Prime ideals and group actions

“Hopf Algebras and Related Topics” – USC 02/15/2009

Martin Lorenz
Temple University, Philadelphia
Recently there has been some interest in so-called “Additivity Principles” [2] which, for a ring extension \( R \subseteq S \) and a prime ideal \( P \) of \( S \), relate the Goldie rank of \( R \) to the Goldie ranks of \( R/Q \), for all primes \( Q \) of \( S \) which are minimal over \( P \cap S \).

In this note, we prove such a theorem for the ring extension \( R^{G} \subseteq S^{G} \) where \( R^{G} \) is the fixed subring of a finite group \( G \) acting as \( \mathbb{Z}/e \) automorphisms of \( R \), such that \(|G|^{-1} \mathbb{Z} \subseteq R^{G} \). Our result improves the bound on Goldie ranks obtained in [4].

We also include a few additional remarks on prime ideals in fixed rings.

We first require, form [4], some facts about the relationship between prime ideals in \( R \) and \( R^{G} \). For \( P \) a prime ideal of \( R \), let

The last two authors wish to thank the University of Keese for its hospitality while this work was being done.

469

Copyright © 1982 by Marcel Dekker, Inc.
Thank you

Prime ideals and group actions
Thank you

Prime ideals and group actions

USC 02/15/2009
• **Background**: enveloping algebras and quantized coordinate algebras
Overview

- **Background**: enveloping algebras and quantized coordinate algebras
- **Tool**: actions of algebraic groups, a brief reminder
Overview

- **Background**: enveloping algebras and quantized coordinate algebras

- **Tool**: actions of algebraic groups, a brief reminder

- **Tool**: the Amitsur-Martindale ring of quotients
Overview

- **Background**: enveloping algebras and quantized coordinate algebras
- **Tool**: actions of *algebraic groups*, a brief reminder
- **Tool**: the Amitsur-Martindale ring of quotients
- **Rational** and primitive ideals
Overview

- **Background**: enveloping algebras and quantized coordinate algebras
- Tool: actions of *algebraic groups*, a brief reminder
- Tool: the *Amitsur-Martindale ring of quotients*
- **Rational** and primitive ideals
- **Stratification** of the prime spectrum
References

- “Group actions and rational ideals”, Algebra and Number Theory 2 (2008), 467-499


Both articles & the pdf file of this talk available on my web page:

http://math.temple.edu/~lorenz/
I will work / base field $k = \overline{k}$
Background
Goal: For $R = U(g)$, the enveloping algebra of a finite-dim’l Lie algebra $g$, describe

$$\text{Prim } R = \{\text{primitive ideals of } R\}$$

kernels of irreducible (generally infinite-dimensional) representations $R \rightarrow \text{End}_k(V)$
Jacques Dixmier (* 1924)

- former secretary of Bourbaki
- Ph.D. advisor of A. Connes, M. Duflo, …
- author of several influential monographs:
  - *Les $C^*$-algèbres et leurs représentations*, Gauthier-Villars, 1969
  - *Algèbres enveloppantes*, Gauthier-Villars, 1974
Dixmier’s Problem 11

from *Algèbres enveloppantes*, 1974:

10. On suppose que $tr ad x = 0$ pour tout $x \in g$. Est-ce que $Z(g) \neq k$ ?

11. Soient $\mathfrak{f}$ un idéal de $g$, $I$ un idéal primitif de $U(g)$. Les propriétés suivantes sont-elles vraies: (a) il existe un idéal primitif de $U(\mathfrak{f})$ générique pour $U(\mathfrak{f}) \cap I$; (b) deux tels idéaux sont conjugués par le groupe adjoint algébrique de $g$; (c) soit $L$ un tel idéal; il existe une représentation simple $\sigma$ de $\mathfrak{f}$ de noyau $L$, et une représentation simple $\rho$ de $st(\sigma, g)$, telles que $\rho | \mathfrak{f}$ soit un multiple de $\sigma$ et que $\text{ind}(\rho, g)$ soit simple de noyau $I$.

