Torus actions on noncommutative algebras

Quantum Groups and Noncommutative Algebraic Geometry
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Overview

- **Background**: algebraic group actions on noncommutative prime spectra

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- **Recent work**: a new approach to the Stratification Thm for torus actions

Notations and hypotheses

Throughout this talk,

$\Bbbk$	denotes an algebraically closed base field
R	is an associative $\Bbbk$-algebra (with 1)
G	is an affine algebraic $\Bbbk$-group acting rationally on R; equivalently, R is a $\Bbbk[G]$-comodule algebra

Example: torus actions

If $G \cong (\mathbb{k}^\times)^d$ is an algebraic torus, then

- $\mathbb{k}[G] = \mathbb{k}\Lambda$, the group algebra of $\Lambda = X(G) \cong \mathbb{Z}^d$
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Thus: a rational G-action on R is equivalent to a $\mathbb{Z}^d$-grading

$$R = \bigoplus_{\lambda \in \mathbb{Z}^d} R_\lambda \,, \qquad R_\lambda R_{\lambda'} \subseteq R_{\lambda+\lambda'}$$

Group actions on noncommutative prime spectra

G-prime ideals

G-action on $R \rightsquigarrow$ G-actions on $\{$ideals of $R\}$, $\operatorname{Spec} R$, $\ldots$

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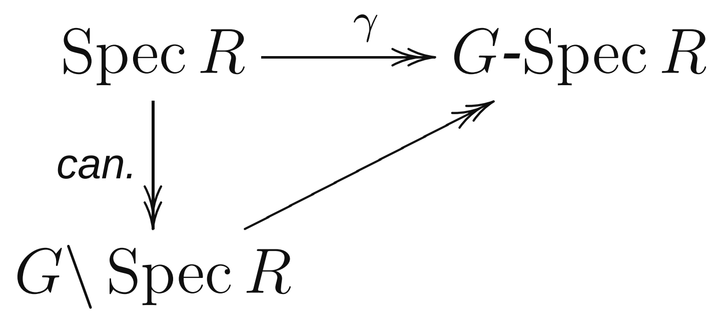
Definition The algebra R is called G-prime if $R \neq 0$ and the product of any two nonzero G-stable (**!**) ideals of R is nonzero.

A G-stable ideal I of R is called G-prime if R/I is G-prime

$$G\text{-}\operatorname{Spec} R = \{G\text{-prime ideals of } R\}$$

Prop^n (a) *The assignment* $\gamma : P \mapsto P:G \underset{def}{=} \bigcap_{g\in G} g.P$ *yields surjections*

“G-core of P”



(b) *If G is connected then all G-primes are in fact prime; so*

$$G\text{-Spec}\, R = \{G\text{-stable prime ideals of } R\}$$

The Goodearl-Letzter stratification

The "strata" of $\operatorname{Spec} R$ are the fibres of $\gamma\colon\ \operatorname{Spec} R \twoheadrightarrow G\text{-}\operatorname{Spec} R$:

$$\operatorname{Spec}_I R \underset{\text{def}}{=} \gamma^{-1}(I) = \{P \in \operatorname{Spec} R \mid P{:}G = I\}$$

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Finiteness question

When are there only finitely many strata ?

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Heuristic fact: For many algebras R, there is a G-action such that G-$\mathrm{Spec}\,R$ is finite — and interesting!

. . . but a general finiteness criterion still needs to be found

Finiteness question

The special case where R is affine commutative with a torus action is "classical". Here is a slightly more general finiteness criterion:

Theorem: *Let R be prime affine PI-algebra / $\Bbbk$ and let G be an algebraic $\Bbbk$-torus acting rationally on R. Then:*

$G\text{-}\operatorname{Spec} R$ is finite if and only if the G-action on the center $\mathcal{Z}(R)$ is multiplicity free: $\dim_{\Bbbk} \mathcal{Z}(R)_\lambda \leq 1$ *for all* $\lambda \in \Lambda = X(G)$

Description of strata

Given $I \in G$**-**$\mathrm{Spec}\, R$**, can one describe the stratum** $\mathrm{Spec}_I\, R$?

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The answer requires some preparations . . .

