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The Korteweg–de Vries equation: Its place in the development of nonlinear physics

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This article puts Korteweg and de Vries's manuscript (published in the *Philosophical Magazine* in 1895) in historical context. The article highlights the importance of the Korteweg–de Vries equation in the development of concepts used in nonlinear physics and also mentions some of their recent applications.

Keywords: nonlinear physics; solitons; breathers; biomolecules; conducting polymers; magnetism; nonlinear optics

1. The early history

Solitons or solitary waves are large-amplitude excitations with finite spatial widths which are found in certain nonlinear systems and which can propagate over long distances without changing their shapes. Solitary waves also have the property that, although they may interact strongly with other solitary waves, they emerge from the interaction intact. The first report of the observation of a solitary wave was written by John Scott Russell. His account is reproduced below:

> J. Scott Russell. Report on Waves, Fourteenth Meeting of the British Association for the Advancement of Science, 1844.
>
> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

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This observation propelled Scott Russell to commence experimental investigations of solitary waves in water tanks. Empirically he found that the velocity of the wave, v, depended on its amplitude. Scott Russell became convinced that, were it not for viscosity, the solitary wave would retain its shape indefinitely. His observations seemed to be in contradiction with the theoretical description of hydrodynamics, since it was thought that such waves should change shape, becoming progressively steeper at the front until the wave eventually breaks. The first step in supporting Scott Russell's description of these waves was taken in 1876 when Lord Rayleigh published a theoretical paper in this journal [1] which supported John Scott Russell's experimental observations. In this paper, Lord Rayleigh considered an incompressible fluid with negligible viscosity and showed that, if the amplitude of the wave, a, is small compared with the depth of the canal h ($h \gg a$), the profile of the wave is given by

$$u(x, t) = a\,\mathrm{sech}^2 \frac{(x - vt)}{\xi} \tag{1}$$

where the wave's spatial extent is given by $\xi^2 \approx \frac{4}{3} h^2 (h/a + 1)$ and the velocity by $v^2 \approx g(h + a)$. The solitary wave's shape is sketched in Figure 1. However, the approximate nature of Rayleigh's treatment and McCowan's [2,3] subsequent treatments of the solitary wave phenomenon suggested that the solitary wave only approximately retained its shape. The tendency for large-amplitude waves to break can be illustrated by examining the (dimensionless) nonlinear partial differential equation [4]

$$\frac{\partial u}{\partial t} + \left(1 + u\right) \frac{\partial u}{\partial x} = 0 \tag{2}$$

which can formally be solved to yield

$$u(x, t) = f(x - (1 + u)t) \tag{3}$$

where f is an arbitrary function. Since this solution describes a wave with a local velocity v that depends on its local amplitude, the nonlinearity shifts its shape. This can be seen by considering an initial condition given by

$$u(x, 0) = f(x) \tag{4}$$

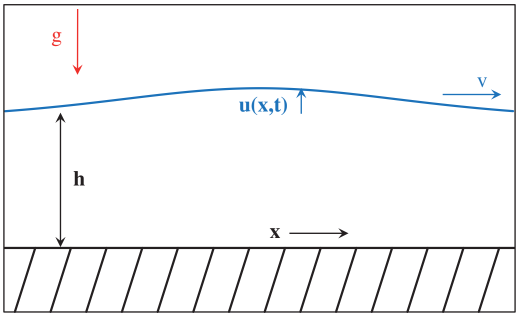

Figure 1. A sketch of the profile of a long solitary wave $u(x, t)$ moving with velocity v, in a very long rectangular channel of depth h.

where f is any positive single-valued function. The solution at finite t can be simply obtained from the $t=0$ value by translating the value at x_0 through a distance $(1+f(x_0))t$. As seen in Figure 2, the function changes its shape and may even not be single-valued for sufficiently large times, signifying that the wave has broken.

