $\beta = 0$, or $z = \gamma y$ for some $\gamma \neq 0$. The set of extremal elements of $\mathscr{K}$ is denoted by Ext $\mathscr{K}$.

By $\tilde{\mathscr{E}}$ we denote the complex extension of $\mathscr{E}$, i.e., $\tilde{\mathscr{E}} = \mathscr{E} \oplus \mathrm{i}\mathscr{E}$ with the norm defined by $\|z\|_{\tilde{\mathscr{E}}} = \sup_{0\leq\theta\leq 2\pi} \|x\cos\theta + y\sin\theta\|_{\mathscr{E}}$ where $z = x + \mathrm{i}y$, $x, y \in \mathscr{E}$. An operator $A \in \mathscr{B}(\mathscr{E})$ can similarly be extended to $\tilde{A} \in \mathscr{B}(\tilde{\mathscr{E}})$ by setting $\tilde{A}z = Ax + \mathrm{i}Ay$, where $z = x + \mathrm{i}y$. Let $A \in \mathscr{B}(\tilde{\mathscr{E}})$, and let $\mu$ be a complex scalar. Let $R(\mu, A) = (\mu I - A)^{-1}$ be the resolvent operator. The complement of the resolvent set $\{\mu \in \mathbb{C} : (\mu I - A)^{-1} \in \mathscr{B}(\tilde{\mathscr{E}})\}$ is called the spectrum of $A$ and denoted $\sigma(A)$ [40, p.264]. We let $r(A) = \max\{|\mu| : \mu \in \sigma(A)\}$ and call it the spectral radius of $A$.

A nonnegative operator $B$ has $r(B) \geq 0$, and there exists an eigenvector $x \geq 0$, called the Frobenius eigenvector, such that $Bx = r(B)x$; see e.g. [17], [42].

We say that the operator $A$ has *Property P* if every $\alpha \in \sigma(A)$ such that $|\alpha| = r(A)$ is a pole of the resolvent operator $R(\mu, A)$. We will assume throughout the paper that the operators studied have *Property P*. It is well known that if $\alpha$ is a pole of the resolvent operator

(1) $$(\mu I - A)^{-1} = \sum_{k=0}^{\infty} A_k(\alpha)(\mu - \alpha)^k + \sum_{k=1}^{q(\alpha)} B_k(\alpha)(\mu - \alpha)^{-k},$$

where $A_k(\alpha)$ and $B_{k+1}(\alpha)$, $k = 0, 1, \cdots$ belong to $\mathscr{B}(\tilde{\mathscr{E}})$, and $q(\alpha) < +\infty$ is the multiplicity of $\alpha$ as a pole of the resolvent operator [40]. This multiplicity $q$ is called the index of $A$ with respect to $\alpha$ and is denoted by $\mathrm{ind}_\alpha A$. Thus, $\mathrm{ind}_\alpha A \geq 1$.

## 2. Iterative aggregation

In this section we consider the solution of the linear system

(2) $$Ax = b,$$

where $x \in \mathscr{E}$, $A \in \mathscr{B}(\mathscr{E})$, and $b \geq 0$. For the solution of (2), it is customary to consider a splitting $A = M - N$ with $M^{-1} \geq 0$,

(3) $$T = M^{-1}N \geq 0,$$

and $r(T) < 1$, in which case $\lim_{k\to\infty} T^k = 0$. Given an initial guess $x^{(0)}$, one can consider an iteration process

(4) $$x^{(k+1)} = Tx^{(k)} + c,$$

where $c = M^{-1}b$. It is well known that the process (4) converges linearly to the solution of (2) with asymptotic convergence factor of $r(T)$; see e.g. Varga [42]. Thus, the system (2) is equivalent to $x - Tx = M^{-1}b$ and, without loss of generality, we consider operators of the form $A = I - B$, $B \geq 0$, where $I$ denotes the identity operator in $\mathscr{E}$.

We discuss now the aggregation and disaggregation maps. Let us consider a nonempty set $\mathscr{D} \subset \mathscr{E}$. Let the aggregation map $R \in \mathscr{B}(\mathscr{E}, \mathscr{F})$ be such that $R\mathscr{K} \subset \mathscr{H}$, i.e., $R$ maps nonnegative elements in $\mathscr{E}$ into nonnegative elements in $\mathscr{F}$, and for all $x \in \mathscr{D}$, let the disaggregation map, which depends on $x$, $S(x) \in \mathscr{B}(\mathscr{F}, \mathscr{E})$ be such that $S(x)\mathscr{H} \subset \mathscr{K}$, i.e., $S(x)$ maps nonnegative elements in $\mathscr{F}$ into nonnegative elements in $\mathscr{E}$. Let $P(x) = S(x)R$ and $I_{\mathscr{F}}$ the identity operator in $\mathscr{F}$. We assume that for every $x \in \mathscr{D}$ the following two relations hold

$$RS(x) = I_{\mathscr{F}}, \tag{5}$$

$$P(x)x = S(x)Rx = x. \tag{6}$$

It follows from (5) that $[P(x)]^2 = P(x)$, i.e., that $P(x)$ is a projection on $\mathscr{E}$. Relations (5) and (6) are quite natural and, as we will point out in the next section, they guarantee that in the case of Markov chains the aggregated matrix is also stochastic. Before describing the iterative aggregation method we present several examples of aggregation and disaggregation maps.

*Example 2.1.* Let $\mathscr{E} = \mathbb{R}^n$, $\mathscr{F} = \mathbb{R}^m$, $1 \leq m \leq n$ and partition the index set into $m$ nonempty and disjoint sets $\{1, \cdots, n\} = G_1 \cup \cdots \cup G_m$. Define $z = Rx$ as $z_i = \sum_{j \in G_i} x_j$, and $[S(x)z]_j = z_i x_j / (\sum_{\ell \in G_i} x_\ell)$ if $j \in G_i$ and zero otherwise. Here $\mathscr{D} = \{x \in \mathscr{E} : x \geq 0, Rx > 0\}$. Thus, $R$ is an $m \times n$ matrix whose columns have a single nonzero entry and this entry is a one, e.g. of the form

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{7}$$

Similarly, $S(x)$ is a $n \times m$ nonnegative matrix with a single nonzero per row. It is easy to see that these maps satisfy (5) and (6). This basic example is Method (a) in Haviv [15] and also appears in other references, e.g. Chatelin [3], Schweitzer [33].

*Example 2.2.* Let $\mathscr{E} = \ell^1$, $\mathscr{F} = \mathbb{R}^n$. Let $\mathbb{N} = \{1, 2, \cdots\}$ be the natural numbers and let $g : \mathbb{N} \to \{1, \cdots, n\}$ be given. Let $G_i = \{j \in \mathbb{N} : g(j) = i\}$. Define $z = Rx$ as $z_i = \sum_{j \in G_i} x_j$, and $[S(x)z]_j = z_i x_j / (\sum_{\ell \in G_i} x_\ell)$ if $j \in G_i$ and zero otherwise. Let $\ell^1_+ = \{x \in \ell^1 : x_j \geq 0, j \in \mathbb{N}\}$. Here $\mathscr{D} = \{x \in \ell^1_+ : \sum_{j \in G_i} x_j > 0, i = 1, \cdots, n\}$.

*Example 2.3.* Let $\mathscr{E} = Q[0, 1]$, where $Q[0, 1]$ is the space of piecewise continuous functions on $[0, 1]$, i.e., of functions $x : [0, 1] \to \mathbb{R}$ such that there exists a finite number of points $0 = t_0 < t_1 < \cdots < t_k = 1$ such that the restriction $x|_{[t_{j-1}, t_j]}$ is continuous and bounded on $(t_{j-1}, t_j)$, $j = 1, \cdots, k = k(x)$, with the norm $\|x\| = \max\{|x(t)|, t \in [0, 1]\}$. Let $\mathscr{F} = \mathbb{R}^n$, and we select $n + 1$ distinct points $0 = s_0 < s_1 < \cdots < s_n = 1$. Define $z = Rx$, $z_i = \int_{s_{i-1}}^{s_i} x(s)ds$, $i = 1, \cdots, n$, and $S(x)w = v$, $v(s) = [1/(Rx)_i] w_i x(s)$ if $s_{i-1} \leq s \leq s_i$. Here $\mathscr{D} = \{x \in \mathscr{E} : x \geq 0,\ Rx > 0\}$. We note that in the integral defining $R$, any positive Radon measure on $[0, 1]$ can be used.

