Group actions and rational ideals

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Rational ideals: definition and connection with irreducible representations
Overview

- **Rational ideals**: definition and connection with irreducible representations

- **Actions of algebraic groups**: brief reminder of some basics
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- The **main result** on rational ideals
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- **Rational ideals**: definition and connection with irreducible representations
- Actions of **algebraic groups**: brief reminder of some basics
- The **main result** on rational ideals
- Some **history** . . . if time
References

- “Group actions and rational ideals”, Algebra and Number Theory 2 (2008), 467-499


Both articles & the pdf file of this talk available on my web page:

http://math.temple.edu/~lorenz/
Part I: Rational Ideals
Under favorable circumstances, **rational** ideals are the same as **primitive** ideals, that is, kernels of irreducible representations.

“In detail . . .”
Definition of rational ideals

Notation: 

$k$ some algebraically closed base field

$R$ an associative $k$-algebra (with 1)
Definition of rational ideals

**Notation:**
- \( \mathbb{K} \): some algebraically closed base field
- \( R \): an associative \( \mathbb{K} \)-algebra (with 1)

**Definition:**
- The **extended centroid** of \( R \) is defined by
  \[
  C(R) = Z Q_r(R)
  \]

Here \( Q_r \) is the right **Amitsur-Martindale quotient ring** and \( Z \) denotes the center.
Definition of rational ideals

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Definition:  
- The extended centroid of \( R \) is defined by
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  \]
  Here \( Q_r \) is the right Amitsur-Martindale quotient ring and \( \mathbb{Z} \) denotes the center.

- If \( P \in \text{Spec} R \) then \( C(R/P) \) is a \( \mathbb{k} \)-field. We call \( P \) rational if \( C(R/P) = \mathbb{k} \).
I will not discuss the definition of $Q_r(R)$. Here are the original sources:

for prime rings $R$:


for general $R$:

Examples

- If $R$ is simple, or a finite product of simple rings, then

$$Q_r(R) = R$$
Examples

- If $R$ is **simple**, or a finite product of simple rings, then

  $$Q_r(R) = R$$

- For $R$ **semiprime right Goldie**,

  $$Q_r(R) = \{ q \in Q_{cl}(R) \mid qI \subseteq R \text{ for some } I \triangleleft R \text{ with } \text{ann}_R I = 0 \}$$

  In particular,

  $$C(R) = ZQ_{cl}(R)$$
Examples

- $R = U(\mathfrak{g})/I$ a semiprime image of the enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$. Then 

$$Q_r(R) = \{ \text{ad } \mathfrak{g}-\text{finite elements of } Q_{cl}(R) \}$$
Given an irreducible representation $\rho: R \to \text{End}_k(V)$, let $P = \text{Ker} \, \rho$ be the corresponding primitive ideal of $R$. 
Given an irreducible representation $\rho: \mathbb{R} \to \text{End}_k(V)$, let $P = \text{Ker} \rho$ be the corresponding primitive ideal of $\mathbb{R}$.

- There always is an embedding of $k$-fields

$$C(R/P) \hookrightarrow \mathbb{Z} (\text{End}_R(V))$$
Given an irreducible representation $\rho: R \to \text{End}_k(V)$, let $P = \text{Ker} \rho$ be the corresponding primitive ideal of $R$.

- There **always** is an embedding of $k$-fields

$$
C(R/P) \rightarrow \mathcal{Z}(\text{End}_R(V))
$$

- **Typically**, $\text{End}_R(V) = k$ ("weak Nullstellensatz"); in this case

```
primitive \Rightarrow \text{rational}
```
Examples

The weak Nullstellensatz holds for

- $R$ any affine $\mathbb{k}$-algebra, $\mathbb{k}$ uncountable
  - Amitsur
- $R$ an affine PI-algebra
  - Kaplansky
- $R = U(\mathfrak{g})$
  - "Quillen’s Lemma"
- $R = \mathbb{k}\Gamma$ with $\Gamma$ polycyclic-by-finite
  - Hall, L.
- many quantum groups: $O_q(\mathbb{k}^n)$, $O_q(M_n(\mathbb{k}))$, $O_q(G)$, ...
Examples