Cf. 4.5.9, 5.4.3, 5.4.4, 5.6.5.
Dixmier’s Problem 11

- Problem 11 for $\mathfrak{t}$ solvable, $\text{char} \ k = 0$


- for noetherian or Goldie rings $R / \text{char} \ k = 0$:

  Mœglin & Rentschler


Dixmier’s Problem 11

- Problem 11 for \( \mathfrak{t} \) solvable, \( \text{char} \, k = 0 \)


  ⇒ existence: (a)


  ⇒ uniqueness: (b)

- for noetherian or Goldie rings \( R / \text{char} \, k = 0 \):

  Möeglin & Rentschler


under weaker Goldie hypotheses / $\text{char } k$ arbitrary:

N. Vonessen


Goal: For $R = \mathcal{O}_q(\mathbb{A}^n), \mathcal{O}_q(M_n), \mathcal{O}_q(G)$ . . . a quantized coordinate ring, describe

$$\text{Spec } R = \{\text{prime ideals of } R\} \supseteq \text{Prim } R$$
Quantum groups

Goal: For $R = \mathcal{O}_q(\mathbb{k}^n), \mathcal{O}_q(M_n), \mathcal{O}_q(G)$ . . . a quantized coordinate ring, describe

$$\text{Spec } R = \{\text{prime ideals of } R\} \supseteq \text{Prim } R$$

Typically, some algebraic torus $T$ acts rationally by $\mathbb{k}$-algebra automorphisms on $R$; so have

$$\text{Spec } R \longrightarrow \text{Spec}^T R = \{T\text{-stable primes of } R\}$$

$$P \quad \mapsto \quad P : T = \bigcap_{g \in T} g.P$$
Quantum groups

\( T \)-stratification of \( \text{Spec} \, R \)

(Goodearl & Letzter, ...; see the monograph by Brown & Goodearl)

\[
\text{Spec} \, R = \bigsqcup_{I \in \text{Spec}^T \, R} \text{Spec}_I \, R
\]

\( \{ P \in \text{Spec} \, R \mid P : T = I \} \)
\( T \)-stratification of \( \text{Spec} \ R \)

( Goodearl & Letzter, \ldots; see the monograph by Brown & Goodearl )
Tool: Algebraic Groups
Definition

There an anti-equivalence of categories

\[
\left\{ \text{affine algebraic groups } / \mathbb{k} \right\} \cong \left\{ \text{comm. affine reduced Hopf } \mathbb{k}\text{-algebras} \right\}
\]

\[
G = \text{Hom}_{\mathbb{k}\text{-alg}}(\mathbb{k}[G], \mathbb{k}) \leftrightarrow \mathbb{k}[G]
\]
There an anti-equivalence of categories

\[ \{ \text{affine algebraic groups } / \mathbb{k} \} \cong \{ \text{comm. affine reduced Hopf } \mathbb{k}\text{-algebras} \} \]

\[ G = \text{Hom}_{\mathbb{k}\text{-alg}}(\mathbb{k}[G], \mathbb{k}) \iff \mathbb{k}[G] \]

Equivalently, affine algebraic groups are precisely the closed subgroups of $\text{GL}_n$ for some $n$:

$\text{GL}_n, \text{SL}_n, \text{T}_n, \text{U}_n, \text{D}_n, \text{O}_n, \ldots$

all finite groups
For any affine algebraic $\mathbb{k}$-group $G$,

\[ G\text{-modules} \equiv \mathbb{k}[G]\text{-comodules} \]

Comodule structure map

\[ \Delta_M : M \rightarrow M \otimes \mathbb{k}[G] \]

\[ m \mapsto \sum m_0 \otimes m_1 \]
For any affine algebraic \( \mathbb{k} \)-group \( G \),

\[
G\text{-modules} \equiv \mathbb{k}[G]\text{-comodules}
\]

Comodule structure map

\[
\Delta_M : M \to M \otimes \mathbb{k}[G] \\
m \mapsto \sum m_0 \otimes m_1
\]

We obtain a linear representation \( \rho_M : G \to \text{GL}(M) \) by

\[
g.m = \sum m_0 \langle g, m_1 \rangle
\]

Representations arising in the way are called rational.
For the remainder of this talk,

\( R \) denotes an associative \( \mathbb{k} \)-algebra (with 1)

\( G \) is an affine algebraic \( \mathbb{k} \)-group acting rationally on \( R \);
so \( R \) is a \( \mathbb{k}[G] \)-comodule algebra.