*Example 2.4.* Let $\Omega = [a,b] \times [c,d] \subset \mathbb{R}^2$. Let $\mathscr{E} = \mathscr{L}^2(\Omega)$, and $\mathscr{F} = \mathscr{L}^2([a,b])$. Define $(Ru)(s) = \int_c^d u(s,t)dt$, and $S(u)z = [1/(Ru)(s)]z(t)u(s,t)$. Here $\mathscr{D} = \{u \in \mathscr{E} : u \geq 0, \int_c^d u(s,t)dt > 0\}$. In this example, let $B \in \mathscr{B}(\mathscr{E})$ be defined by a kernel $a(s,t,s',t')$, i.e., $Bu = \int_\Omega a(s,t,s',t')u(s',t')ds'dt'$, and we have that $B\mathscr{L}^2 \subset Q[\Omega]$, where $Q[\Omega]$ is the space of piecewise continuous functions on $\Omega$. We note that $\Omega$ need not be a rectangle, we can consider $\Omega = \Omega_1 \times \Omega_2$, $\Omega_i \subset \mathbb{R}$, $i = 1,2$, and that we can also consider $\Omega_1 \subset \mathbb{R}^2$, $\Omega_2 \subset \mathbb{R}$, $\Omega \subset \mathbb{R}^3$, namely a "cylinder".

Let $B_{\mathscr{F}}(x) = RBS(x)$ be the aggregated operator and assume further that $B_{\mathscr{F}}(x) \geq 0$, i.e., that $B_{\mathscr{F}}(x)\mathscr{H} \subset \mathscr{H}$, and that $r(B_{\mathscr{F}}(x)) < 1$ for all $x \in \mathscr{D}$. We define the following

**Algorithm 2.5 (Iterative Aggregation).** Given a nonnegative operator $B$, $b \geq 0$, an initial guess $x^{(0)}$, and a convergence parameter $\varepsilon > 0$, let $k = 0$.

1. Solve the (aggregated) equation

   (8) $$z - B_{\mathscr{F}}(x)z = Rb.$$

   with $x = x^{(k)}$ and call the solution $z^{(k)}$.
2. Disaggregate and iterate according to the formula $x^{(k+1)} = BS(x^{(k)})z^{(k)} + b$.
3. Test if $\|x^{(k+1)} - x^{(k)}\| < \varepsilon$ (or other convergence test). If yes, STOP; otherwise $k := k + 1$ and go to 1.

In the case of (3), the operator in (8) is $T_{\mathscr{F}}(x) = RM^{-1}NS(x)$. If $\mathscr{E} = \mathbb{R}^n$, $\mathscr{F} = \mathbb{R}^m$, $1 \leq m \leq n$, $T_{\mathscr{F}}(x)$ can be computed with $m$ solutions of a linear system with coefficient matrix $M$. These solutions can be efficiently computed, depending on the choice of $M$, and this has to be taken into account when evaluating the suitability of the method.

In order to analyze the convergence of the iterative aggregation method, we can express each iteration in terms of the elements of the original space $\mathscr{E}$ as

(9) $$x^{(k+1)} = BU(x^{(k)}) + b,$$

where

(10) $$U(x) = S(x)[I_{\mathscr{F}} - B_{\mathscr{F}}(x)]^{-1}Rb.$$

Let $x^\star$ be the solution of (2). Thus, $R(I-B)x^\star = Rb$, and using the relations (5) and (6), it follows that

(11) $$U(x^\star) = x^\star.$$

It is immediate that the method is consistent, i.e., that $x^\star = BU(x^\star)+b$. In the following theorem, the local convergence properties of the iterative aggregation method are shown. The proof resembles that of Marek [24]; cf. also Mandel and Sekerka [22]. In the proof here, and in the other convergence theorems in the paper, we use the fact that the spectral radius of the Jacobian of the map is less than unity in a whole neighborhood around the solution.

**Theorem 2.6.** *Let $R \in \mathscr{B}(\mathscr{E},\mathscr{F})$ , $R\mathscr{K} \subset \mathscr{H}$. Let $\mathscr{W} \subset \mathscr{E}$ be a subspace such that $\|x\| \leq \|x\|_{\mathscr{W}}$, for all $x \in \mathscr{W}$. Let $\mathscr{D}$ be a nonempty $\mathscr{W}$-open set such that $x^\star \in \mathscr{D}$, $S(x) \in \mathscr{B}(\mathscr{F},\mathscr{E})$, $S(x)\mathscr{H} \subset \mathscr{K}$, $S(x)$ is $\mathscr{W}$-continuous, for all $x \in \mathscr{D}$, and (5) and (6) hold for all $x \in \mathscr{D}$. Let $r(B) < 1$ and $r(B_{\mathscr{F}}(x)) < 1$ for all $x \in \mathscr{D}$.*

*Moreover, let $U$ defined in (10) be such that $BU(x) + b \in \mathscr{D}$ whenever $x \in \mathscr{D}$. Let there be a positive number $\beta$ such that $r(J(x^\star)) \le \beta < 1$, where*

$$J(x) = J(B, x) = B[I - P(x)B]^{-1}[I - P(x)]. \tag{12}$$

*Then, the iterative aggregation method, i.e., the iteration process (9), is $\mathscr{W}$-locally convergent, i.e., there exists a $\mathscr{W}$-open neighborhood $\mathscr{U}$ of $x^\star$ such that*

$$\lim_{k\to\infty} \|x^{(k)} - x^\star\| = 0 \ \ \textit{for any } x^{(0)} \in \mathscr{U}. \tag{13}$$

*The speed of convergence is characterized by the estimates*

$$\|x^{(k)} - x^\star\| \le \|x^{(k)} - x^\star\|_{\mathscr{W}} \le \kappa\rho^k \tag{14}$$

*where $\rho = r(J(x^\star)) + \eta < 1$, for some $\eta > 0$ and $\kappa$ independent of $k$.*

*Proof.* The condition $r(B_{\mathscr{F}}(x)) < 1$, together with (5) and (6) imply the following relations

$$[I_{\mathscr{F}} - B_{\mathscr{F}}(x)]^{-1} = \sum_{k=0}^{\infty}[RBS(x)]^k = R\sum_{k=0}^{\infty}[P(x)B)]^k S(x), \tag{15}$$

$$S(x)[I_{\mathscr{F}} - B_{\mathscr{F}}(x)]^{-1}R = [I - P(x)B]^{-1}P(x), \tag{16}$$

$$[I - P(x)B]^{-1} = I + [I - P(x)B]^{-1}P(x)B,$$

$$\begin{aligned} J(x) &= B\{I - [I - P(x)B]^{-1}P(x)(I - B)\} \\ &= B\{I - S(x)[I_{\mathscr{F}} - RBS(x)]^{-1}R(I - B)\}. \end{aligned} \tag{17}$$

It follows from the last equation, (5) and (6) that

$$J(x)x = 0. \tag{18}$$

We replace $b = (I - B)x^\star$ in (10) and use (11), (17) and (18) to write

$$\begin{aligned} BU(x) - BU(x^\star) &= BS(x)[I_{\mathscr{F}} - RBS(x)]^{-1}R(I - B)x^\star - Bx^\star = -J(x)x^\star \\ &= J(x)(x - x^\star). \end{aligned} \tag{19}$$

Let $\zeta > 0$ be arbitrary. It is well known that there is a norm $\|.\|_\zeta$ equivalent to the norm $\|.\|_{\mathscr{W}}$ such that

$$\|J(x^\star)\|_\zeta \ \le \ r(J(x^\star)) + \zeta \ ; \tag{20}$$

see, e.g., [29, p.55]. The $\mathscr{W}$-continuity of $S(x)$ implies the $\mathscr{W}$-continuity of $J(x)$. Thus, for a given $\varepsilon > 0$ there is a $\delta > 0$ such that $\|J(x) - J(x^\star)\|_\zeta < \varepsilon$ as soon as $\|x - x^\star\|_\zeta < \delta$. According to (19) and (20) this implies that

$$\|BU(x) - BU(x^\star)\|_\zeta \ \le \ [r(J(x^\star)) + \varepsilon + \zeta] \ \|x - x^\star\|_\zeta \ . \tag{21}$$

Thus the map $BU(x)$ is contractive in a neighborhood $\mathscr{U}$ of $x^\star$. Since all iterates $x^{(k)} \in \mathscr{D} \subset \mathscr{W}$, the iterative process (9) is $\mathscr{W}$-convergent. The equivalence of the norms $\|.\|_{\mathscr{W}}$ and $\|.\|_\zeta$ together with (21) imply the desired result (14). □

We point out that in Algorithm 2.5, step 2, more than one relaxation step can be taken, and the convergence proof can be carried out in an analogous way.