A breakthrough came about when Korteweg and de Vries [5] (see the facsimile reproduction following this paper), starting from a hydrodynamic description, derived a nonlinear partial differential equation which had solutions in which the nonlinearity is counterbalanced by a dispersive term which stabilizes the shape of the solitary wave, thereby vindicating Scott Russell's observations. The partial differential equation arrived at by Korteweg and de Vries's reasoning had the same form as one previously studied by de Boussinesq [6]. The partial differential equation written in terms of dimensionless variables with the form

$$\frac{\partial u}{\partial t} + \left(1 + u\right)\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{5}$$

has been named in honor of Korteweg and de Vries (KdV). The single-soliton solution can be found by assuming that the solution travels with (dimensionless) velocity v and does not change shape. This amounts to the assumption that the solution depends on space and time only through the combination $x - vt$, which makes the equation a perfect differential. The equation can be integrated a second time by introducing an integrating factor $\partial u/\partial x$. The single-soliton solution found by Korteweg and de Vries is given by

$$u(x,t) = a\,\mathrm{sech}^2\frac{(x - vt)}{\xi} \tag{6}$$

in which the amplitude, a, of the wave and its the spatial extent, ξ, depend on the velocity and are given by

$$a = 3(v - 1) \tag{7}$$

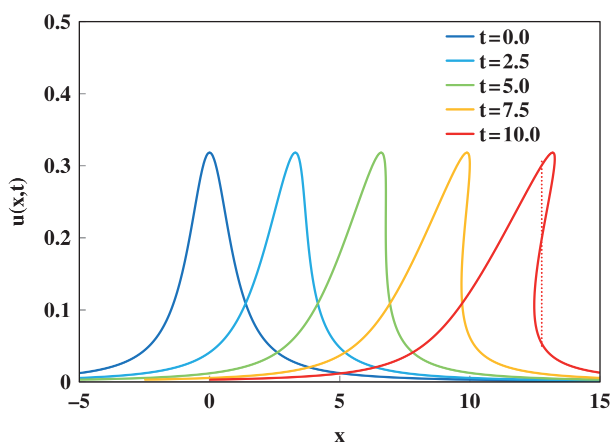

Figure 2. The time evolution of a nonlinear wave described by Equation (2). The solution becomes multi-valued at large times.

and

$$\xi = \frac{2}{\sqrt{v-1}} \tag{8}$$

in agreement with Lord Rayleigh's analysis [1] to within terms of order a/h. Korteweg and de Vries's mathematical description of non-dispersive solitary waves in continuous media [5] was a major milestone in the development of nonlinear excitations. It showed that there exist essentially nonlinear excitations which cannot be obtained by simply perturbing the excitations of the linearized system.

2. The Fermi–Pasta–Ulam problem and the KdV equation

Another major milestone was reached in 1955 when Fermi, Pasta and Ulam [7], with the help of M. Tsingu who coded and ran the computations, numerically investigated the vibrations of a linear chain of up to 32 atoms coupled by weakly nonlinear interactions. For a chain with purely harmonic interactions, the distribution of energy amongst the harmonic degrees of freedom would remain independent of time. It was expected that, if the system was started in a single approximate normal mode, the anharmonicity would cause the system to gradually relax into a state of thermal equilibrium after the atoms had undergone a sufficiently large number of oscillations. In particular, it was expected that the energy of the system would flow out from the initial mode and eventually be distributed amongst the other approximate normal modes of oscillation according to a Boltzmann-like distribution function. However, what they found was surprising. Although the system initially shared its energy with other modes, as expected from an analysis originally performed by Rayleigh, this process did not persist at longer times. The motion that Fermi, Pasta and Ulam observed was quasi-periodic in that it regularly almost recovered its initial state. The system did not relax into a state of thermodynamic equilibrium and did not show the property of mixing that would lead to the approximate equipartition of the energy that they had expected. In 1965 Zabusky and Kruskal showed that, in the continuum limit, the Fermi–Pasta–Ulam problem mapped onto the KdV equation [8]. Furthermore, they found that although the equation was nonlinear, the solitary waves described by the KdV equation appeared as if they did not interact with each other. That is, after two solitons collide, they emerge with their shapes and velocities unchanged as seen in Figure 3. The only signature of the collision was a shift in their phases. The shifts in the space-time trajectories are sketched in Figure 4. Owing to these particle-like attributes of the wave pulse excitations of the KdV equation, Zabusky and Kruskal [8] first penned the term "soliton" to describe them.

