How large can a finite group of matrices be?

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Martin Lorenz
Temple University, Philadelphia

Overview

- Groups . . . and some of their uses

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- Groups . . . and some of their uses
- The size of finite matrix groups

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- Groups . . . and some of their uses
- The size of finite matrix groups
- The Minkowski sequence: two mysteries

Reference

R. M. Guralnick & L.: *Orders of finite groups of matrices*, Contemp. Math. **420**, 141–162 (2006), arXiv:math.GR/0511191.

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☞ pdf file of **this talk** on my web page

Part I: Groups

The definition of a “group”

. . . here it is (from **Bourbaki**):

DÉFINITION 1. — *On appelle* groupe *un ensemble muni d'une loi de composition associative, possédant un élément neutre et pour laquelle tout élément est inversible.*

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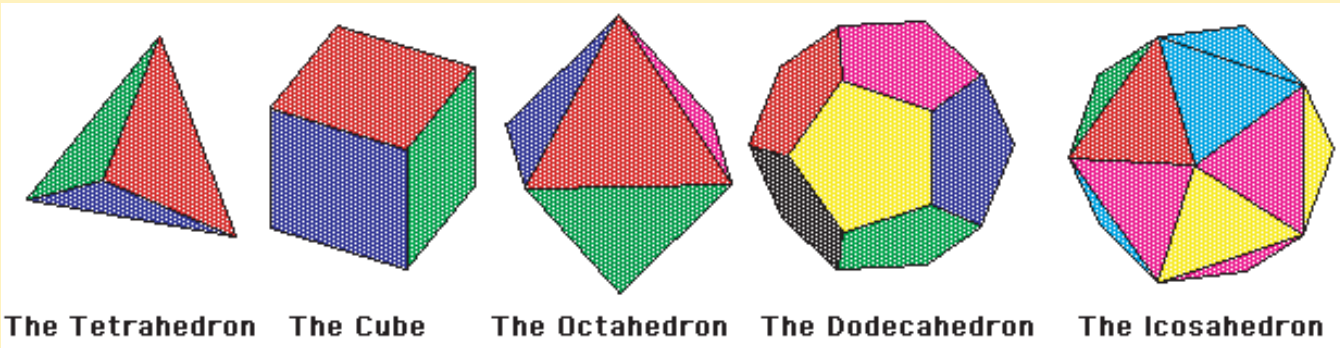
Groups have been around long before they were defined in the above terms . . .

Geometry: symmetry groups

Given $X \subset \mathbb{R}^n$ (e.g., a Platonic solid in $\mathbb{R}^3$), any distance preserving transformation of $\mathbb{R}^n$ that maps X to itself is called a symmetry of X.



(from MacTutor History of Mathematics archive)

Example: the dodecahedron

$\rightsquigarrow$ symmetry group

A_5

It has **60** symmetries which form the so-called alternating group of degree 5.

Physics: gauge groups

Theories in physics are often described by "Lagrangians" which are invariant under certain symmetry transformation groups, called gauge groups.

These are infinite groups such as the orthogonal group

$$\begin{aligned}\mathbf{O}_n(\mathbb{R}) &= \{A \in \mathrm{M}_n(\mathbb{R}) \mid \|Ax\| = \|x\| \quad \forall x \in \mathbb{R}^n\} \\ &= \{A \in \mathrm{M}_n(\mathbb{R}) \mid A \cdot A^{\mathrm{tr}} = \mathbf{1}_{n\times n}\}\end{aligned}$$

$n \times n$-matrices

transpose matrix

Algebra: Galois groups

Recall: the **quadratic polynomial** equation

$$ax^2 + bx + c = 0$$

has solutions

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Algebra: Galois groups

Similar formulas (more complicated and requiring higher roots) are known for **cubic** and **quartic** polynomials.

Tartaglia, Ferrari, Cardano; 16^{th} century

Algebra: Galois groups

Similar formulas (more complicated and requiring higher roots) are known for **cubic** and **quartic** polynomials.