Mandel and Sekerka [22] showed that for the Example 2.1, the hypothesis $r(J(x^\star)) \leq \beta < 1$ holds. The same arguments can be used to extend the result to Example 2.2. In what follows we show the same for Example 2.4. In this case we have $\mathscr{E}' = \mathscr{E}$ and $B'u = \int_\Omega a(s,t,s',t')u(s,t)\ ds\ dt$. We assume that there is $u_0 \in \mathscr{K}^{\mathrm{d}}$ (the d-interior of the cone $\mathscr{K}$) and $0 < \alpha < 1$ such that

$$B'u_0 \leq \alpha u_0. \tag{22}$$

It follows that $r(B) \leq \alpha < 1$; see e.g. [26]. Let us assume first that $x^\star \in \mathscr{K}^{\mathrm{d}}$. We define $D \in \mathscr{B}(\mathscr{E})$ as $(Df)(s,t) = \sqrt{u_0(s,t)/x^\star(s,t)}f(s,t)$. Let $v = Dx^\star = D^{-1}u_0$, $G = DJ(x^\star)D^{-1}$, $E = DBD^{-1}$, and $Q = DP(x^\star)D^{-1}$. Of course, $Q^2 = Q$. We show now that $Q' = Q$ which implies that it is an orthogonal projection. To that end, first note that since $P(x^\star)x^\star = x^\star$, we have that $P(x^\star)D^{-2} = D^{-2}$. Given $x, y \in \mathscr{E}$ arbitrary, let $\psi = Dx$, $\eta = Dy$. Then $\langle Qx, y\rangle = \langle DP(x^\star)D^{-2}\psi, D^{-1}\eta\rangle = \langle D^{-1}\psi, D^{-1}\eta\rangle = \langle x, Qy\rangle$. We introduce the norm $\|T\|_v = \inf\{\lambda \geq 0 : \lambda v - Tv \in \mathscr{K}\}$. It follows from (22) that

$$E'v = D^{-1}B'DD^{-1}u_0 \leq \alpha D^{-1}u_0 = \alpha v$$

and thus $\|E'\|_v \leq \alpha$. Since $b \geq 0$, and $Bx^\star + b = x^\star$, we have that $Bx^\star \leq x^\star$ and thus $Ev \leq v$ which implies that $\|E\|_v \leq 1$. Therefore

$$\|E\|^2 \leq r(E'E) \leq \|E'E\|_v \leq \|E'\|_v\|E\|_v \leq \alpha.$$

The fact that $r(J(x^\star)) = r(G) \leq \|E\| \leq \alpha^{1/2} < 1$ is a consequence of the following lemma, whose proof can be found in Mandel and Sekerka [22].

**Lemma 2.7.** *Let $Q \in \mathscr{B}(\mathscr{E})$ be an orthogonal projection. Let $E \in \mathscr{B}(\mathscr{E})$ such that $\|E\| < 1$, and $G = E(I - QE)^{-1}(I - Q)$. Then $r(G) \leq \|E\|$.*

Using the continuity of $J$, the case where $x^\star \notin \mathscr{K}^{\mathrm{d}}$, $x^\star \in \mathscr{K}$, can be treated as a limiting case.

## 3. Iterative aggregation for Markov operators

We say that an operator $B \in \mathscr{B}(\mathscr{E})$, $B \geq 0$ is a Markov operator corresponding to $\hat{x}' \in \mathscr{K}'$ strictly positive if for all $x \in \mathscr{K}$ the following equality holds

$$\langle Bx, \hat{x}'\rangle = \langle x, \hat{x}'\rangle. \tag{23}$$

**Proposition 3.1.** *Let $B$ be a Markov operator corresponding to $\hat{x}'$. Let 1 be a pole of the resolvent operator, then $r(B) = 1$ and $\mathrm{ind}_1 B = 1$.*

*Proof.* Since $B$ is a Markov operator corresponding to the strictly positive element $\hat{x}' \in \mathscr{K}'$, it follows that $\langle u, \hat{x}'\rangle > 0$ for all $u \in \mathscr{K}$, $u \neq 0$ and $\langle Bx, \hat{x}'\rangle = \langle x, \hat{x}'\rangle$, for all $x \in \mathscr{K}$. Let $x_0 \neq 0$, $x_0 \in \mathscr{K}$ be the Frobenius eigenvector of $B$. Then,

$$0 < r(B)\langle x_0, \hat{x}'\rangle = \langle Bx_0, \hat{x}'\rangle = \langle x_0, \hat{x}'\rangle,$$

thus $r(B) = 1$.
Let $q > 1$ be the multiplicity of the pole of the resolvent operator. Then there exists $y \in \mathscr{K}$, $y \neq 0$, such that $0 \leq (I - B)^{q-1} y \neq 0$; see e.g. Marek [23]. Then

$$0 < \langle (I - B)^{q-1} y, \hat{x}' \rangle = \langle (I - B)^{q-2} y, \hat{x}' - B'\hat{x}' \rangle = 0.$$

The contradiction implies $q = 1$. □

We note that the essential condition in Proposition 3.1 is that there exists $u \neq 0$, $u \in \mathscr{K}$, such that $Bu \leq \alpha u$, for some $\alpha > 0$, and some $x'_0 \in \mathscr{K}'$, $x'_0 = Bx'_0$, such that $\langle u, x'_0 \rangle > 0$. Sufficient conditions for this to hold are (i) that $u \in \mathscr{K}^{\mathrm{d}}$, or (ii) that $x'_0$ be strictly positive.

When $\mathscr{E} = \mathbb{R}^n$, $\mathscr{K} = \mathbb{R}^n_+$, $\hat{x}'$ is usually taken as the vector $e = (1, 1, \cdots, 1)^{\mathrm{T}}$ and the Markov operator corresponding to $e$ is usually a transition matrix of a finite Markov chain. In this case, the condition (23) is simply $B^{\mathrm{T}} e = e$, i.e., the matrix $B$ is column stochastic.

In this section we consider a Markov operator $B$ corresponding to $\hat{x}'$ and the problem of finding a stationary distribution of $B$, i.e., we wish to solve

$$Bx = x, \qquad \langle x, \hat{x}' \rangle = 1. \tag{24}$$

The second equation in (24) can be seen as the normalization of $x$ corresponding to $\hat{x}'$. In the case of finite Markov chains, i.e., in Example 2.1, with $\hat{x}' = e$, the element $x$ represents probabilities and $\langle x, e \rangle = \sum_{j=1}^{n} x_j = 1$. Similarly, in infinite Markov chains (Example 2.2) $\langle x, \hat{e} \rangle = \sum_{j=1}^{\infty} x_j = 1$, where $\hat{e}_j = 1$, $j = 1, 2, \cdots$

Before describing the iterative aggregation method for Markov operators we need to understand what is the appropriate aggregated problem to solve in the (smaller) space $\mathscr{F}$. The following lemma states that given a Markov operator with respect to $\hat{x}'$, under certain conditions, one can choose an element $\tilde{x}'$ in the dual space of the (smaller) space $\mathscr{F}$ in such a way that the aggregated operator is a Markov operator with respect to $\tilde{x}'$. The condition required is that

$$\langle P(u)x, \hat{x}' \rangle = \langle x, \hat{x}' \rangle, \tag{25}$$

for all $u \in \mathscr{D}$ and for all $x \in \mathscr{K}$.

We first show that for finite and infinite Markov chains (Examples 2.1 and 2.2), condition (25) holds. From the definitions of the aggregation and disaggregation maps it follows that

$$[S(u)Rx]_j = \frac{u_j \sum_{\ell \in G_i} x_\ell}{\sum_{\ell \in G_i} u_\ell}, \quad \text{if } j \in G_i,$$

thus $\sum_{j \in G_i} [S(u)Rx]_j = \sum_{\ell \in G_i} x_\ell$, and therefore

$$\langle S(u)Rx, e \rangle = \sum_j [S(u)Rx]_j = \sum_{i=1}^{m} \sum_{j \in G_i} [S(u)Rx]_j = \sum_j x_j = \langle x, e \rangle,$$

where the $\sum_j$ is from 1 to $n$ in Example 2.1 and from 1 to $\infty$ in Example 2.2 (and $e$ should be interpreted as $\hat{e}$ in the latter case).