Zabusky and Kruskal found the properties of the two-soliton solutions by numerical methods. Their amazing discovery sparked a period of intensive investigation aimed at finding multi-soliton solutions of the KdV equation analytically. This period was marked by the development of powerful and elegant mathematical methods, such as the inverse scattering method. The properties of the two-soliton solutions found by Zabusky and Kruskal were proved analytically by Lax [9]. In 1967 Gardner et al. [10] found that it was possible to construct solutions

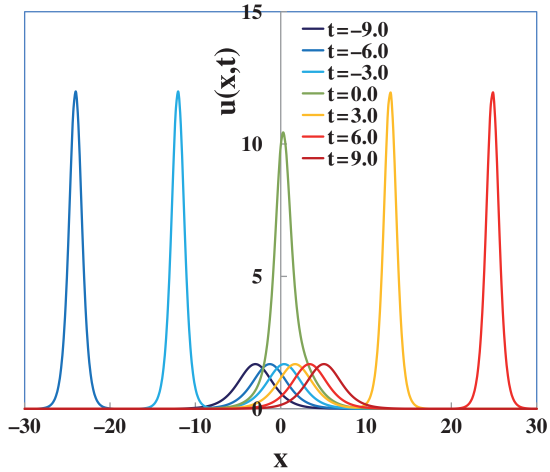

Figure 3. The wave profile of two colliding KdV solitons, at various times before and after the collision. The fast-moving large-amplitude soliton collides with the slow-moving small-amplitude soliton. Note the decrease in amplitude that occurs at the time of interaction.

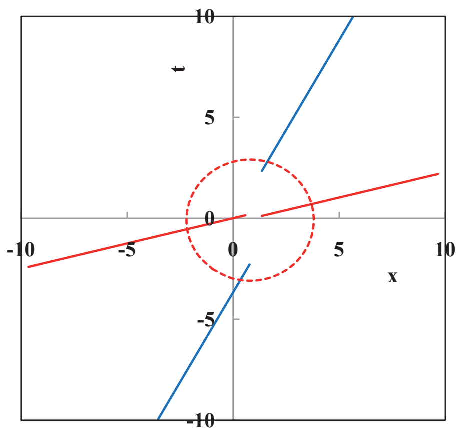

Figure 4. A sketch of the asymptotic world-lines of colliding solitons, indicating the type of shifts caused by the interaction. The interaction region where the collision takes place is contained within the dashed circle.

to this equation which describe finite numbers of solitons and continuous small-amplitude "radiation" that emerge from arbitrary initial conditions. Thus, the KdV equation was the first nonlinear field theory that was found to be exactly integrable. The unusual properties of collisions of two solitions were found to extend to the multi-soliton case. In 1971 Hirota [11], after reducing the KdV evolution equation to a homogeneous equation of degree 2, discovered the N-soliton solution. Just as exact integrability for systems with N degrees of freedom implies the existence of N

conservation laws, one might expect that an exactly integrable continuous system implies the existence of an infinite number of conservation laws. Miura et al. discovered that, in addition to satisfying conservation laws such as conservation of mass, energy and momentum, the KdV equation satisfied eight more conservation laws [12]. Furthermore, they also established a method for constructing an infinite number of conservation laws [12]. The exact integrability of the KdV equation was subsequently proved to be connected with it, satisfying the infinite number of conservation laws [13,14]. The infinite number of conservation laws puts extreme constraints on the form of solitons, their interactions and the regions of the high-dimensional phase space accessible to the system. The lack of approach to equilibrium for finite Fermi–Pasta–Ulam systems remains a subject of research [15,16].