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For **degree 5**, however, this is no longer possible!

Niels Henrik Abel, 1824

Algebra: Galois groups

The ultimate reason for the "non-solvability of the quintic equation by radicals" is **group theoretical**:

A_5 is non-abelian simple

FINAL WORD: associated to a polynomial eq^n of **any degree**, there is a finite group, the Galois group of the eq^n.

Evariste Galois
1811 – 1832

The eq^n is solvable by radicals

$\Updownarrow$

its Galois group has no non-abelian simple "pieces"

The Enormous Theorem

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They are now all known; the Enormous Theorem (about 1983) gives a complete classification:

There are several series of finite simple groups (e.g. A_5, A_6, A_7, ...) and **26** isolated ones, known as the sporadic groups.

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- The original “proof” of the Enormous Theorem was spread over a vast number journal articles and various unpublished manuscripts, some of which are incomplete.

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- The largest sporadic group is known as the Monster; it has size

$$808017424794512875886459904961710757005754368000000000$$

Part II: Matrix Groups

We will be looking at matrices in

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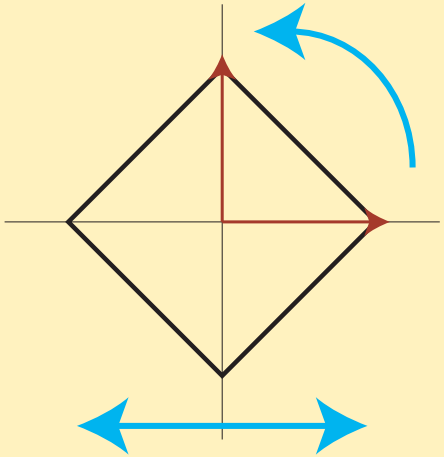
Def$^{\mathbf{n}}$: A finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ is a finite (non-empty) collection of matrices $\mathcal{G} \subseteq \mathrm{GL}_n(\mathbb{C})$ satisfying

$$A, B \in \mathcal{G} \qquad \Rightarrow \qquad A \cdot B \in \mathcal{G}$$

The number of matrices in $\mathcal{G}$ is called the order of $\mathcal{G}$.

Example: symmetries of the square ($n = 2$)



$\rightsquigarrow$ **matrices**:

$90°$ rotation: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

reflection across y-axis: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

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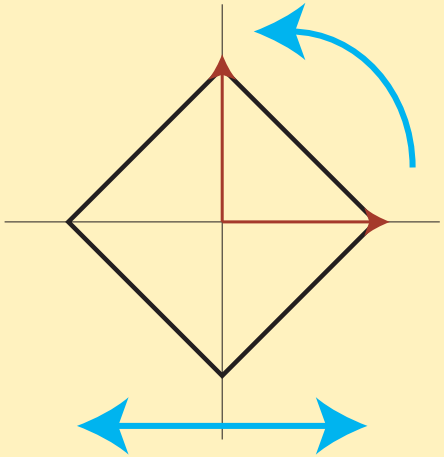

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The group $\mathcal{G} = \{$ all symmetries of the square $\} \subseteq \mathrm{GL}_2(\mathbb{C})$ has order

$$|\mathcal{G}| = 4 \cdot 2 = 2^n \, n! \; .$$

Basic Q$^{\mathbf{n}}$: Given n, what are the possible orders of finite subgroups of $\mathrm{GL}_n(\mathbb{C})$?

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Answer: They can be anything you want! Indeed, pick s and form the scalar matrix

$$A = \begin{pmatrix} e^{2\pi i/s} & & \\ & \ddots & \\ & & e^{2\pi i/s} \end{pmatrix}_{n\times n}$$

Then $\mathcal{G} = \{\mathbf{1}_{n\times n}, A, A^2, \ldots A^{s-1}\}$ is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ of order s.