**Lemma 3.2.** *Let $\mathscr{D} \subset \mathscr{E}$. Let $R \in \mathscr{B}(\mathscr{E},\mathscr{F})$, $R\mathscr{K} \subset \mathscr{H}$, $S(x) \in \mathscr{B}(\mathscr{F},\mathscr{E})$, $S(x)\mathscr{H} \subset \mathscr{K}$, for all $x \in \mathscr{D}$, and (5) and (6) hold for all $x \in \mathscr{D}$. Assume further that (25) holds for all $u \in \mathscr{D}$ and all $x \in \mathscr{K}$. Let $B$ be a Markov operator corresponding to $\hat{x}'$. Then there exists $\tilde{x}' = \tilde{x}'(u) \in \mathscr{F}'$ strictly positive on $\mathscr{H}$ such that $B_{\mathscr{F}}(u) = RBS(u)$ is a Markov operator corresponding to $\tilde{x}'$. Thus there exists a stationary distribution $\tilde{w} \in \mathscr{H}$, i.e., $RBS(u)\tilde{w} = \tilde{w}$, and $\langle \tilde{w}, \tilde{x}'\rangle = \langle S(u)\tilde{w}, \hat{x}'\rangle = 1$.*

*Proof.* Fix $u \in \mathscr{D}$. Let $\tilde{x}' = S(u)'\hat{x}'$. Then for any $w \neq 0$, $w \in \mathscr{H}$, $\langle w, \tilde{x}'\rangle = \langle S(u)w, \hat{x}'\rangle > 0$. Using the Markov property of $B$ and (25) we have that for all $w \in \mathscr{F}$

$$\begin{aligned}\langle w, \tilde{x}'\rangle &= \langle S(u)w, \hat{x}'\rangle = \langle S(u)w, B'\hat{x}'\rangle = \langle P(u)BS(u)w, \hat{x}'\rangle \\ &= \langle RBS(u)w, S(u)'\hat{x}'\rangle = \langle B_{\mathscr{F}}(u)w, \tilde{x}'\rangle.\end{aligned}$$ □

We note that if $\tilde{x}' = S(u)'\hat{x}'$, then it follows from (25) that $R'\tilde{x}' = \hat{x}'$.

In the case of a finite Markov chain, given a vector $u$, Lemma 3.2 says that the aggregated matrix $RBS(u)$ is a Markov operator with respect to $\tilde{e} = S(u)^{\mathrm{T}}e$. But

$$\tilde{e}_j = [S(u)^{\mathrm{T}}e]_j = \sum_{\ell\in G_i} u_\ell / \sum_{\ell\in G_i} u_\ell = 1, \quad \text{if } j \in G_i.$$

In other words, if $B$ is column stochastic in $\mathbb{R}^n$, the aggregated matrix $RBS(u)$ is column stochastic in $\mathbb{R}^m$, for all $u \in \mathscr{D}$. This is in contrast to some aggregation or disaggregation maps other than the ones defined in Example 2.1, where the aggregated matrix is not necessarily column stochastic; see e.g. [6], [16], [33], [34].

Let $w$ be the solution of (24). One can find this stationary distribution by solving the problem with another operator $T$, $T \geq 0$, $T$ a Markov operator having $w$ as a stationary distribution. For example, this is the case if $T$ is a polynomial in $B$ with nonnegative coefficients. Also, if such operator $T$ is given, any power of $T$ plays the same role. In this case, the aggregated problem is

$$T_{\mathscr{F}}(x)z = z, \qquad \langle z, \tilde{x}'\rangle = 1, \tag{26}$$

where $T_{\mathscr{F}}(x) = RTS(x)$, cf. (8). In light of Lemma 3.2, (26) has a solution for $\tilde{x}' = S(x)'\hat{x}'$. Thus we have the following

**Algorithm 3.3 (Iterative Aggregation for Markov Operators).** Given a Markov operator $T$ with respect to $\hat{x}'$, an initial guess $x^{(0)}$, and a convergence parameter $\varepsilon > 0$, let $k = 0$.

1. Solve the (aggregated) equation (26) with $x = x^{(k)}$ and call the solution $z^{(k)}$.
2. Disaggregate and iterate according to the formula $x^{(k+1)} = TS(x^{(k)})z^{(k)}$.
3. Test if $\|x^{(k+1)} - x^{(k)}\| < \varepsilon$ (or other convergence test). If yes, STOP; otherwise $k := k + 1$ and go to 1.

Since $r(T) = 1$ and $r(T_{\mathscr{F}}(x)) = 1$ we cannot write (10), and the proof of the local convergence of the iterative aggregation method for Markov operators is more complicated than that in the previous section. We associate with $T$ (or $B$) an operator $V$, which we call *core* operator, satisfying $V \geq 0$ and $r(V) < 1$, and elements $b^{(j)} \in \mathscr{K}$, $j = 1, \cdots, k$ such that for some power $p$, we have the following decomposition

$$T^p x = Vx + \sum_{j=1}^{k} \langle x, \hat{x}'_j\rangle b^{(j)}, \quad \hat{x}'_j \in \mathscr{K}', \ \hat{x}' = \sum_{j=1}^{k} \hat{x}_j{}'. \tag{27}$$

This is a theoretical development and even though this construction may be possible in practice, it is not used in Algorithm 3. The elements $b^{(j)} \in \mathscr{K}$ have to be small enough so that $V \geq 0$. Polák [32] showed that for Example 2.1 and $T = B$ one can choose $p = 1$ and $k = 1$ by choosing $b = b^{(1)}$ such that $b_j = \min_i (Be_i)_j$, where $e_i = (0, \cdots, 1, \cdots, 0)^{\mathrm{T}}$. If $w \in \mathscr{K}$ is a stationary distribution of $T$, i.e. if

$$Tw = w, \qquad \langle w, \hat{x}' \rangle = 1, \tag{28}$$

then it follows from (27), that given $V$, the element $b$ can be found simply by $b = w - Vw$; see further Lemmas 5.1 and 5.3.

The decomposition (27) allows us to work with the core operator $V$, whose powers converge to zero and leave aside the other part of $T$ (or $T^p$), which is essentially a finite rank update. Additionally, for some cases we can explicitly find $C \in \mathscr{B}(\mathscr{F}, \mathscr{E})$ such that

$$T = V + CR(I - V). \tag{29}$$

For example, in the rank-one case, following Polák [32], we can define

$$Cz = \frac{1}{\langle b, \hat{x}' \rangle} \langle z, \tilde{x}' \rangle b. \tag{30}$$

The vector $b$ plays the role of the right hand side corresponding to a solution of a linear system of the form

$$(I - V)w = b \tag{31}$$

and the theory developed in Sect. 2 can be tacitly applied. In Sect. 5 we show explicitly the decomposition (27) for stochastic matrices, i.e. for Example 2.1. The matrix $C$ for stochastic matrices can be defined also as in (30); cf. Lemma 5.3.

We will see later that the analogous operator to $J(x)$ in (12) corresponding to Algorithm 3.3 is

$$\bar{J}(x) = T[I - P(x)V]^{-1}[I - P(x)]. \tag{32}$$

In the following lemma, we show that the operator $T$ in (32) can be replaced by the associated core operator $V$. We note that from the identities (15)–(17) it follows that the condition $r(P(x)V) < 1$ is equivalent to the natural condition $r(RVS(x)) < 1$.

**Lemma 3.4.** *Let $V \in \mathscr{B}(\mathscr{E})$, $V \geq 0$, $r(V) < 1$. Let $\mathscr{D} \subset \mathscr{E}$. Let $R \in \mathscr{B}(\mathscr{E}, \mathscr{F})$, $R\mathscr{K} \subset \mathscr{H}$, $S(x) \in \mathscr{B}(\mathscr{F}, \mathscr{E})$, $S(x)\mathscr{H} \subset \mathscr{K}$, for all $x \in \mathscr{D}$, and (5) and (6) hold for all $x \in \mathscr{D}$. Assume further that $r(P(x)V) < 1$ for all $x \in \mathscr{D}$, and that there exists $C \in \mathscr{B}(\mathscr{F}, \mathscr{E})$ such that (29) holds. Then*

$$\bar{J}(x) = V[I - P(x)V]^{-1}[I - P(x)]. \tag{33}$$

*Proof.* It follows from (29) that

$$\begin{aligned} &V[I - P(x)V]^{-1}[I - P(x)] \\ &\quad = [T - CR(I - V)][I - P(x)V]^{-1}[I - P(x)]. \end{aligned}$$

However, since by (5) $R[I - P(x)] = 0$ we have that

$$\begin{aligned} &R(I - V)[I - P(x)V]^{-1}[I - P(x)] \\ &\quad = R[I - P(x)V + P(x)V - V][I - P(x)V]^{-1}[I - P(x)] \\ &\quad = R[I - P(x)] = 0, \end{aligned}$$

and the proof is complete. □

By comparing (33) with (12) we see that $\bar{J}(x) = J(V, x)$, that is, the Jacobian corresponding to $T$ (or $B$) with $r(T) = 1$ is the same as that of the iterative aggregation for (31) with $r(V) < 1$.

In the following lemma, we establish a relation between the disaggregated iterate and the element $b$, cf. (9) and (10). The element $w^o$ corresponds to the last iterate and $w^n$ to the new one.