Equivalently, we have a rational representation

\[ \rho = \rho_R : G \to \text{Aut}_{\mathbb{k}-\text{alg}}(R) \subseteq \text{GL}(R) \]
Tool: The Amitsur-Martindale ring of quotients
In brief,

\[ Q_r(R) = \lim_{I \in \mathcal{E}} \text{Hom}(I_R, R_R) \]

where \( \mathcal{E} = \mathcal{E}(R) \) is the filter of all \( I \subseteq R \) such that \( 1 \cdot \text{ann}_R I = 0 \).
In brief,

\[ Q_r(R) = \lim_{\substack{\longrightarrow \\ I \in \mathcal{E}}} \text{Hom}(I_R, R_R) \]

where \( \mathcal{E} = \mathcal{E}(R) \) is the filter of all \( I \leq R \) such that \( 1 \cdot \text{ann}_R I = 0 \).

Explicitly, elements of \( Q_r(R) \) are equivalence classes of right \( R \)-module maps

\[ f : I_R \to R_R \quad (I \in \mathcal{E}) , \]

the map \( f \) being equivalent to \( f' : I'_R \to R_R \quad (I' \in \mathcal{E}) \) if \( f = f' \) on some \( J \subseteq I \cap I', J \in \mathcal{E} \).
In brief,

\[ Q_r(R) = \lim_{\longrightarrow} \text{Hom}(I_R, R_R) \]

where \( \mathcal{E} = \mathcal{E}(R) \) is the filter of all \( I \leq R \) such that \( 1.\text{ann}_R I = 0 \).

Addition and multiplication of \( Q_r(R) \) come from addition and composition of \( R \)-module maps.
In brief,

\[ Q_r(R) = \lim_{I \in \mathcal{E}} \text{Hom}(I_R, R_R) \]

where \( \mathcal{E} = \mathcal{E}(R) \) is the filter of all \( I \subseteq R \) such that \( l \cdot \text{ann}_R I = 0 \).

Sending \( r \in R \) to the equivalence class of \( \lambda_r : R \to R, x \mapsto rx \), yields an embedding of \( R \) as a subring of \( Q_r(R) \).
Def\textsuperscript{s} & Facts:

- The \textbf{extended centroid} of $R$ is defined by

\[ C(R) = \mathcal{Z} Q_r(R) \]

Here $\mathcal{Z}$ denotes the center. If $R$ is prime then $C(R)$ is a $\mathbb{k}$-field.
Def's & Facts:  

- The **extended centroid** of $R$ is defined by

$$C(R) = Z \ Q_r(R)$$

Here $Z$ denotes the center. If $R$ is prime then $C(R)$ is a $k$-field.

- Put $\tilde{R} = R C(R) \subseteq Q_r(R)$. The algebra $R$ is said to be **centrally closed** if $R = \tilde{R}$. If $R$ is semiprime then $\tilde{R}$ is centrally closed.
for prime rings $R$:


for general $R$:

Examples

- If $R$ is simple, or a finite product of simple rings, then

$$Q_r(R) = R$$
• If $R$ is **simple**, or a finite product of simple rings, then

$$Q_r(R) = R$$

• For $R$ **semiprime right Goldie**,

$$Q_r(R) = \{ q \in Q_{cl}(R) \mid qI \subseteq R \text{ for some } I \triangleleft R \text{ with } \text{ann}_R I = 0 \}$$

In particular,

$$C(R) = \mathbb{Z}Q_{cl}(R)$$

(classical quotient ring of $R$)
Examples

- $R = U(g)/I$ a semiprime image of the enveloping algebra of a finite-dimensional Lie algebra $g$. Then

$$Q_r(R) = \{ \text{ad}_g\text{-finite elements of } Q_{cl}(R) \}$$
Rational Ideals
**Definition**

**Want:** an *intrinsic* characterization of “primitivity”, ideally in detail . . .
Definition: Recall that $C(R/P)$ is a $\mathbb{k}$-field for any $P \in \text{Spec } R$. We call $P$ rational if $C(R/P) = \mathbb{k}$. 
Definition: Recall that $C(R/P)$ is a $\mathbb{k}$-field for any $P \in \text{Spec } R$. We call $P$ rational if $C(R/P) = \mathbb{k}$.