Modified Q$^{\mathbf{n}}$: What about $\mathrm{GL}_n(\mathbb{Q})$ instead?

Minkowski's Theorem

Hermann Minkowski
1864 – 1909

- "geometry of numbers"
- relativity: "space-time"
- quadratic forms

Minkowski's Theorem

Minkowski (1887): *The least common multiple of the orders of all finite subgroups of* $\mathrm{GL}_n(\mathbb{Q})$ *is given by*

$$M(n) = \prod_{\ell \text{ prime}} \ell^{\left\lfloor \frac{n}{\ell-1} \right\rfloor + \left\lfloor \frac{n}{\ell(\ell-1)} \right\rfloor + \left\lfloor \frac{n}{\ell^2(\ell-1)} \right\rfloor + \cdots}$$

Here $\lfloor \quad \rfloor$ is the greatest integer ("floor") function

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The ℓ-factor for $\ell > n+1$ equals 1; so the product is finite.
E.g.,

$$M(1) = 2^1 = 2 \qquad M(2) = 2^{2+1}\,3^1 = 24$$
$$M(3) = 2^{3+1}\,3^1 = 48 \qquad M(4) = 2^{4+2+1}\,3^2\,5^1 = 5760$$

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Recursion: $M(2n+1) = 2\,M(2n)$

$$M(2n) = 2\,M(2n-1) \prod_{\ell \text{ prime},\ \ell-1 \mid 2n} \ell n_\ell$$

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$M(2n) = 2\,M(2n-1)$ $\underbrace{\prod_{\ell \text{ prime},\ \ell-1 \mid 2n} \ell^{n_\ell}}_{= \text{denominator of } \frac{B_{2n}}{n}}$

Bernoulli numbers: $\frac{x}{e^x-1} = \sum_{n \geq 0} \frac{B_n x^n}{n!}$

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Using the identity $(m!)_\ell = \ell^{\left\lfloor \frac{m}{\ell} \right\rfloor + \left\lfloor \frac{m}{\ell^2} \right\rfloor + \cdots}$, one can rewrite the ℓ-factor:

ℓ-part

$$M(n)_\ell = \ell^{\left\lfloor \frac{n}{\ell-1} \right\rfloor} \left(\left\lfloor \frac{n}{\ell-1} \right\rfloor ! \right)_\ell$$

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More on the Minkowski numbers $M(n)$ later ...

Example: $\mathrm{GL}_n(\mathbb{Q})$ contains the subgroup of "monomial matrices": exactly one entry ± 1 in each row and column, all other entries are 0

$$\mathbf{O}_n(\mathbb{Z}) = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \rtimes S_n$$

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This group has order $|\mathbf{O}_n(\mathbb{Z})| = 2^n\, n!$ $\quad\rightsquigarrow\quad$ $|\mathbf{O}_n(\mathbb{Z})|_2 = M(n)_2$

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Similarly: for each prime ℓ, there is a subgroup $\mathcal{G} \subseteq \mathrm{GL}_n(\mathbb{Q})$ such that $|\mathcal{G}|_\ell = M(n)_\ell$.

Hence, the main content of Minkowski's Theorem is:

> *The order of any finite subgroup of* $\mathrm{GL}_n(\mathbb{Q})$ *divides* $M(n)$

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How does one go about proving this?

Idea of proof

Given: $\mathcal{G}$ a finite subgp. of $\mathrm{GL}_n(\mathbb{Q})$
ℓ a prime

Want: $|\mathcal{G}|_\ell$ divides $M(n)_\ell$

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Step 1 (group theory & linear algebra): After a base change, $\mathcal{G} \subseteq \mathrm{GL}_n(\mathbb{Z})$ (!)

