Multiplicative Invariant Theory

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Introduction to multiplicative invariants: definitions, examples, ...
Overview

- **Introduction** to multiplicative invariants: definitions, examples, ... 

- **Regularity**: reflection groups and semigroup algebras
Overview

- **Introduction** to multiplicative invariants: definitions, examples, ...

- **Regularity**: reflection groups and semigroup algebras

- The **Cohen-Macaulay property**: reminders on CM rings and some results on multiplicative invariants
Part I: Introduction
Given: a group $G$ and a $G$-lattice $L \cong \mathbb{Z}^n$; so

$$G \rightarrow GL(L) \cong GL_n(\mathbb{Z})$$

an integral representation of $G$
Given: a group $G$ and a $G$-lattice $L \cong \mathbb{Z}^n$; so

$$G \rightarrow \text{GL}(L) \cong \text{GL}_n(\mathbb{Z})$$

Choose a base ring $k$ and form the group algebra

$$k[L] = \bigoplus_{m \in L} kx^m \cong k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], \quad x^m x^{m'} = x^{m+m'}$$

The $G$-action on $L$ extends uniquely to a "multiplicative" action by $k$-algebra automorphisms on $k[L]$:

$$g(x^m) = x^{g(m)} \quad (g \in G, m \in L)$$
Multiplicative Invariants

- Given: a group $G$ and a $G$-lattice $L \cong \mathbb{Z}^n$; so
  \[ G \to \text{GL}(L) \cong \text{GL}_n(\mathbb{Z}) \]

- Choose a base ring $k$ and form the group algebra
  \[ k[L] = \bigoplus_{m \in L} kx^m \cong k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], \quad x^m x^{m'} = x^{m+m'} \]

  The $G$-action on $L$ extends uniquely to a “multiplicative” action by $k$-algebra automorphisms on $k[L]$.

- The multiplicative invariant algebra is
  \[ k[L]^G = \{ f \in k[L] \mid g(f) = f \ \forall g \in G \} \]
Example #1

Multiplicative inversion in rank 2: 

\( k = \mathbb{Z} \)

\[ G = \langle g \mid g^2 = 1 \rangle \]

\[ L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \]

action: \( g(e_i) = -e_i \)
Example #1

**Multiplicative inversion in rank 2:**

\((k = \mathbb{Z})\)

\[ G = \langle g \mid g^2 = 1 \rangle \]

\[ L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \]

**action:** \(g(e_i) = -e_i\)

Putting \(x_i = x^{e_i}\) we have:

\[ \mathbb{Z}[L] = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}] \quad \text{with} \quad g(x_i) = x_i^{-1} \]
Example #1

Multiplicative inversion in rank 2: \( k = \mathbb{Z} \)

\[
G = \langle g \mid g^2 = 1 \rangle \\
L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \\
\text{action: } g(e_i) = -e_i
\]

Straightforward calculation gives

\[
\mathbb{Z}[L]^G = \mathbb{Z}[\xi_1, \xi_2] \oplus \eta \mathbb{Z}[\xi_1, \xi_2]
\]

with \( \xi_i = x_i + x_i^{-1} \) and \( \eta = x_1 x_2 + x_1^{-1} x_2^{-1} \); they satisfy

\[
\eta \xi_1 \xi_2 = \eta^2 + \xi_1^2 + \xi_2^2 - 4
\]
Example #1

Multiplicative inversion in rank 2: 
($k = \mathbb{Z}$)

$G = \langle g \mid g^2 = 1 \rangle$
$L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$

action: $g(e_i) = -e_i$

Hence:

$\mathbb{Z}[L]^G \cong \mathbb{Z}[x, y, z]/(x^2 + y^2 + z^2 - xyz - 4)$
Example #1': linear version

Linear inversion in rank 2:

\[ G = \langle g \mid g^2 = 1 \rangle \]
\[ L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \]

action: \( g(e_i) = -e_i \)
Example #1’: linear version

Linear inversion in rank 2:

\[ G = \langle g \mid g^2 = 1 \rangle \]
\[ L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \]
action: \( g(e_i) = -e_i \)

Now:
\[ S(L) = \mathbb{Z}[x_1, x_2] \text{ with } g(x_i) = -x_i \]
Example #1': linear version

Linear inversion in rank 2:

\[ G = \langle g \mid g^2 = 1 \rangle \]
\[ L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \]
action: \( g(e_i) = -e_i \)