**Lemma 3.5.** *Let $V \in \mathscr{B}(\mathscr{E})$, $V \geq 0$, $r(V) < 1$. Let $\mathscr{D} \subset \mathscr{E}$. Let $R \in \mathscr{B}(\mathscr{E}, \mathscr{F})$, $R\mathscr{K} \subset \mathscr{H}$, $S(x) \in \mathscr{B}(\mathscr{F}, \mathscr{E})$, $S(x)\mathscr{H} \subset \mathscr{K}$, for all $x \in \mathscr{D}$, and (5), (6) and $r(P(x)V) < 1$ hold for all $x \in \mathscr{D}$. Let $B$ be a Markov operator corresponding to $\hat{x}'$. Let $w \in \mathscr{K}$ satisfy (24), i.e. let $w$ be a stationary distribution of $B$, and let $b = w - Vw$. Assume that if $RBS(u)\tilde{w} = \tilde{w}$, $\tilde{w} \in \mathscr{H}$, $\langle S(u)\tilde{w}, \hat{x}\rangle = 1$, for all $u \in \mathscr{D}$, i.e., assume that if $\tilde{w}$ is a stationary distribution of the aggregated operator, cf. Lemma 3.2, then*

$$\tilde{w} - RVS(u)\tilde{w} = Rb. \tag{34}$$

*Then for any $w^o \in \mathscr{D}$ and $w^n$ defined as $w^n = TS(w^o)\tilde{w}$, the following identity holds*

$$w^n = T[I - P(w^o)V]^{-1}P(w^o)b.$$

*Proof.* We write $\tilde{w} = [I - RVS(w^o)]^{-1}Rb$ and by using (16) with the appropriate operators, the lemma follows. □

The next lemma sets the stage for the local convergence proof of Algorithm 3.3.

**Lemma 3.6.** *Let the hypotheses of Lemma 3.5 hold. Assume further that $Tw = w$ for some $T \in \mathscr{B}(\mathscr{E})$, $T \geq 0$, $r(T) = 1$. Then*

$$w = T[I - P(x)V]^{-1}P(x)b + T[I - P(x)V]^{-1}[I - P(x)]w, \tag{35}$$

*and*

$$w^n - w = T[I - P(w^o)V]^{-1}[I - P(w^o)](w^o - w).$$

*Proof.* We have that

$$\begin{aligned} w &= Tw = T[I - P(x)V]^{-1}[w - P(x)Vw] \\ &= T[I - P(x)V]^{-1}[P(x)(I - V)w + w - P(x)w] \\ &= T[I - P(x)V]^{-1}\{P(x)b + [w - P(x)w]\} \\ &= T[I - P(x)V]^{-1}P(x)b + T[I - P(x)V]^{-1}[I - P(x)]w, \end{aligned}$$

which is the required identity (35). By Lemma 3.5, the identity (35), and the relation (5) with $x = w^o$,

$$\begin{aligned} w^n - w &= \quad T[I - P(w^o)V]^{-1}P(w^o)b + T[I - P(w^o)V]^{-1}[I - P(w^o)]w^o \\ &\quad -T[I - P(w^o)V]^{-1}P(w^o)b - T[I - P(w^o)V]^{-1}[I - P(w^o)]w \\ &= T[I - P(w^o)V]^{-1}[I - P(w^o)](w^o - w), \end{aligned}$$

and the proof is complete. □

**Theorem 3.7.** *Let* $R \in \mathscr{B}(\mathscr{E}, \mathscr{F})$ , $R\mathscr{K} \subset \mathscr{H}$, *and* $S(x) \in \mathscr{B}(\mathscr{F}, \mathscr{E})$. *Let* $\mathscr{W} \subset \mathscr{E}$ *be a subspace such that* $\|x\| \leq \|x\|_{\mathscr{W}}$ *for all* $x \in \mathscr{W}$. *Let* $\mathscr{D}$ *be a nonempty* $\mathscr{W}$*-open set such that* $S(x)\mathscr{H} \subset \mathscr{K}$, $S(x)$ *is* $\mathscr{W}$*-continuous for all* $x \in \mathscr{D}$, *and (5) and (6) hold for all* $x \in \mathscr{D}$. *Let* $B \in \mathscr{B}(\mathscr{E})$ *be a Markov operator with respect to* $\hat{x}' \in \mathscr{K}'$, *with a stationary distribution* $w$ *satisfying (24), such that* $Rw \in \mathscr{H}^{\mathrm{d}}$. *Let* $V \in \mathscr{B}(\mathscr{E})$, $V \geq 0$, $r(V) < 1$. *Let* $b = w - Vw$, $b \in \mathscr{K}$ *be such that* $0 \neq b \leq Tx$ *for all* $x \in \mathrm{Ext}\,\mathscr{K}$, $\langle x, \hat{x}' \rangle = 1$. *Assume that* $r(P(x)V) \leq \alpha < 1$, *for some* $\alpha \in \mathbb{R}$, $0 < \alpha$, *and that* $RBS(x)\tilde{w} \in \mathscr{D}$ *whenever* $x \in \mathscr{D}$ *and* $\tilde{w} = RBS(x)\tilde{w} \in \mathscr{H}$. *Moreover, assume* $b + VS(x)[I - R\underline{V}S(x)]^{-1}Rb \in \mathscr{D}$ *whenever* $x \in \mathscr{D}$. *Let there be a* $\beta \in \mathbb{R}$, $0 < \beta < 1$, *such that* $r(\bar{J}(w)) \leq \beta$, *where* $\bar{J}(w)$ *is defined in (32). Then, Algorithm 3.3 is* $\mathscr{W}$*-locally convergent, i.e., there exists a* $\mathscr{W}$*-open neighborhood* $\mathscr{U}$ *of* $w$ *such that*

$$\lim_{k\to\infty} \|x^{(k)} - w\| = 0 \quad \text{for } x^{(0)} \in \mathscr{U}.$$

*The speed of convergence is characterized by the estimates*

$$\|x^{(k)} - w\| \leq \|x^{(k)} - w\|_{\mathscr{W}} \leq \kappa\rho^k$$

*where* $\rho = r(\bar{J}(w)) + \eta < 1$, *for some* $\eta > 0$ *and* $\kappa$ *independent of* $k$.

*Proof.* By the way we define $b$, we can write $Bx = Vx + \langle x, \hat{x}' \rangle b$. We see that all hypotheses of Lemmas 3.5 and 3.6 are satisfied for $T = B$. Let $\zeta > 0$ be arbitrary. It is well known that there is a norm $\|.\|_\zeta$ equivalent to the norm $\|.\|_{\mathscr{W}}$ such that

$$\|\bar{J}(w)\|_\zeta \;\leq\; r(\bar{J}(w)) + \zeta \;; \tag{36}$$

see, e.g., [29, p.55]. By Lemma 3.6 we have that

$$\begin{aligned} \|x^{k+1} - w\|_\zeta &\leq \|\bar{J}(x^{k+1})\|_\zeta \; \|x^k - w\|_\zeta \leq \cdots \leq \\ &\leq \|\bar{J}(x^{k+1})\|_\zeta \cdots \|\bar{J}(x^0)\|_\zeta \; \|x^0 - w\|_\zeta . \end{aligned}$$

The $\mathscr{W}$-continuity of $S(x)$ implies the $\mathscr{W}$-continuity of $\bar{J}(x)$. Thus, there exists a $\mathscr{W}$-open neighborhood $\mathscr{U}$ of $w$ such that if $x^{(0)} \in \mathscr{U}$, then

$$\|\bar{J}(x^k)\|_\zeta \;\leq\; r(\bar{J}(w)) + \zeta \; + \eta$$

with some $\eta > 0$ independent of $k$, and, as in Theorem 2.6, the theorem follows. □

The hypotheses of Theorem 3.7 usually hold in practice. In particular, in Sect. 5, this is shown for Example 2.1.

## 4. Inexact correction

In practice, the system (8) is often solved iteratively. A splitting $I_{\mathscr{F}} - B_{\mathscr{F}}(x) = F(x) - G(x)$ is used, where $H(x) = F(x)^{-1}G(x) \geq 0$ and $r(H(x)) < 1$; see e.g. the different options in the package by W. Stewart [36] or in the paper [28] and the references given therein. A certain number, say $s$, of iterations is performed and (8) is replaced by

$$\begin{aligned} z_0^{(k)} &= Rx^{(k)} \\ \text{for } j &= 0, \cdots, s-1 \\ & \quad z_{j+1}^{(k)} = H(x^{(k)})z_j^{(k)} + F(x^{(k)})^{-1}Rb \end{aligned} \tag{37}$$

Thus,

$$z_s^{(k)} = H(x^{(k)})^s R x^{(k)} + \sum_{j=0}^{s-1} H(x^{(k)})^j F(x^{(k)})^{-1} Rb.$$

The iterative aggregation method with inexact correction for the solution of the linear system (2) can then be expressed in terms of the elements of $\mathscr{E}$ as

$$x^{(k+1)} = BU^{(s)}(x^{(k)}) + b, \tag{38}$$

where

$$U^{(s)}(x) = S(x)[H(x)^s Rx + \sum_{j=0}^{s-1} H(x)^j F(x)^{-1} Rb]; \tag{39}$$

cf. [10], [19]. From the consistency of the iterative process (37) it follows the consistency of the method (38), see, e.g. Varga [42], and in particular

$$U^{(s)}(x^\star) = x^\star. \tag{40}$$

We remark that the splitting $F(x) - G(x)$ can be taken as $F(x) = I_{\mathscr{F}}$ and $G(x) = B_{\mathscr{F}}(x)$ and all our results apply also to this particular case.