The weak Nullstellensatz holds for

- \( R \) any affine \( \mathbb{k} \)-algebra, \( \mathbb{k} \) uncountable \hspace{1cm} \text{Amitsur}
- \( R \) an affine PI-algebra \hspace{1cm} \text{Kaplansky}
- \( R = U(\mathfrak{g}) \)
- \( R = \mathbb{k}\Gamma \) with \( \Gamma \) polycyclic-by-finite \hspace{1cm} \text{Hall, L.}
- many quantum groups: \( \mathcal{O}_q(\mathbb{k}^n), \mathcal{O}_q(M_n(\mathbb{k})), \mathcal{O}_q(G), \ldots \)

In fact, in all these examples except the first, it has been shown that the \textbf{Dixmier-Mœglin equivalence} holds (under mild restrictions on \( \mathbb{k} \) or \( q \)):

\[
\text{primitive } \iff \text{ rational}
\]
Part II: Algebraic Groups
Definition

There is an anti-equivalence of categories

\[
\begin{array}{c}
\left\{ \text{affine algebraic groups \ensuremath{\mathcal{O}/k}} \right\} \\
\cong \\
\left\{ \text{comm. affine reduced Hopf \ensuremath{k}} \text{-algebras} \right\}
\end{array}
\]

\[G = \text{Hom}_{\text{\ensuremath{k}} \text{-alg}}(\mathcal{O}[G], \mathcal{O}) \iff \mathcal{O}[G]\]
There an anti-equivalence of categories

\[
\left\{ \text{affine algebraic groups } \mathbb{A}/k \right\} \quad \sim \quad \left\{ \text{comm. affine reduced Hopf } k\text{-algebras} \right\}
\]

\[
G = \text{Hom}_{k\text{-alg}}(k[G], k) \quad \leftrightarrow \quad k[G]
\]

Equivalently, affine algebraic groups are precisely the closed subgroups of \( GL_n \) for some \( n \):

\[
GL_n, SL_n, T_n, U_n, D_n, O_n, \ldots
\]

all finite groups
For any affine algebraic $\mathbb{k}$-group $G$,

\[ G\text{-modules} \equiv \mathbb{k}[G]\text{-comodules} \]

Comodule structure map

\[ \Delta_M : M \to M \otimes \mathbb{k}[G] \]

\[ m \mapsto \sum m_0 \otimes m_1 \]
For any affine algebraic $\mathbb{k}$-group $G$, we have:

$$G\text{-modules} \equiv \mathbb{k}[G]\text{-comodules}$$

Comodule structure map

$$\Delta_M : M \rightarrow M \otimes \mathbb{k}[G]$$

$$m \mapsto \sum m_0 \otimes m_1$$

We obtain a linear representation $G \rightarrow \text{GL}(M)$ by

$$g.m = \sum m_0 m_1(g)$$

Representations arising in the way are called rational.
• Rational reps are **locally finite**. In fact, they can also be characterized as the linear reps $G \to \text{GL}(M)$ that are

(a) locally finite and

(b) for each finite-dimensional $G$-stable subspace $V \subseteq M$, the resulting map $G \to \text{GL}(V)$ is a morphism of alg. groups.
Rational representations

- Rational rep\(^s\) are **locally finite**. In fact, they can also be characterized as the linear rep\(^s\) \(G \to \text{GL}(M)\) that are
  
  (a) locally finite and

  (b) for each finite-dimensional \(G\)-stable subspace \(V \subseteq M\), the resulting map \(G \to \text{GL}(V)\) is a morphism of alg. groups.

- Tensor products of rational rep\(^s\) of \(G\) are again rational. Similarly for sums, subrep\(^s\) and homomorphic images.
Part III: Main Result on Rational Ideals
Let \( G \) be an arbitrary group acting on \( R \) by \( \kappa \)-algebra auto\(^s\).
Let $G$ be an arbitrary group acting on $R$ by $k$-algebra automorphisms.