One obtains:

\[ S(L)^G = \mathbb{Z}[\xi_1, \xi_2] \oplus \eta \mathbb{Z}[\xi_1, \xi_2] \]
\[ (\xi_i = x_i^2, \ \eta = x_1 x_2) \]

Relation:

\[ \eta^2 = \xi_1 \xi_2 \]
Example #1′: linear version

Linear inversion in rank 2:

\[ G = \langle g \mid g^2 = 1 \rangle \]
\[ L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \]

action: \( g(e_i) = -e_i \)

Hence:

\[ \mathbb{Z}[L]^G \cong \mathbb{Z}[x, y, z]/(z^2 - xy) \]
Back to general multiplicative actions:

$L$  a $G$-lattice
$k$  a commutative base ring
$k[L]$  the group algebra
Multiplicative invariants have a $\mathbb{Z}$-structure:

A $k$-basis of $k[L]^G$ is given by the distinct orbit sums

$$\text{orb}(m) := \sum_{m' \in G(m)} x^{m'} \quad (m \in L)$$

$$k[L]^G = k \otimes_{\mathbb{Z}} \mathbb{Z}[L]^G$$
It suffices to consider \textbf{finite groups}: 

Each $\text{orb}(m)$ is supported on 

$$L_{\text{fin}} = \{ m \in L \mid [G : G_m] < \infty \}$$

The stabilizer of $m \in L$.

$G$ acts on $L_{\text{fin}}$ through the finite quotient $G = G / \text{Ker}_G(L_{\text{fin}})$. Thus:

$$\mathbb{K}[L]^G = \mathbb{K}[L_{\text{fin}}]^G$$
In particular, $\mathbb{k}[L]^G$ is always affine/$\mathbb{k}$
(Hilbert # 14 ok).

On the other hand . . .
In general, $\mathbb{k}[L]$ has no grading (connected) that is preserved by the action of $G$.

$\implies$ computational theory not yet highly developed

$\exists$ some GAP & MAGMA-programs (L., Marc Renault)
Jordan (1880): $\text{GL}_n(\mathbb{Z})$ has only finitely many finite subgroups up to conjugacy.
**Finite Linear Groups**

**Jordan (1880):** $GL_n(\mathbb{Z})$ has only finitely many finite subgroups up to conjugacy.

$\Rightarrow$ there are only finitely many multiplicative invariant algebras $\mathbb{k}[L]^G$ (up to $\cong$) with $\text{rank } L$ bounded
Jordan (1880): $\text{GL}_n(\mathbb{Z})$ has only finitely many finite subgroups up to conjugacy.

Minkowski (1887): The least common multiple of their orders is given by

$$M_n = \prod_{p} p^{\left\lfloor \frac{n}{p-1} \right\rfloor + \left\lfloor \frac{n}{p(p-1)} \right\rfloor + \left\lfloor \frac{n}{p^2(p-1)} \right\rfloor + \ldots}$$
## Finite Linear Groups

<table>
<thead>
<tr>
<th>$n$</th>
<th># fin. $G \leq \text{GL}_n(\mathbb{Z})$ (up to conj.)</th>
<th># max’l $G$ (up to conj.)</th>
<th>$M_n$</th>
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Bourbaki: “Invariants exponentiels” (Chap. VI § 3 of Groupes et algèbres de Lie, 1968)

\[ R(\mathfrak{g}) \cong \mathbb{Z}[\Lambda]^\mathcal{W} \cong \mathbb{Z}[x_1, \ldots, x_{\text{rank} \mathfrak{g}}] \]

where \( R(\mathfrak{g}) \) = representation ring of a semisimple Lie algebra \( \mathfrak{g} \), \( \Lambda = \) weight lattice of \( \mathfrak{g} \), and \( \mathcal{W} = \) Weyl group.
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Steinberg, Richardson (1970s)
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Steinberg, Richardson (1970s)

“\( \Delta \)-methods” for group rings: Passman, Zalesskiï, Roseblade, Dan Farkas \( \rightsquigarrow \) “multiplicative invariants” (mid 1980s)
Part II: Regularity
Regularity at 1

Notations:

- $G$ a finite group
- $L \cong \mathbb{Z}^n$ a faithful $G$-lattice
- $k = \overline{k}$ a field with $\text{char } k \nmid |G|$
Regularity at 1