- Put $\text{Rat } R = \{ P \in \text{Spec } R \mid P \text{ is rational} \}$; so $\text{Rat } R \subseteq \text{Spec } R$.
Given an irreducible representation $f : R \to \text{End}_k(V)$, let $P = \text{Ker } f$ be the corresponding primitive ideal of $R$. 
Given an irreducible representation \( f: R \to \text{End}_k(V) \), let \( P = \ker f \) be the corresponding primitive ideal of \( R \).

- There **always** is an embedding of \( \mathbb{k} \)-fields

\[
\mathcal{C}(R/P) \hookrightarrow \mathcal{Z}(\text{End}_R(V))
\]
Given an irreducible representation $f : R \to \text{End}_k(V)$, let $P = \text{Ker } f$ be the corresponding primitive ideal of $R$.

- There **always** is an embedding of $\mathbb{k}$-fields
  $$\mathcal{C}(R/P) \hookrightarrow \mathcal{Z}(\text{End}_R(V))$$

- **Typically**, $\text{End}_R(V) = \mathbb{k}$ ("weak Nullstellensatz"); in this case
  $$\text{Prim } R \subseteq \text{Rat } R$$
Examples

The weak Nullstellensatz holds for

- $R$ any affine $\mathbb{k}$-algebra, $\mathbb{k}$ uncountable  
  \quad \text{Amitsur}

- $R$ an affine PI-algebra  
  \quad \text{Kaplansky}

- $R = U(g)$  
  \quad \text{“Quillen’s Lemma”}

- $R = \mathbb{k}\Gamma$ with $\Gamma$ polycyclic-by-finite  
  \quad \text{Hall, L.}

- many quantum groups: $\mathcal{O}_q(\mathbb{k}^n)$, $\mathcal{O}_q(M_n(\mathbb{k}))$, $\mathcal{O}_q(G)$, \ldots
Examples

The weak Nullstellensatz holds for

- \( R \) any affine \( \mathbb{k} \)-algebra, \( \mathbb{k} \) uncountable (Amitsur)
- \( R \) an affine PI-algebra (Kaplansky)
- \( R = U(\mathfrak{g}) \) ("Quillen's Lemma")
- \( R = \mathbb{k}\Gamma \) with \( \Gamma \) polycyclic-by-finite (Hall, L.)
- many quantum groups: \( \mathcal{O}_q(\mathbb{k}^n) \), \( \mathcal{O}_q(M_n(\mathbb{k})) \), \( \mathcal{O}_q(G) \), \ldots

In fact, in all these examples except the first, it has been shown that, under mild restrictions on \( \mathbb{k} \) or \( q \),

\[ \text{Prim } R = \text{Rat } R \]
The $G$-action on $R$ yields actions on \{ ideals of $R$ \}, $\text{Spec } R$, $\text{Rat } R$, . . . . As usual, $G\backslash ?$ denotes the orbit sets in question.

**Definition:** A proper $G$-stable ideal $I \triangleleft R$ is called $G$-prime if $A, B \triangleleft R$, $AB \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$. Put $G\text{-Spec } R = \{ G$-prime ideals of $R \}$.
Proposition \(\text{The assignment } \gamma: P \mapsto P: G = \bigcap_{g \in G} g \cdot P \text{ yields surjections}\)
Definition: Let $I \in G\text{-Spec } R$. The group $G$ acts on $C(R/I)$ and the invariants $C(R/I)^G$ are a $\mathbb{k}$-field. We call $I$ $G$-rational if $C(R/I)^G = \mathbb{k}$. Put

$$G\text{-Rat } R = \{G\text{-rational ideals of } R\}$$
**Definition:** Let \( I \in G\text{-Spec } R \). The group \( G \) acts on \( \mathcal{C}(R/I) \) and the invariants \( \mathcal{C}(R/I)^G \) are a \( \mathbb{k} \)-field. We call \( I \) \( G \)-rational if \( \mathcal{C}(R/I)^G = \mathbb{k} \). Put

\[
G\text{-Rat } R = \{ \text{G-rational ideals of } R \}
\]