We also point out that the number of iterations in (37) can vary from one step to the next, i.e., $s = s(k)$. The following local convergence result applies to this general case as long as $s(k) \geq \hat{s}$, $k = 0, \cdots$ In other words, we show that if the number of inner iterations is large enough, the overall methods converges; cf. Nichols [31]. In Lanzkron, Rose and Szyld [19], and in Frommer and Szyld [10], [11], similar situations are studied. In those references, unlike here, conditions for convergence for any number of inner iterations are prescribed. The splitting induced by the operator $J(x)$ is not necessarily regular, cf. [19], and therefore such conditions do not hold here; nevertheless see the comment before Algorithm 4.2.

**Theorem 4.1.** *Let the hypotheses of Theorem 2.6 hold. Let $I_{\mathscr{F}} - B_{\mathscr{F}}(x) = F(x) - G(x)$ be a splitting, i.e., $F(x)^{-1} \in \mathscr{B}(\mathscr{E})$, and let $H(x) = F(x)^{-1}G(x) \geq 0$ with*

$$r(H(x)) \leq \mu < 1 \quad \text{for all} \quad x \in \mathscr{D}. \tag{41}$$

*Then, there is a $\hat{s}$ such that if $s \geq \hat{s}$, the iteration process (38) is $\mathscr{W}$-locally convergent, i.e., there exists a $\mathscr{W}$-open neighborhood $\mathscr{U}$ of $x^\star$ such that (13) holds. The speed of convergence is characterized by the estimates (14) where $\rho = r(J(x^\star)) + \eta < 1$, for some $\eta > 0$ and $\kappa$ independent of $k$.*

*Proof.* Let

$$J^{(s)}(x) = B\left[I - S(x)H(x)^s Rx + S(x)\sum_{j=0}^{s-1} H(x)^j F(x)^{-1} R(I-B)\right]. \tag{42}$$

Note that $J^{(s)}(x) = \gamma(x) + \Lambda(x)$ is an affine operator in the sense that $J^{(s)}(x)u = \gamma(x) + \Lambda(x)u$. It follows, using (5) and (6), that

$$\begin{aligned} J^{(s)}(x)x &= -BS(x)H(x)^s Rx + B\left[x - S(x)\sum_{j=0}^{s-1} H(x)^j F(x)^{-1} R(I-B)S(x)Rx\right] \\ &= -BS(x)H(x)^s Rx + B[x - S(x)(I_{\mathscr{F}} - H(x)^s)Rx] = 0. \qquad (43) \end{aligned}$$

We replace $b = (I - B)x^\star$ in (39) and use (40), (42) and (43) to write

$$BU^{(s)}(x) - BU^{(s)}(x^\star) \;=\; BS(x)\left[H(x)^s Rx + \sum_{j=0}^{s-1} H(x)^j F(x)^{-1} R(I-B)x^\star - Bx^\star\right]$$

$$= \; -J^{(s)}(x)x^\star = J^{(s)}(x)(x - x^\star). \qquad (44)$$

The hypothesis (41) implies that

$$\sum_{j=0}^{\infty} H(x)^j = [I_{\mathscr{F}} - B_{\mathscr{F}}(x)]^{-1} F(x). \qquad (45)$$

Comparing (42) with (17), we obtain

$$\lim_{s\to\infty} J^{(s)}(x) = J(x). \qquad (46)$$

Let $\zeta$ and $\xi$ be arbitrary. From (41) it follows that the rate of convergence of the partial sums in (45), as well as of the sequence $H(x)^s \to 0$, can be bounded independent of $x$. Thus, there exists a $\hat{s} = \hat{s}(\xi)$ independent of $x$ such that

$$\|J^{(s)}(x) - J(x)\|_\zeta < \xi \quad \text{for } s \geq \hat{s},$$

where $\|.\|_\zeta$ is the norm equivalent to the norm $\|.\|_{\mathscr{W}}$ for which (20) holds. Thus, using the same arguments as in Theorem 2.6 we have that

$$\|BU^{(s)}(x) - BU^{(s)}(x^\star)\|_\zeta \;\leq\; [r(J(x^\star)) + \varepsilon + \xi + \zeta]\; \|x - x^\star\|_\zeta$$

and the theorem is proved. □

In a way analogous to the one described earlier in the section, often in practice, the aggregated Markov problem (26) is not solved exactly. Instead, an iterative method is used, and the (inner) process is stopped after a certain number of (inner) iterations, or, equivalently, after certain (inner) convergence criteria is satisfied. Here, we choose to stop the number of inner iterations if the residual is small enough, this is done, e.g. in Elman and Golub [8] and Golub and Overton [13], [14]. Thus we have the following *Iterative Aggregation with Inexact Correction Algorithm for Markov Operators*

**Algorithm 4.2.** Given a Markov operator $T$ with respect to $\hat{x}'$, an initial guess $x^{(0)}$, inner convergence parameters $\varepsilon_k > 0$, $k = 0, 1, \cdots$, and a global convergence parameter $\varepsilon > 0$, let $k = 0$.

1. Let $j = 0$ and $z_0^{(k)} = Rx^{(k)}$.
2. Compute $z_{j+1}^{(k)} = H(x^{(k)})z_j^{(k)}$, $\quad \langle S(x^{(k)})z_{j+1}^{(k)}, \hat{x}'\rangle = 1$
   where $H(x) = F(x)^{-1}G(x) \geq 0$, and $I_{\mathscr{F}} - RTS(x) = F(x) - G(x)$.
3. Test if $\|z_{j+1}^{(k)} - H(x^{(k)})z_{j+1}^{(k)}\| < \varepsilon_k$. If yes, let $z^{(k)} = z_{j+1}^{(k)}$,
   otherwise, let $j := j + 1$ and go to 2.
4. Disaggregate and iterate according to the formula $x^{(k+1)} = TS(x^{(k)})z^{(k)}$.
5. Test if $\|x^{(k+1)} - x^{(k)}\| < \varepsilon$ (or other convergence test). If yes, STOP; otherwise $k := k + 1$ and go to 1.

**Theorem 4.3.** *Let the hypotheses of Theorem 3.7 hold. In addition, let* $I_{\mathscr{F}} - RBS(x) = F(x) - G(x)$ *be a regular splitting, i.e.,* $F(x)^{-1} \geq 0$ *and* $G(x) \geq 0$. *Let* $H(x) = F(x)^{-1}G(x)$ *be such that* $r(H(x)) < 1$ *for all* $x \in \mathscr{D}$. *Let* $\varepsilon_k > 0$ *be such that* $\|z^{(k)} - H(x^{(k)})z^{(k)}\| < \varepsilon_k$ *implies*

$$\|u^{(k)}\|_\zeta < \nu\|x^{(k)} - w\|_\zeta \tag{47}$$

*where* $Ru^{(k)} = z^{(k)} - RVS(x^{(k)})z^{(k)}$, *the norm* $\|.\|_\zeta$ *is the norm equivalent to* $\|.\|_{\mathscr{W}}$ *such that (36) holds, and where* $\nu > 0$ *is such that* $\beta + \nu < 1$. *Then, Algorithm 4.2 is* $\mathscr{W}$*-locally convergent, i.e., there exists a* $\mathscr{W}$*-open neighborhood* $\mathscr{U}$ *of* $w$ *such that*

$$\lim_{k\to\infty} \|x^{(k)} - w\| = 0 \quad \text{for } x^{(0)} \in \mathscr{U}.$$

*The speed of convergence is characterized by the estimates*

$$\|x^{(k)} - w\| \leq \|x^{(k)} - w\|_{\mathscr{W}} \leq \kappa\rho^k$$

*where* $\rho = r(\bar{J}(w)) + \eta + \nu < 1$, *for some* $\eta > 0$ *and* $\kappa$ *independent of* $k$.