3. Solitons and breathers in other integrable systems

The work of Lax [9] made it clear that there may be other exactly integrable one-dimensional continuous systems. It is now known that the properties of the KdV equation are generic, in that they are characteristic of a whole class of exactly integrable systems. The simplest members of this class include the KdV equation, the modified KdV equation, the sine–Gordon equation and the nonlinear Schrödinger equation. The sine–Gordon equation has a relativistically covariant form and obeys the equation of motion

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + m^2 \sin u = 0 \tag{9}$$

which can be derived from the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2 + 2m^2 \cos u\right] \tag{10}$$

and in which m defines a characteristic mass scale. The energy $\mathcal{H}$ and momentum $\mathcal{P}$ densities can be found by standard methods and are given by

$$\mathcal{H} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + 2m^2(1 - \cos u)\right] \tag{11}$$

and

$$\mathcal{P} = \left(\frac{\partial u}{\partial t}\right)\left(\frac{\partial u}{\partial x}\right). \tag{12}$$

The approximate small-amplitude plane-wave-like vibrations of the sine–Gordon equation have the dispersion relation

$$\omega^2 - k^2 = m^2 \tag{13}$$

similar to the excitations of the Klein–Gordon equation. The sine–Gordon equation also has large-amplitude soliton and anti-soliton solutions of the form

$$u(x,t) = 4\tan^{-1}\left(\exp\left[\pm\frac{\gamma(x-vt)}{\xi}\right]\right) \tag{14}$$

where the length scale is given by $\xi = m^{-1}$ and the factor γ

$$\gamma = (1-v^2)^{-\frac{1}{2}} \tag{15}$$

produces a Lorentz contraction. These soliton excitations are particle-like (with a particle mass of $8m$) since their energy and momentum densities have a pulse form that is the same as the sech^2 shape of the KdV soliton. Simple analytic two-soliton and soliton–anti-soliton solutions were discovered by Perring and Skyrme [17] in 1962 after numerical integration of the equation of motion. In addition to the soliton and anti-soliton excitations, the sine–Gordon equation also has large-amplitude breather solutions given by

$$u(x,t) = 4\tan^{-1}\left(\frac{\sin \omega\gamma(t-vx)}{\omega\xi\cosh\frac{\gamma(x-vt)}{\xi}}\right) \tag{16}$$

where

$$\omega^2 + \xi^{-2} = m^2. \tag{17}$$

The wave profile of the breather is shown in Figure 5. The breather is an oscillating excitation with a finite spatial width, and travels with velocity v. The breather modes are quite distinct from the small-amplitude vibrational modes of weakly interacting or non-interacting translationally invariant systems, where the vibrations extend over distances which are quite large and can be comparable to the size of the entire system. In linearized systems, localized modes can only occur around impurities that

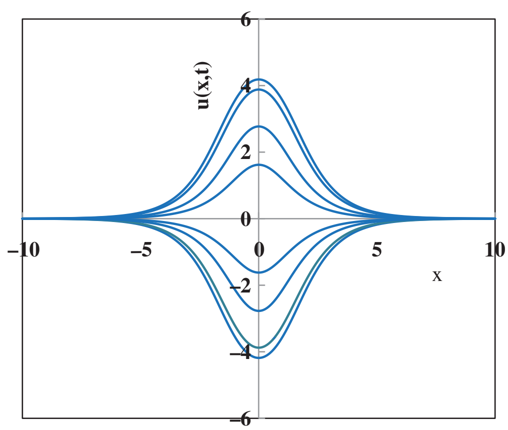

Figure 5. The time evolution of a nonlinear wave described by Equation (16) in its inertial reference frame. The breather is an oscillatory excitation of finite spatial extent.