The following result solves Dixmier’s Problem # 11 (a),(b) for arbitrary algebras.

**Theorem 1**

\[
G\backslash \text{Rat } R \underset{\text{bij.}}{\longrightarrow} G\text{-Rat } R
\]

\[
\cup G.P \quad \mapsto \quad \bigcap_{g \in G} g.P
\]
Noncommutative spectra

\[ \text{Spec } R \to \text{Rat } R \to G \backslash \text{Spec } R \]

\[ \gamma : P \mapsto P : G = \bigcap_{g \in G} g \cdot P \]

\[ G \backslash \text{Spec } R \to G \backslash \text{Rat } R \to G \text{-Spec } R \]

\[ \sim \]

Prime ideals and group actions  USDA 02/15/2009
\textit{Spec} $R$ carries the \textbf{Jacobson-Zariski topology}: closed subsets are those of the form $V(I) = \{ P \in \text{Spec} R \mid P \supseteq I \}$ where $I \subseteq R$. 
Noncommutative spectra

\[ Spec \, R \]

\[ \rightarrow \]

\[ can. \]

\[ \leftarrow \]

\[ \gamma: P \mapsto \overline{P} : G = \bigcap_{g \in G} g.P \]

\[ G \backslash Spec \, R \]

\[ \Rightarrow \]

\[ G \backslash Rat \, R \]

\[ \ni \]

\[ \cong \]

\[ G \text{-} Spec \, R \]

\[ \rightarrow \]

\[ G \text{-} Rat \, R \]

is a surjection whose target has the final topology,
is an inclusion whose source has the induced topology, and
is a homeomorphism, from Thm 1
Next, we turn to $\text{Spec } R$ and the map $\gamma \ldots$
Stratification of the prime spectrum
Recall: $\gamma: \text{Spec } R \to G\text{-Spec } R$, $P \mapsto P : G = \bigcap_{g \in G} g.P$, yields the $G$-stratification of $\text{Spec } R$

\[
\text{Spec } R = \bigsqcup_{I \in G\text{-Spec } R} \text{Spec}_I R
\]

Goal #1: describe the $G$-strata

\[
\text{Spec}_I R = \gamma^{-1}(I) = \{ P \in \text{Spec } R \mid P : G = I \} 
\]
For simplicity, I assume $G$ to be **connected**; so $\mathbb{k}[G]$ is a domain. In particular,

$$G\text{-Spec } R = \text{Spec}^G R = \{G\text{-stable primes of } R\}$$
The rings $T_I$

For a given $I \in G\text{-Spec } R$, put

$$T_I = C(R/I) \otimes k(G)$$

This is a **commutative** domain, a tensor product of two fields.
For a given $I \in G\text{-Spec } R$, put

$$T_I = \mathcal{C}(R/I) \otimes \mathbb{k}(G)$$

This is a **commutative** domain, a tensor product of two fields.

$G$-actions:
- on $\mathcal{C}(R/I)$ via the given action $\rho : G \to \text{Aut}_{\mathbb{k}\text{-alg}}(R)$
- on $\mathbb{k}(G)$ by the right and left regular actions $\rho_r : (x.f)(y) = f(yx)$ and $\rho_\ell : (x.f)(y) = f(x^{-1}y)$
- on $T_I$ by $\rho \otimes \rho_r$ and $\text{Id} \otimes \rho_\ell$ ← commute
The rings $T_I$

For a given $I \in G\text{-Spec } R$, put

$$T_I = \mathcal{C}(R/I) \otimes \mathbb{k}(G)$$

This is a **commutative** domain, a tensor product of two fields.