Have induced $G$-actions on

- $\{\text{ideals of } R\}$
- $\text{Spec } R = \{\text{prime ideals of } R\}$
- $\text{Rat } R = \{\text{rational ideals of } R\}$
- $\text{Prim } R = \{\text{primitive ideals of } R\}$
- $Q_r(R)$ and $C(R) = \mathbb{Z} Q_r(R)$

**Notation:** $G \backslash \text{Spec } R$ will denote the set of $G$-orbits in $\text{Spec } R$, and similarly for other $G$-sets.
Definition: A proper $G$-stable ideal $I \triangleleft R$ is called

- **$G$-prime** if $AB \subseteq I$ for $G$-stable ideals $A, B \triangleleft R$ implies that $A \subseteq I$ or $B \subseteq I$.

- **$G$-rational** if $I$ is $G$-prime and $C(R/I)^G = \mathbb{k}$. 


**Definition:** A proper $G$-stable ideal $I \triangleleft R$ is called

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- **$G$-rational** if $I$ is $G$-prime and $C(R/I)^G = \mathbb{k}$.

Put $G\text{-Spec } R = \{G$-prime ideals of $R\}$ and similarly for $G\text{-Rat } R$.

**Prop**$^n$: The assignment $P \mapsto \bigcap_{g \in G} g.P$ yields maps $G\backslash \text{Spec } R \to G\text{-Spec } R$ and $G\backslash \text{Rat } R \to G\text{-Rat } R$. 

G-prime and G-rational ideals
Now assume that $G$ is an affine alg. $\mathbb{k}$-group acting rationally on the $\mathbb{k}$-algebra $R$; so we have a rational rep $\rho$

$$G \to \text{Aut}_{\mathbb{k} \text{-alg}}(R) \subseteq \text{GL}(R)$$

Equivalently, $R$ is a $\mathbb{k}[G]$-comodule algebra.
Now assume that $G$ is an affine alg. $\mathbb{k}$-group acting rationally on the $\mathbb{k}$-algebra $R$; so we have a rational rep

$$G \to \text{Aut}_{\mathbb{k}\text{-alg}}(R) \subseteq GL(R)$$

Equivalently, $R$ is a $\mathbb{k}[G]$-comodule algebra.

**Theorem:**

$$G \backslash \text{Rat } R \; \overset{\text{bij.}}{\longrightarrow} \; G\text{-Rat } R$$

$$\bigcup_{g \in G} g.P$$
Some comments

- $G \setminus \text{Spec } R \rightarrow G\text{-Spec } R$ is surjective (easy, even under more general circumstances) but it is rarely injective (exactly if all primes of $R$ are stable under the connected component of $G$).

- The Thm easily reduces to the case where $G$ is connected. In this case, there is the following result on algebraic groups, due to Vonessen (1998) and Abe & Kanno (1959).

\[\text{Prop}^n: \text{Let } G \text{ act on } \mathbb{k}(G) \text{ via } \rho_r \text{ and let } F \text{ be a } G\text{-stable } \mathbb{k}\text{-subfield of } \mathbb{k}(G). \text{ Let } \text{Hom}_G(F, \mathbb{k}(G)) \text{ denote the collection of all } G\text{-equivariant } \mathbb{k}\text{-algebra homomorphisms } \phi: F \rightarrow \mathbb{k}(G). \text{ Then the } G\text{-action on } \text{Hom}_G(F, \mathbb{k}(G)) \text{ that is given by } g.\phi = \rho_\ell(g) \circ \phi \text{ is transitive.}\]

Here, $\rho_r$ and $\rho_\ell$ denote the right and left regular actions of $G$ on its function field $\mathbb{k}(G)$.