Theorem 1 TFAE

(1) \( k[L]^G \) is regular at \( \pi(1) \)
(2) \( G \) acts as a reflection group on \( L \)
(3) \( k[L]^G = k[M] \) is a semigroup algebra with \( \varepsilon(M) \subseteq k^* \)
Theorem 1 TFAE

(1) $k[L]^G$ is regular at $\pi(1)$
(2) $G$ acts as a reflection group on $L$
(3) $k[L]^G = k[M]$ is a semigroup algebra with $\varepsilon(M) \subseteq k^*$

Here,

$$X = \text{Spec } k[L] \xrightarrow{\pi} X/G = \text{Spec } k[L]^G$$
$$\cup$$
$$1 = \text{Ker } \varepsilon$$

where $\varepsilon: k[L] \longrightarrow k$ is the counit: $\varepsilon(x^m) = 1$ for all $m \in L$
Regularity at 1

Theorem 1 TFAE

1. \( k[L]^G \) is regular at \( \pi(1) \)
2. \( G \) acts as a reflection group on \( L \)
3. \( k[L]^G = k[M] \) is a semigroup algebra with \( \varepsilon(M) \subseteq k^* \)

(1) \( \Rightarrow \) (2) uses linearization: Put \( \mathcal{E} = \text{Ker} \, \varepsilon \). Then

\[
L_k = L \otimes k \\
m \otimes 1 \\
\mapsto \\
\mathcal{E}/\mathcal{E}^2 \\
x^m - 1 + \mathcal{E}^2
\]

leads to \( S(L_k)^G_{\pi(0)} \cong k[L]^G_{\pi(1)} \). Now use the S-T-C Theorem.
Regularity at $1$

**Theorem 1** TFAE

1. $k[L]^G$ is regular at $\pi(1)$
2. $G$ acts as a reflection group on $L$
3. $k[L]^G = k[M]$ is a semigroup algebra with $\varepsilon(M) \subseteq k^*$

$(2) \Rightarrow (3)$ uses root systems: $\exists$ root system $\Phi$ so that

$$\mathbb{Z}\Phi \subseteq L \subseteq \Lambda(\Phi) \quad \text{with} \quad G = \mathcal{W}(\Phi)$$

Use **Bourbaki's Thm**: $\mathbb{Z}[\Lambda(\Phi)]^{\mathcal{W}(\Phi)}$ is a polynomial algebra.
Theorem 1 TFAE

(1) \( k[L]^G \) is regular at \( \pi(1) \)

(2) \( G \) acts as a reflection group on \( L \)

(3) \( k[L]^G = k[M] \) is a semigroup algebra with \( \varepsilon(M) \subseteq k^* \)

(3) \( \Rightarrow \) (1) uses torus actions:

\( (3) \Leftrightarrow X/G = \text{Spec } k[L]^G \) is an affine toric variety so that \( \pi(1) \) belongs to the open torus orbit

This implies (1).
Example #1 revisited

Recall: **multiplicative inversion**

\[ G = \langle g \mid g^2 = 1 \rangle \]

\[ L = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \]

**action:** \[ g(e_i) = -e_i \]

\[ k[L]^G \cong k[x, y, z]/(x^2 + y^2 + z^2 - xyz - 4) \]

\[ k[L]^G \text{ is not a semi-group algebra:} \]
Example #2: $U_n$ and the root lattice $A_{n-1}$

Notation:

\[ U_n = \bigoplus_{i=1}^{n} \mathbb{Z} e_i \cong \mathbb{Z}^n \]
\[ A_{n-1} = \{ \sum_i z_i e_i \in U_n \mid \sum_i z_i = 0 \} \cong \mathbb{Z}^{n-1} \]
\[ S_n\text{-action: } \sigma(e_i) = e_{\sigma(i)} \ (\sigma \in S_n) \]

Note: $S_n$ acts as a reflection group; transpositions are reflections
Example #2: $U_n$ and the root lattice $A_{n-1}$

**Notation:**

$U_n = \bigoplus_1^n \mathbb{Z}e_i \cong \mathbb{Z}^n$

$A_{n-1} = \{ \sum_i z_i e_i \in U_n \mid \sum_i z_i = 0 \} \cong \mathbb{Z}^{n-1}$

$S_n$-action:

$\sigma(e_i) = e_{\sigma(i)} \ (\sigma \in S_n)$

Put $x_i = x^{e_i} \in k[U_n]$; so $\sigma(x_i) = x_{\sigma(i)}$ for $\sigma \in S_n$. Then

$$k[U_n] = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = k[x_1, \ldots, x_n][s_n^{-1}] ,$$

where $s_n = \prod_1^n x_i$ is the $n^{th}$ elementary symmetric polynomial.
Example #2: $U_n$ and the root lattice $A_{n-1}$

Notation:

$U_n = \bigoplus_1^n \mathbb{Z}e_i \cong \mathbb{Z}^n$

$A_{n-1} = \{ \sum_i z_i e_i \in U_n \mid \sum_i z_i = 0 \} \cong \mathbb{Z}^{n-1}$

$S_n$-action: $\sigma(e_i) = e_{\sigma(i)}$ ($\sigma \in S_n$)

\[
\therefore \quad \mathbb{k}[U_n]^{S_n} = \mathbb{k}[x_1, \ldots, x_n][s_n^{-1}]^{S_n}
\]

\[
= \mathbb{k}[x_1, \ldots, x_n]^{S_n}[s_n^{-1}]
\]

\[
= \mathbb{k}[s_1, \ldots, s_{n-1}, s_n^{-1}]
\]

$\cong \mathbb{k}[\mathbb{Z}^{n-1}_+ \oplus \mathbb{Z}]$

elem. symmetric poly’s
Example #2: $U_n$ and the root lattice $A_{n-1}$

Notation:

$U_n = \bigoplus_1^n \mathbb{Z}e_i \cong \mathbb{Z}^n$

$A_{n-1} = \{ \sum_i z_i e_i \in U_n \mid \sum_i z_i = 0 \} \cong \mathbb{Z}^{n-1}$

$S_n$-action: $\sigma(e_i) = e_{\sigma(i)} \ (\sigma \in S_n)$

Now,

$\mathbb{k}[A_{n-1}] = \mathbb{k}[U_n]_0$

the degree 0-component for the ($S_n$-stable) “total degree”
grading of $\mathbb{k}[U_n] = \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. 

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Example #2: $U_n$ and the root lattice $A_{n-1}$

Notation:

\[
U_n = \bigoplus_1^n \mathbb{Z} e_i \cong \mathbb{Z}^n
\]

\[
A_{n-1} = \{ \sum_i z_i e_i \in U_n \mid \sum_i z_i = 0 \} \cong \mathbb{Z}^{n-1}
\]

$S_n$-action:

\[
\sigma(e_i) = e_{\sigma(i)} \ (\sigma \in S_n)
\]

Get $k[A_{n-1}]_{S_n} = k[U_n]_{S_n} = k[s_1, \ldots, s_{n-1}, s_n^{\pm 1}]_0$; so

\[
k[A_{n-1}]_{S_n} \cong k[M]
\]

with

\[
M = \left\{ (t_1, \ldots, t_{n-1}) \in \mathbb{Z}_{+}^{n-1} \mid \sum it_i \in n\mathbb{Z} \right\}
\]
Example #2: $U_n$ and the root lattice $A_{n-1}$

**Notation:**

$U_n = \bigoplus_1^n \mathbb{Z}e_i \cong \mathbb{Z}^n$

$A_{n-1} = \{ \sum_i z_i e_i \in U_n \mid \sum_i z_i = 0 \} \cong \mathbb{Z}^{n-1}$

$S_n$-action: $\sigma(e_i) = e_{\sigma(i)} \ (\sigma \in S_n)$

$\mathbb{C}[A_{n-1}]^{S_n}$ is not regular:

$(n > 2; \text{ picture for } n = 3)$
Here is the global version of Theorem 1

(same notations and hypotheses)
Theorem 1' TFAE

1. \(k[L]^G\) is regular

2. \(G\) acts as a reflection group on \(L\) and \(H^1(G/D, L^D) = 0\)

3. \(k[L]^G \cong \mathbb{k}[\mathbb{Z}_+^r \oplus \mathbb{Z}^s]\)

4. \(\exists\) root system \(\Phi\) s.t. \(L/L^G \cong \Lambda(\Phi)\) and \(G = W(\Phi)\)