Put

$$\text{Spec}^G T_I = \{ (\rho \otimes \rho_r)(G) \text{-stable primes of } T_I \}$$
Theorem 2  \textit{Given } I \in G\text{-Spec } R, \textit{there is a bijection}

\[ c: \text{Spec}_I R \longrightarrow \text{Spec}^G T_I \]

\textit{having the following properties:}

(a) \textit{G-equivariance: } \[c(g.P) = (\text{Id} \otimes \rho_\ell)(g)(c(P));\]
(b) \textit{inclusions: } \[P \subseteq P' \iff c(P) \subseteq c(P');\]
(c) \textit{hearts: } \[\mathcal{C} (T_I/c(P)) \cong \mathcal{C} (R/P \otimes \mathbb{k}(G)) \text{ as } \mathbb{k}(G)\text{-fields};\]
(d) \textit{rationality: } \[P \text{ is rational } \iff T_I/c(P) = \mathbb{k}(G).\]
Theorem 2  There is a bijection

\[ c: \text{Spec}_I R \rightarrow \text{Spec}^G T_I \]

having the following properties:

(a) \text{G-equivariance}: \ c(g.P) = (\text{Id} \otimes \rho_e)(g)(c(P));
(b)\text{ inclusions}: \ P \subseteq P' \iff c(P) \subseteq c(P');
(c) \text{hearts}: \ C(T_I/c(P)) \cong C(R/P \otimes \mathbb{k}(G)) \text{ as } \mathbb{k}(G)-\text{fields};
(d) \text{rationality}: \ P \text{ is rational} \iff T_I/c(P) = \mathbb{k}(G).

\textbf{Cor:} Rational ideals are maximal in their strata.
Assume that $R$ sat$^\ast$ the **Nullstellensatz**: weak Nullstellensatz & Jacobson property

e.g., $R$ affine noetherian / uncountable $\mathbb{k}$, affine PI, ...
Assume that $R$ satisfies the **Nullstellensatz**: weak Nullstellensatz & Jacobson property

**Prop**

The following are equivalent:

(a) $G$-$\text{Spec } R$ is finite;

(b) $G$ has finitely many orbits in $\text{Rat } R$;

(c) $R$ satisfies (1) ACC for $G$-stable semiprime ideals, (2) the Dixmier-Mœglin equivalence, and (3) $G$-$\text{Rat } R = G$-$\text{Spec } R$.

locally closed = primitive = rational
Goal #2: Finiteness of $G$-Spec $R$

**Example:** If $G$ is an algebraic torus then a sufficient condition for the equality $G$-Spec $R = G$-Rat $R$ is

$$\dim_{\mathbb{k}} R_{\lambda} \leq 1 \quad \text{for all } \lambda \in X(G)$$

For a commutative domain $R$, this is also necessary.

**Cor** (classical) Let $R$ be an affine commutative domain / $\mathbb{k}$ and let $G$ be an algebraic $\mathbb{k}$-torus acting rationally on $R$. Then:

$$G$$-Spec $R$ is finite $\iff \dim_{\mathbb{k}} R_{\lambda} \leq 1 \quad \text{for all } \lambda \in X(G).$$
Recall: locally closed = open \cap \text{closed}

**Theorem 3**  \( \text{If } P \in \text{Rat } R \text{ then:} \)

\[
\{P\} \text{ loc. cl. in } \text{Spec } R \iff \{P:G\} \text{ loc. cl. in } G-\text{Spec } R
\]
Recall: locally closed = open ∩ closed

**Theorem 3**  \( If \ P \in \text{Rat} \ R \ then:\)

\[
\{P\} \text{ loc. cl. in } \text{Spec} R \iff \{P:G\} \text{ loc. cl. in } G-\text{Spec} R
\]

**Cor**  \( If \ P \in \text{Rat} \ R \text{ is loc. closed in } \text{Spec} R \ then \ the \ orbit \ G.P \ is \ open \ in \ its \ closure \ in \ \text{Rat} \ R. \)
Prime ideals and group actions