Description of strata

For simplicity, I assume G to be **connected**; so $\Bbbk[G]$ is a domain. In particular,

$$G\text{-}\operatorname{Spec} R = \{G\text{-stable primes of } R\}$$

Description of strata

For a given $I \in G\text{-}\mathrm{Spec}\, R$, put

“heart of I”
extended centroid of R/I

$\mathrm{Fract}\, \Bbbk[G]$

$$T_I = \mathcal{C}(R/I) \otimes \Bbbk(G)$$

This is a commutative domain, a tensor product of two fields.

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G-actions:

- on $\mathcal{C}(R/I)$ via the given action on R,
  $\rho\colon G \to \mathrm{Aut}_{\Bbbk\text{-alg}}(R)$
- on $\Bbbk(G)$ by the right and left regular actions
  $\rho_r\colon (x.f)(y) = f(yx)$ and $\rho_\ell\colon (x.f)(y) = f(x^{-1}y)$
- on T_I by $\rho \otimes \rho_r$ and $\mathrm{Id} \otimes \rho_\ell$ $\longleftarrow$ commute!

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Put

$$\operatorname{Spec}^G T_I = \{(\rho \otimes \rho_r)(G)\text{-stable primes of } T_I\}$$

Description of strata

General Stratification Theorem: *Given* $I \in G\text{-}\operatorname{Spec} R$*, there is a bijection*

$$c\colon \operatorname{Spec}_I R \longrightarrow \operatorname{Spec}^G T_I$$

having the following properties:

(a) G-equivariance: $c(g.P) = (\operatorname{Id} \otimes \rho_\ell)(g)(c(P))$;

(b) inclusions: $P \subseteq P' \iff c(P) \subseteq c(P')$;

(c) hearts: $\mathcal{C}\,(T_I/c(P)) \cong \mathcal{C}\,(R/P \otimes \Bbbk(G))$ *as* $\Bbbk(G)$*-fields;*

(d) "rationality": $\mathcal{C}(R/P) = \Bbbk \iff T_I/c(P) = \Bbbk(G)$.

Description of strata

. . . but what is

$$\operatorname{Spec}^G T_I$$

commutative!

?

Stratification Theorem for torus actions

Statement of the theorem

Let G be an arbitrary affine algebraic $\Bbbk$-group for now. For a given $I \in G\text{-}\operatorname{Spec} R$, put

$$Z_I = \{c \in \mathcal{C}(R/I) \mid \dim_{\Bbbk} \langle G.c \rangle_{\Bbbk} < \infty\}$$

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Properties:

- Z_I is a G-stable $\Bbbk$-subalgebra of $\mathcal{C}(R/I)$
- Z_I contains the field of G-invariants, $\mathcal{C}(R/I)^G$
- the G-action on Z_I is rational

Statement of the theorem

Torus Stratification Theorem: *Let G be an algebraic torus and let $I \in G\text{-}\operatorname{Spec} R$. Then:*

(a) *There is an isomorphism of algebras*

$$Z_I \cong \mathcal{C}(R/I)^G \Gamma_I \ ,$$

the group algebra of the lattice $\Gamma_I = X(G/\operatorname{Ker}_G(Z_I))$ over the field $\mathcal{C}(R/I)^G$.

(b) *There is a G-equivariant order isomorphism*

$$\gamma \colon \operatorname{Spec}_I R \xrightarrow{\sim} \operatorname{Spec}(Z_I) \ .$$

Concluding remarks

- The original result, due to Goodearl & Letzter and Goodearl & Stafford ($\sim$ 2000), is for noetherian R. Now R can be arbitrary.

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- The definition of the algebra Z_I is easily stated in terms of the given G-action on R, for any G.
- The group algebra (Laurent polynomial algebra) structure of Z_I arises naturally from the fact that G is a torus.

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- The definition of the algebra Z_I is easily stated in terms of the given G-action on R, for any G.
- The group algebra (Laurent polynomial algebra) structure of Z_I arises naturally from the fact that G is a torus.
- The proof uses the "General Stratification Theorem".

References

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Papers & **this talk** available at http://math.temple.edu/~lorenz/