*Proof.* Let $v^{(k)} = z^{(k)} - H(x^{(k)})z^{(k)}$. Then

$$Fz^{(k)} - Gz^{(k)} = Fv^{(k)} = z^{(k)} - RBS(x^{(k)})z^{(k)}.$$

Therefore we see that $z^{(k)} - RVS(x^{(k)})z^{(k)} - Rb = Fv^{(k)}$, and also that

$$x^{(k+1)} = B[I - P(x^{(k)})V]^{-1}[P(x^{(k)})b + S(x^{(k)})Fv^{(k)}].$$

It follows that

$$x^{(k+1)} - w = \bar{J}(x^{(k)})(x^{(k)} - w) + B[I - P(x^{(k)}V]^{-1}S(x^{(k)})Fv^{(k)}.$$

Since $Ru^{(k)} = Fv^{(k)}$, we deduce that

$$\|x^{(k+1)} - w\|_\zeta \leq \|\bar{J}(x^{(k)})\|_\zeta \; \|x^{(k)} - w\|_\zeta + \|u^{(k)}\|_\zeta.$$

Since by our hypothesis, as in Theorem 3.7, $\|\bar{J}(x^{(k)})\|_\zeta \leq \rho(\bar{J}(x^{(k)})) + \zeta + \eta$ with some $\eta > 0$ independent of $k$, and by (47), the theorem follows. □

## 5. Applications to stochastic matrices

In this section we study in more detail stochastic matrices, i.e., Example 2.1. We exhibit an associated zero convergent core matrix $V$, and illustrate the iterative aggregation method with some numerical experiments.

Let $\mathscr{E} = \mathbb{R}^n$, $\mathscr{K} = \mathbb{R}^n_+$ and let $B$ be a stochastic matrix, i.e., a Markov operator corresponding to the vector $e \in \mathbb{R}^n$, $e^{\mathrm{T}} = (1, \cdots, 1)$. Let

$$B = E \begin{bmatrix} F_0 & 0 & . & . & . & 0 \\ G_1 & F_1 & . & . & . & 0 \\ . & . & . & . & . & . \\ G_r & 0 & . & . & . & F_r \end{bmatrix} E^{\mathrm{T}} \tag{48}$$

be a representation of $B$ in which the square diagonal blocks $F_j$ are of order $n_j$ with $\sum_{j=0}^{r} n_j = n$. The matrices $F_j$ are irreducible for $1 \leq j \leq r$, $r(F_0) < 1$, and $E$ is a permutation matrix; see e.g. Gantmacher [12]. If $w \in \mathbb{R}^n_+$ is any stationary probability vector of $B$, i.e., if $Bw = w$, $\langle w, e \rangle = 1$, it follows immediately that $\hat{w} = E^{\mathrm{T}} w$ is a stationary probability vector of $E^{\mathrm{T}} BE$. Moreover, since $r(F_0) < 1$, then, this stationary probability vector has the form

(49) $$\hat{w}^{\mathrm{T}} = (0, \hat{w}_1, \cdots, \hat{w}_r) \ ,$$

where $\hat{w}_j \in \mathbb{R}^{n_j}$, $j = 1, \cdots, r$. The first $n_0$ components of $\hat{w}$, which are zero, are said to correspond to the transient state. The other components are said to correspond to the ergodic states.

Let $\hat{F}_j = \frac{1}{2}(I + F_j)$, $0 \leq j \leq r$. Let $\hat{t}$ be any positive integer such that

(50) $$T = \left[\frac{1}{2}(I + B)\right]^{\hat{t}} = E \begin{bmatrix} \hat{F}_0^{\hat{t}} & 0 & . & . & . & 0 \\ \hat{G}_1 & \hat{F}_1^{\hat{t}} & . & . & . & 0 \\ . & . & . & . & . & . \\ \hat{G}_r & 0 & . & . & . & \hat{F}_r^{\hat{t}} \end{bmatrix} E^{\mathrm{T}}$$

has its diagonal blocks $\hat{F}_j^{\hat{t}}$ strictly positive for $1 \leq j \leq r$. It is well known that we may choose $\hat{t} = \max\{t_j : 1 \leq j \leq r\}$, where $1 \leq t_j \leq n_j^2 - 2n_j + 2$; see e.g. Varga [42, p.42]. It is easy to see that $Bw = w$ if and only if $Tw = w$, i.e., $w \in \mathbb{R}^n_+$ is a probability stationary vector for $B$ if and only if it is for $T$. Therefore, the permuted probability stationary vector of $T$, $E^{\mathrm{T}} w = \hat{w}$ has the same form (49).

The following lemma provides the existence of a core matrix $V$ associated with $T$ of the form (50), with the appropriate choice of the exponent $\hat{t}$.

**Lemma 5.1.** *Let $B$ be a column stochastic matrix represented as in (48). Let $T$ be defined as in (50), and let $w$ be a stationary probability vector of $B$ and thus of $T$, i.e.,*

(51) $$Tw = w, \quad \langle w, e \rangle = 1,$$

*where $e^{\mathrm{T}} = (1, \cdots, 1)$. For every $\epsilon > 0$ there exists a matrix $V_\epsilon$ and a vector $b \in \mathbb{R}^n_+$, which depends on $w$ and $\epsilon$, such that*

(52) $$Tw = V_e w + \langle w, e \rangle b$$

*and*

(53) $$r(V_\epsilon) = \epsilon \ .$$

*Moreover, (51) holds if and only if $w - V_\epsilon w = b$ .*

*Proof.* Consider for each $j = 1, \cdots, r$, the decomposition of the diagonal blocks of $B$ in (48) into their eigenprojections, i.e., $F_j = P_j + Z_j$, $1 \leq j \leq r$, where $P_j^2 = P_j$, $P_j Z_j = Z_j P_j = 0$, $1 \notin \sigma(Z_j) \subset \{|\lambda| \leq 1\}$. Then for any $t$, $\hat{F}_j^t = [\frac{1}{2}(I + F_j)]^t = [\frac{1}{2}(I + P_j + Z_j)]^t = [P_j + \frac{1}{2}(I + Z_j)(I - P_j)]^t$ and therefore

(54) $$\hat{F}_j^t = P_j + \left[\frac{1}{2}(I + Z_j)\right]^t (I - P_j), \quad 1 \leq j \leq r.$$

For each $j = 1, \cdots, r$, let $\hat{x}_j \in \mathbb{R}_+^{n_j}$ be the unique stationary probability vector of the irreducible matrix $F_j$, i.e.,

$$F_j \hat{x}_j = \hat{x}_j = P_j \hat{x}_j, \ \langle \hat{x}_j, e_j \rangle = 1, \tag{55}$$

where $e_j \in \mathbb{R}^{n_j}$, $e_j^{\mathrm{T}} = (1, \cdots, 1)$.
Since for every $x \in \mathbb{R}_+^{n_j}$

$$\lim_{t\to\infty} \frac{\hat{F}_j^t x}{\langle x, e_j \rangle} = \frac{1}{\langle x, e_j \rangle} P_j x,$$

and the rate of convergence is independent of $x$, we can find a positive integer $p_j$ such that for $t \geq p_j$

$$\hat{F}_j^t x \geq (1-\epsilon) P_j x, \quad j = 1, ..., r. \tag{56}$$

By hypothesis, $r(F_0) < 1$ and thus, there is a positive integer $p_0$ such that

$$r(\hat{F}_0^{p_0}) < \epsilon. \tag{57}$$

Let

$$\hat{t} = \max\{p_j, \ j = 0, 1, \cdots, r\} \tag{58}$$

and define $U_j = \hat{F}_j^{\hat{t}} - (1-\epsilon)P_j$; see (54). Thus,

$$U_j = \epsilon P_j + \left[\frac{1}{2}(I + Z_j)\right]^{\hat{t}} (I - P_j). \tag{59}$$

We provide now the core matrix

$$V_\epsilon = E \begin{bmatrix} \left[(1/2)(I + F_0)\right]^{\hat{t}} & 0 & . & . & . & 0 \\ \hat{G}_1 & U_1 & . & . & . & 0 \\ . & . & . & . & . & . \\ \hat{G}_r & 0 & . & . & . & U_r \end{bmatrix} E^{\mathrm{T}}, \tag{60}$$

where $\hat{G}_j$ are the blocks in (50). It follows from (56) that $r(U_j) = \epsilon$, $1 \leq j \leq r$. Thus, from (57) and (58) we have that

$$r(V_\epsilon) = \max[r(\hat{F}_0^{\hat{t}}), \epsilon] = \epsilon \tag{61}$$

which is the desired condition (53).
We provide now the vector $b = b(w, \epsilon)$. Let $w \in \mathbb{R}_+^n$ be any stationary probability vector of $B$. It is easy to show that the permuted vector $\hat{w} = E^{\mathrm{T}} w$ is a convex combination of the vectors $\overline{x}_j$ defined as $\overline{x}_j^{\mathrm{T}} = (0, \cdots, \hat{x}_j, \cdots, 0)$, $1 \leq j \leq r$, where $\hat{x}_j \in \mathbb{R}_+^{n_j}$ are as defined in (55). In other words, there exist numbers $\mu_j$,

$$0 \leq \mu_j \leq 1\ , \quad j = 1, \cdots, r, \quad \sum_{j=1}^{r} \mu_j = 1,$$

such that

$$w = \sum_{j=1}^{r} \mu_j \bar{x}_j,$$

i.e., $\hat{w}^{\mathrm{T}} = (0, \mu_1 \hat{x}_1, \cdots, \mu_r \hat{x}_r)$; cf. (49). We define $b = Ec$, where $c^{\mathrm{T}} = \left(0,\ c_1^{\mathrm{T}}, \cdots, c_r^{\mathrm{T}}\right)$, and $c_j = \mu_j(1-\epsilon)\hat{x}_j$, $j = 1, \cdots, r$. The identity (52) follows.
Using the identities (51) in (52) one obtains $w - V_\epsilon w = b$, and conversely. The proof is complete. □

