Total Derivatives
and the Geometry of Line Bundles
or
the porcupine’s coat

Consider a surface in $\mathbb{R}^3$, and a real-valued function on that surface. It’s not hard to say what a directional derivative of such a function is. You take a curve in the chosen direction and compute the infinitesimal variation of the function along the curve.

You can think of the total derivative of a function $f$ as the collection of all its directional derivatives. That means, it’s a map

$$\frac{df}{dx} : \text{vectors tangent to the surface} \rightarrow \mathbb{R}.$$ 

You give a vector to $\frac{df}{dx}$, and it gives you back the derivative of $f$ in the direction of that vector. The $\frac{df}{dx}$ by itself can be thought of as the total derivative map: it takes a function to its total derivative.

Now we will generalize a bit. So far, we have dealt with a function whose value at each point lies in the same one-dimensional vector space, namely $\mathbb{R}$. This is the ordinary notion of a function. But it is desirable to deal with a different sort of “function” as well: one where the value at each point lies in a different one-dimensional vector space, or line. Picture a line attached to each point of the surface, the result being something like a porcupine. The collection of all these lines (the porcupine’s coat) is called a line bundle. The “function” whose values are spread out all among the different lines (or quills) is called a cross-section of the line bundle. The value at a point lies in the line-quill emanating from that point.

I should warn you before we go further that a line bundle can be more complicated than the average porcupine’s coat. It’s probably better to think of a very agile and contortionistic porcupine with his quills arranged in some unusual way.

We want to study the geometry of a given line bundle by looking at its cross-sections and doing analysis, i.e. taking derivatives. Can we compute the directional derivative of a cross-section like we did for a function? If you try it, you’ll see that you are obliged to attempt “value there minus value here, over $h$.” But “value there” lies in one line, while “value here” lies in another. Although the two lines are isomorphic as vector spaces, there is no natural or canonical isomorphism by which to identify them, so your difference quotient will depend on an arbitrary choice.

So, in order to make sense of the difference quotients for cross-sections, we have to make a choice at the outset, precisely how to identify nearby lines. This is the same as choosing a systematic way of computing the total derivative of any cross-section. Such a systematic choice is called a connection for the line bundle. A connection is a total derivative map for cross-sections. Connections always exist and are never unique.

These lectures will be an introduction to the study of line bundles using connections. By regarding a connection as nothing but a total derivative map for cross-sections, and keeping the porcupine clearly in mind, we’ll try to avoid all the technical definitions of differential geometry. We’ll see how far geometrical
imagination, along with a little analysis, can take us. I intend to discuss questions like the following ones, as time permits:

1. Is there a reason for studying line bundles and connections at all, other than for their own sake? Are they useful?

2. How can one categorize the different classes of bundles which a given surface carries?

3. For a given bundle over a given surface: is there a “best” connection? This would mean there is after all a natural way of defining derivatives of cross-sections.

4. A connection seems to be a somewhat local sort of an object at first glance. It tells how to identify nearby lines of a bundle. So how can it give information about the global structure of the bundle?