break the translational invariance of the system. However, the existence of breather excitations in the sine–Gordon and the KdV field theories demonstrates that nonlinearity can produce oscillatory modes with finite spatial extents. The breather solutions share the same properties as solitons in that they emerge from breather–breather collisions with their forms intact. The inverse scattering method indicates that the breather can be considered as a bound state of a soliton–anti-soliton pair. In this bound state, the separation between the soliton and anti-soliton is oscillating. However, the one-dimensional quantum sine–Gordon system is also one of the known exactly integrable quantum systems [18,19] and, in the quantum description, the hierarchy of quantized breather excitations can be described as the bound states of multiple small-amplitude oscillations [20–22]. The two lowest members of this hierarchy are the (quantized) excitations of the linearized system and the bound states of two such linear excitations. The lowest members of the hierarchy of quantum breathers have been shown to exist in discrete lattice systems.

Soliton excitations are not restricted to occur only in exactly integrable one-dimensional continuous field theories, but have also been shown to occur in certain discrete one-dimensional lattices with exponential interactions [23]. Although most one-dimensional nonlinear field theories are not exactly integrable, there is great physical interest in these equations because many nonlinear problems can be approximated by soliton-bearing nonlinear equations, and also because of the very important topological implications of solitons (which we only mention in passing). In phases of condensed matter where symmetry is spontaneously broken, solitons may act as defects which partially break the topological symmetry of the low-temperature ordered state [24]. An excellent introduction to the topological theory of defects in ordered media is given by Mermin [25]. In low dimensions, a true transition to a phase with spontaneously broken symmetry may not exist [26]; however, there could exist a cross-over to a state with quasi-order. In the low-temperature state, the quasi-order may be partially broken due to the presence of solitons which have important ramifications for the physical properties. The solitons of these non-integrable theories may not have the precise mathematical properties of the exactly integrable systems but, nevertheless, have very similar properties.

4. Recent observations of solitons

Since the discovery of the many unusual and intriguing properties of solitons, soliton excitations have been found in numerous condensed matter, molecular, biological, and optical systems, of which we shall mention only a few. Already by 1953 Seeger et al. [27] had shown that the domain walls in ferromagnets are solitonic structures. The central peaks observed in scattering experiments on quasi-one-dimensional materials that exhibit structural transitions have been attributed [28] to solitons in the form of domain walls. The nonlinear current–voltage produced by pinned charge density wave materials has been attributed to solitons [29]. Soliton excitations have been found in quasi-one-dimensional isotropic Heisenberg magnets [30], in one-dimensional easy-plane ferromagnets [31] and antiferromagnets [32] and in Josephson junctions [33]. Solitons have also been found in polyacetylene [34,35] and biological molecules [36] including DNA [37,38]. Solitons also have surprising manifestations in optics. McCall and Hahn [39] showed that if the frequency of an intense coherent pulse of radiation incident on an absorbing medium is close to the medium's resonance frequency, the linear theory of absorption completely breaks down. In particular, the medium absorbs energy from the leading edge of the pulse, but re-radiates leading to the formation and propagation of a soliton. This results in an anomalously low energy loss of the optical pulse, a phenomenon known as self-induced transparency. An important technical application of solitons is found in optical communications, where the balance between the nonlinearity in the dielectric constant and dispersion can be used to transmit signals without degradation by using optical solitons [40,41]. Solitons have also been found in highly anisotropic Bose–Einstein condensates of dilute atomic gasses [42]. Sievers and Takeno have speculated [43] that the breather excitations may also be quite widespread throughout nature, perhaps even occurring in the vibrational spectrum of three-dimensional ionic crystals and α-uranium [44,45]. The observations of Scott Russell and their full theoretical description by Korteweg and de Vries signaled the birth of the field of nonlinear physics. Nonlinear physics is a subject that is thriving and will continue to thrive, but perhaps not as a distinct field. This subject may become integrated in the other branches of physics since interactions form essential parts of real physical systems and, with increasing frequency, we comprehend that interactions cannot be treated as small perturbations.

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