We note that during the proof of Lemma 5.1, we have provided, for any $\epsilon > 0$, the decomposition $T = V_\epsilon + (1-\epsilon)EPE^{\mathrm{T}}$, where $V_\epsilon$ is given in (60) and

$$P = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 0 & P_1 & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & P_r \end{bmatrix}.$$

Let us now check that all the requirements of Theorem 3.7 are fulfilled. We choose $R$ and $S(x)$ as in Example 2.1. We have already shown that (5), (6), and (25) hold for all $x \in \mathscr{D} = \{x \in \mathbb{R}_+^n : Rx > 0\} \subset \mathrm{Int}\mathbb{R}_+^m$. Since $\mathscr{W} = \mathscr{E} = \mathbb{R}^n$, the operator-function $S(x)$ is $\mathscr{W}$-continuous on $\mathscr{D}$. The existence of a convergent core $V$ is guaranteed by the previous lemma with $V = V_\epsilon$. The convergence properties of $P(x)V$ and $J(x)$ can be shown as in [22]. Thus, all hypotheses of Theorem 3.7 are fulfilled and we have the following result.

**Theorem 5.2.** *Let $B$ be an irreducible stochastic matrix. Then Algorithm 3.3 is locally convergent to the unique stationary probability vector of $B$, and its rate of convergence is given by the estimate $\|\hat{x} - x^{(k)}\| \leq \kappa^k \|\hat{x} - x^{(0)}\|$, where $\kappa \leq \sqrt{\epsilon}$.*

It is worth mentioning that the rate of convergence can be made arbitrarily fast.

For a general stochastic (reducible) matrix $B$, partitioned as in (48), Algorithm 3.3 can be appropriately adapted to obtain the same result as in Theorem 5.2. The aggregation map $R$ is constructed as a direct sum of aggregation maps $R_j : \mathbb{R}^{n_j} \to \mathbb{R}^{m_j}$, $j = 1, \cdots, r$, for some appropriate numbers $m_j$, with $\sum_{j=1}^{r} m_j \leq m$, i.e.,

$$R = 0 \oplus R_1 \oplus R_2 \oplus ... \oplus R_r,$$

where 0 denotes the zero-map. This means that the matrix (7) has a diagonal block structure. Similarly, the disaggregation operators $S = S(u)$ is defined as

$$S = 0 \oplus S_1(u_1) \oplus ... \oplus S_r(u_r),$$

where $u^{\mathrm{T}} = (0, u_1^{\mathrm{T}}, ..., u_r^{\mathrm{T}})$. Moreover, by a proper choice of $R_j$, $j = 1, \cdots, r$, one can obtain all extreme stationary probability vectors $B$. It follows that in this case, Algorithm 3.3 can be easily implemented for parallel computations.

The construction of a convergent core $V$ of the operator under consideration $B$ is based on the primitivity concept of the irreducible components of the associated matrix $T$. Lemma 5.1 shows that there is always possible to reach a very fast convergence, i.e., when the appropriate power $\hat{t}$ is used in (50). This is of course a theoretical result, in practice, this may be in some cases rather costly. This happens if the transient block $F_0$ is large and has its spectral radius close to 1, and if the number of ergodic blocks is small and the blocks are sparse. In either of these two cases, $\hat{t}$ is large. The same argument implies that, in light of (61) and (58), if the transient part $F_0$ is absent in the representation (48) of $B$, and the diagonal blocks in (48), i.e., the ergodic blocks, are essentially smaller than the size of $B$ itself, then the core operator is obtained with few computations and thus fast rate of convergence is easily and cheaply achievable.

Thus, our Lemma 5.1 offers a possible explanation why the aggregation/disaggregation algorithms converge fast in the nearly decomposable case as observed in practical computations and as shown by Cao and Stewart [2]. In addition, the lemma suggests that fast convergence can be achieved in other situations, namely, when the diagonal blocks are small in size, and the off-diagonal blocks are sparse with elements which need not to be small, i.e., in the case of stochastic matrices representing Markov chains with many states small in size, but not necessarily uncoupled.

The following lemma offers another less expensive way how to guarantee a convergent core $V$ for general stochastic matrix $B$.

**Lemma 5.3.** *Let the exponent $\hat{t}$ in (50) be such that at least one column of $\hat{F}_j^{\hat{t}}$ (not necessarily all of them) is strictly positive, $j = 1, \cdots, r$. Let $c_j^{\mathrm{T}} = (\gamma_1^j. \cdots, \gamma_{n_j}^j)$ with $\gamma_k^j = \min\{\hat{f}_{kl}^j : l = 1, \cdots, n_j\}$, and let $b^{\mathrm{T}} = \left(0, c_1^{\mathrm{T}}, \cdots, c_r^{\mathrm{T}}\right) E^{\mathrm{T}}$ and $V = T - \langle ., \ e\rangle b$, $e^{\mathrm{T}} = (1, \cdots, 1)$. Then $r(V) < 1$, $b \geq 0$, and*

$$Bw = Tw = w, \ \langle w, e\rangle = 1,$$

*holds if and only if $w - Vw = b$.*

In view of (48) there are actually $r$ noted stationary probability vectors and namely those uniquely determined by the irreducible blocks $F_j$, $j = 1, \cdots, r$. These stationary probability vectors are called *extremal*. A systematic way to compute all the extremal stationary probability vectors, say by iterative or semiiterative methods, consists just of computing successive $n$ (in general) iteration sequences with the standard basis elements as starting vectors respectively; see Marek and Szyld [28], Tanabe [39]. In conjunction with applying aggregation/disaggregation algorithms an efficient way to determine the block structure of $B$ as shown in (48) is needed. To this purpose Tarjan's algorithm as implemented by Duff and Reid [7] is recommended.

In the rest of the section we present some numerical experiments which illustrate the convergence of the iterative aggregation method for stochastic matrices. Consider the $8 \times 8$ stochastic matrix given in Courtois [6, Appendix 3]. There, an aggregation to $\mathbb{R}^3$ is suggested, where $G_1 = \{1, 2, 3\}$, $G_2 = \{4, 5\}$, and $G_3 = \{6, 7, 8\}$. The largest eigenvalue of the stochastic matrix which is less than one has value .9998 and thus the power method is extremely slow. Similarly, for the Gauss-Seidel iteration operator, the largest eigenvalue less than one has value .99878. In contrast, the iterative aggregation method with the aggregation just mentioned and inner residual tolerance of $10^{-5}$ converged to the stationary probability distribution to within $10^{-4}$ in 14 outer iterations, using a total of 34 inner iterations.

The sample matrix is nearly decomposable, and has about three zero elements in each row. In order to test the iterative aggregation method in more general stochastic matrices, we increased the value of each off-diagonal entry by a fixed number $\alpha$, and subtracted from the diagonal entry the corresponding amount, so the matrix remains stochastic. In Table 1 we report the number of outer and inner iterations for different values of $\alpha$. In all cases, the inner iterations were stopped when the change from the previous iterate was below $10^{-5}$. The method was stopped when the current iterate did not change by more than $10^{-4}$.

**Table 1.** Iterative aggregation method for stochastic matrices

| $\alpha$ | outer iter. | total inner iter. |
|---|---|---|
| 0.0 | 14 | 34 |
| $3.16 \times 10^{-7}$ | 14 | 34 |
| $1.0 \times 10^{-6}$ | 14 | 34 |
| $3.16 \times 10^{-6}$ | 14 | 34 |
| $1.0 \times 10^{-5}$ | 15 | 34 |
| $3.16 \times 10^{-5}$ | 15 | 35 |
| $1.0 \times 10^{-4}$ | 15 | 36 |
| $3.16 \times 10^{-4}$ | 15 | 34 |
| $1.0 \times 10^{-3}$ | 15 | 29 |
| $3.16 \times 10^{-3}$ | 15 | 23 |

## 6. Conclusion

We have provided a proof of convergence for an iterative aggregation method for general stochastic matrices, not necessaryly nearly decomposable matrices. Our proofs are more general than that, they applied to Markov processes in general Banach spaces. The proofs are based on the idea of associating to the Markov operator a core operator which is zero-convergent, and studying the convergence of the iterative aggregation method for the new associated system.

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