- The fibre over any $I \in G\text{-Rat } R$ is in $G\text{-equivariant bijection with } \text{Hom}_G(C(R/I), \mathbb{k}(G))$, and this set is nonempty. The Thm follows.
Some comments

- $G \setminus \text{Spec } R \rightarrow G\text{-Spec } R$ is surjective \((\text{easy, even under more general circumstances})\) but it is rarely injective \((\text{exactly if all primes of } R \text{ are stable under the connected component of } G)\).

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**Prop**\textsuperscript{n}: Let $G$ act on $k(G)$ via $\rho_R$ and let $F$ be a $G$-stable $k$-subfield of $k(G)$. Let $\text{Hom}_G(F, k(G))$ denote the collection of all $G$-equivariant $k$-algebra homomorphisms $\phi : F \rightarrow k(G)$. Then the $G$-action on $\text{Hom}_G(F, k(G))$ that is given by $g.\phi = \rho_\ell(g) \circ \phi$ is transitive.

Here, $\rho_R$ and $\rho_\ell$ denote the right and left regular actions of $G$ on its function field $k(G)$.

- The fibre over any $I \in G\text{-Rat } R$ is in $G$-equivariant bijection with $\text{Hom}_G(\mathcal{C}(R/I), k(G))$, and this set is nonempty. The Th\textsuperscript{m} follows.
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Here, $\rho_\ell$ and $\rho_\ell$ denote the right and left regular actions of $G$ on its function field $k(G)$.

- The **fibre** over any $I \in G\text{-Rat } R$ is in $G$-equivariant bijection with $\text{Hom}_G(C(R/I), k(G))$, and this set is nonempty. The Th$m$ follows.
Part IV: Some History
Jacques Dixmier (* 1924)

- former member of Bourbaki
- Ph.D. advisor of A. Connes
- author of several highly influential monographs:
  
  * Les algèbres d’opérateurs dans l’espace hilbertien: algèbres de von Neumann, Gauthier-Villars, 1957
  * Les $C^*$-algèbres et leurs représentations, Gauthier-Villars, 1969
  * Algèbres enveloppantes, Gauthier-Villars, 1974

from P. Halmos, “I Have A Photographic Memory”
Some milestones

Dixmier’s Problem # 11 (from: Algèbres enveloppantes, 1974)

10. On suppose que $tr \ ad x = 0$ pour tout $x \in g$. Est ce que $Z (g) \neq k$ ?

11. Soient $\mathfrak{f}$ un idéal de $g$, $I$ un idéal primitif de $U (g)$. Les propriétés suivantes sont-elles vraies : (a) il existe un idéal primitif de $U (\mathfrak{f})$ générique pour $U (\mathfrak{f}) \cap I$; (b) deux tels idéaux sont conjugués par le groupe adjoint algébrique de $g$; (c) soit $L$ un tel idéal; il existe une représentation simple $\sigma$ de $\mathfrak{f}$ de noyau $L$, et une représentation simple $\rho$ de $st (\sigma, g)$, telles que $\rho | \mathfrak{f}$ soit un multiple de $\sigma$ et que $\text{ind} (\rho, g)$ soit simple de noyau $I$. Cf. 4.5.9, 5.4.3, 5.4.4, 5.6.5.
Some milestones

• Problem 11 for \( \mathfrak{g} \) solvable, \( \text{char } k = 0 \)


\( \Rightarrow \) existence


\( \Rightarrow \) uniqueness

• Theorem 1 under Goldie hypotheses, \( \text{char } k = 0 \):

Mœglin & Rentschler


Some milestones

- Problem 11 for $\mathfrak{t}$ solvable, $\text{char} \mathbb{k} = 0$


- Theorem 1 under Goldie hypotheses, $\text{char} \mathbb{k} = 0$

  Mœglin & Rentschler


Some milestones

- Theorem 1 under Goldie hypotheses, \( \text{char } \mathbb{k} \) arbitrary:
  
  N. Vonessen

  *Actions of algebraic groups on the spectrum of rational ideals*,

  *Actions of algebraic groups on the spectrum of rational ideals. II*,