Here, \(D\) is the subgroup of \(G\) that is generated by the “diagonalizable” reflections, conjugate in \(GL(L)\) to

\[
d = \begin{pmatrix}
-1 & 1 & \cdots \\
1 & 1 & \cdots \\
& & \ddots & 1
\end{pmatrix}
\]
### Regularity

<table>
<thead>
<tr>
<th>$n$</th>
<th># finite $G \leq \text{GL}_n(\mathbb{Z})$ (up to conjugacy)</th>
<th># reflection groups $G$ (up to conjugacy)</th>
<th># cases with $\mathbb{k}[L]^G$ regular</th>
</tr>
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<tr>
<td>2</td>
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<td>4</td>
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<td>102</td>
<td>51</td>
</tr>
</tbody>
</table>
Part III: The Cohen-Macaulay Property
Hypotheses: $R$ a comm. noetherian ring
$\alpha$ an ideal of $R$
Reminder: CM Rings

- Hypotheses: $R$ a comm. noetherian ring
  $\mathfrak{a}$ an ideal of $R$

- Always:

  \[
  \text{height } \mathfrak{a} \geq \text{depth } \mathfrak{a} = \inf \{i \mid H^i_{\mathfrak{a}}(R) \neq 0\}
  \]
Reminder: CM Rings

- **Hypotheses:**
  - $R$ a comm. noetherian ring
  - $\mathfrak{a}$ an ideal of $R$

- **Always:**

\[
\text{height } \mathfrak{a} \geq \text{depth } \mathfrak{a} = \inf \{i \mid H^i_{\mathfrak{a}}(R) \neq 0\}
\]

- **Def:** $R$ is **Cohen-Macaulay** iff equality holds for all (maximal) ideals $\mathfrak{a}$

(Zariski) topology, dimension theory, (homological) algebra
Some Examples of CM Rings

**Standard example:** $R$ an affine domain/PID $\mathbb{k}$, finite / some polynomial subalgebra $P = \mathbb{k}[x_1, \ldots, x_n]$. Then:

\[ R \text{ CM } \iff R \text{ is free over } P \]
Some Examples of CM Rings

- **Standard example:** $R$ an affine domain/PID $k$, finite / some polynomial subalgebra $P = k[x_1, \ldots, x_n]$. Then:

\[
R \text{ CM } \Leftrightarrow R \text{ is free over } P
\]

- **Hierarchy:**

regular $\Rightarrow$ complete $\cap$ $\Rightarrow$ Gorenstein $\Rightarrow$ CM

- catenary

- dim 0

- dim 1 reduced

- dim 2 normal
Invariant Rings

Hypotheses:  
- $R$ a CM ring
- $G$ a finite group acting on $R$
Hypotheses: \( R \) a CM ring
\( G \) a finite group acting on \( R \)

If the trace map \( R \rightarrow R^G, r \mapsto \sum_{g} g(r) \), is epi ("non-modular case") then \( R^G \) is CM; otherwise usually not.
Invariant Rings

Hypotheses:  
\( R \)  a CM ring  
\( G \)  a finite group acting on \( R \)

If the trace map \( R \to R^G, \ r \mapsto \sum_G g(r) \), is epi ("non-modular case") then \( R^G \) is CM; otherwise usually not.

Here is a necessary condition . . .
Invariant Rings

Hypotheses:

\( \mathcal{R} \) a CM ring
\( \mathcal{G} \) a finite group acting on \( \mathcal{R} \)
\( \mathcal{R}_k = \{ k\text{-reflections on } \mathcal{R} \} \)
Assume \( \mathcal{R} \) noetherian

Proposition (L. - Pathak)

\[ \mathcal{R}^\mathcal{G} \text{ CM} & \quad H^i(\mathcal{G}, \mathcal{R}) = 0 \ (0 < i < k) \]

\[ \Rightarrow \quad \text{res}: H^k(\mathcal{G}, \mathcal{R}) \hookrightarrow \prod_{\mathcal{H} \subseteq \mathcal{R}_{k+1}} H^k(\mathcal{H}, \mathcal{R}) \]
Hypotheses: \( R \) a CM ring
\( G \) a finite group acting on \( R \)
\( \mathcal{R}_k = \{ k\text{-reflections on } R \} \)
Assume \( R \) noetherian

Proposition (L. - Pathak)

\[
R^G \text{ CM } \wedge \quad H^i(G, R) = 0 \quad (0 < i < k) \\
\Rightarrow \quad \text{res: } H^k(G, R) \rightarrow \prod_{\mathcal{H} \subseteq \mathcal{R}_{k+1}} H^k(\mathcal{H}, R)
\]

Note: The \((H^i = 0)\)-cond\(n\) is vacuous for \( k = 1 \) \( \leadsto \) bi-reflections.
Notations: \( G \) is a finite group \( \neq 1 \)
\( L \) a \( G \)-lattice, WLOG faithful
Multiplicative Invariants: CM-property

Notations:   \[ G \] is a finite group \( \neq 1 \]

\[ L \] a \( G \)-lattice, WLOG faithful

So \( G \hookrightarrow \text{GL}(L), g \mapsto g_L \). In this setting,

\[ g \in G \text{ is a } k\text{-reflection on } \mathbb{k}[L] \iff \text{rank}(g_L - \text{Id}_L) \leq k \]

"\text{g is a } k\text{-reflection on } L" — or on \( L \otimes_{\mathbb{Z}} \mathbb{Q} \)
Multiplicative Invariants: CM-property

Notations: $G$ is a finite group $\neq 1$
$L$ a $G$-lattice, WLOG faithful

Theorem 2
(L, TAMS '06)

If $\mathbb{Z}[L]^G$ is CM then all $G_m/\mathcal{R}^2(G_m)$ for $m \in L$ are perfect groups, but not all $G_m$ are.

subgroup gen. by bireflections on $L$
Multiplicative Invariants: CM-property

Notations: $G$ is a finite group $\neq 1$
$L$ a $G$-lattice, WLOG faithful

Theorem 2  
(L, TAMS '06)  
If $\mathbb{Z}[L]^G$ is CM then all $G_m/R^2(G_m)$ for $m \in L$ are perfect groups, but not all $G_m$ are.

Corollary ("3-copies conjecture")  
$\mathbb{Z}[L^\oplus r]^G$ is never CM for $r \geq 3$. 
Multiplicative Invariants: CM-property

**Notations:**
- $G$ is a finite group $\neq 1$
- $L$ a $G$-lattice, WLOG faithful

**Theorem 2** *(L, TAMS '06)*

If $\mathbb{Z}[L]^G$ is CM then all $G_m/R^2(G_m)$ for $m \in L$ are perfect groups, but **not** all $G_m$ are.

Note that the conclusions of Theorem 2 only refer to the rational type of $L$. In fact ...
Multiplicative Invariants: CM-property

Notations: \( G \) is a finite group \( \neq 1 \)
\( L \) a \( G \)-lattice, WLOG faithful

Theorem 2  
(\( L \), TAMS '06)  
If \( \mathbb{Z}[L]^G \) is CM then all \( G_m/R^2(G_m) \) for \( m \in L \) are perfect groups, but not all \( G_m \) are.

Proposition  
If \( k[L]^G \) is CM then so is \( k[L']^G \) for any \( G \)-lattice \( L' \) so that \( L' \otimes \mathbb{Q} \cong L \otimes \mathbb{Q} \).
Example: $S_n$-lattices

What are the $S_n$-lattices $L$ such that $\mathbb{Z}[L]^{S_n}$ is CM?
Example: $S_n$-lattices

We know:

- only the structure of $L_\mathbb{Q} = L \otimes_{\mathbb{Z}} \mathbb{Q}$ matters (Proposition)

- $S_n$ must act as a bireflection group on $L$ (Theorem 2), and hence on all simple constituents of $L_\mathbb{Q}$
Example: $S_n$-lattices

Classification results of irreducible finite linear groups containing a bireflection (Huffman and Wales, 70s) imply, for $n \geq 7$:

$$L_\mathbb{Q} \cong \mathbb{Q}^r \oplus (\mathbb{Q}^-)^s \oplus (A_{n-1})^t_\mathbb{Q} \quad (s + t \leq 2)$$

sign representation of $S_n$
Example: $S_n$-lattices

In all cases, $\mathbb{Z}[L]^{S_n}$ is indeed CM, with the possible exception of

$$L = A_{n-1}^2$$

This case reduces to

Problem (open for $p \leq n/2$)

Are the "vector invariants"

$$\mathbb{F}_p[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n}$$

CM?
Let $L$ be a $\mathcal{G}$-lattice, where $\mathcal{G}$ is a finite group.