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Step 1 (group theory & linear algebra): After a base change, $\mathcal{G} \subseteq \mathrm{GL}_n(\mathbb{Z})$ (!) So we can reduce entries modulo another prime p :

$$\begin{array}{ccc} \mathrm{GL}_n(\mathbb{Z}) & \longrightarrow & \mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z}) \\ \cup & & \cup \\ A = (a_{ij}) & \mapsto & \overline{A} = (a_{ij} \bmod p) \end{array}$$

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As long as $p \neq 2$, this map is **1-to-1** on $\mathcal{G}$ (!) Hence $|\mathcal{G}| \leq |\mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})|$.

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As long as $p \neq 2$, this map is **1-to-1** on $\mathcal{G}$ (!) Hence $|\mathcal{G}| \leq |\mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})|$. In fact, by Lagrange's Theorem (!)

$$|\mathcal{G}| \textbf{ divides } |\mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})|$$

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Counting bases of n-space over $\mathbb{Z}/p\mathbb{Z}$ one obtains

$$|\,\mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})| = (p^n-1)(p^n-p)(p^n-p^2)\ldots(p^n-p^{n-1})$$
$$= p^{n(n-1)/2}\prod_{i=1}^{n}(p^i-1)$$

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To summarize: if $p \neq 2$ then the order $|\mathcal{G}|$ divides this number; so

$$p \neq 2, p \neq \ell \quad \Rightarrow \quad |\mathcal{G}|_\ell \;\Big|\; \textstyle\prod_{i-1}^{n}(p^i-1)_\ell$$

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$$\therefore \qquad |\mathcal{G}|_\ell \;\Big|\; M(n)_\ell \qquad \text{at least for } \ell \neq 2$$

The prime $\ell = 2$ requires more work, as usual ...

Part III: Two Mysteries

Entering the first six terms of $M(n)$,

$$2, 24, 48, 5760, 11520, 2903040$$

into the *On-Line Encyclopedia of Integer Sequences* by Neil Sloane (AT&T) brings up sequence **A053657**.

A053657 has two alternative descriptions . . .

Sequence $M(n)$ – First Alternative

- **Integer-valued polynomials on primes:** Consider all polynomials $f(x) \in \mathbb{Q}[x]$ of degree $\leq n$ such that $f(p) \in \mathbb{Z}$ for all primes p.

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Minkowski's formula is identical with the one proved independently for $a(n+1)$ by Chabert et. al.; so

$$a(n+1) = M(n)$$

Reference: Jean-Luc Chabert, Scott T. Chapman, and William W. Smith, *A basis for the ring of polynomials integer-valued on prime numbers,* Factorization in integral domains (Iowa City, IA, 1996), Lecture Notes in Pure and Appl. Math., vol. 189, Dekker, New York, 1997, pp. 271–284.

Sequence $M(n)$ – Second Alternative

- **Logarithm power expansion**: Let $P(n, z) \in \mathbb{Q}[z]$ be the coefficient of x^n in the Taylor series for $\left(\frac{-\ln(1-x)}{x}\right)^z$.

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In detail: put $\xi = \frac{-\ln(1-x)}{x} - 1 = \sum_{k=1}^{\infty} \frac{x^k}{k+1}$ to get

$$\left(\frac{-\ln(1-x)}{x}\right)^z = (1+\xi)^z = \sum_{m\geq 0} \binom{z}{m} \xi^m = 1 + \sum_{n\geq 1} P(n,z)x^n$$

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Examples:

$$P(1,z) = \frac{z}{2} \qquad P(2,z) = \frac{5z+3z^2}{24}$$

$$P(3,z) = \frac{6z+5z^2+z^3}{48} \qquad P(4,z) = \frac{502z+485z^2+150z^3+15z^4}{5760}$$

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Putting $b(n) =$ denominator of $P(n, z)$, it has recently been proved that

$$b(n) = a(n+1)$$

So $b(n) = M(n)$.

Reference: Jean-Luc Chabert, *Integer-valued polynomials on prime numbers and logarithm power expansion*, European Journal of Combinatorics **28** (2007), 754–761.

Coincidence ?

. . . if time

Large matrix groups (again)

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Feit (unpubl., $\approx$ 1998):

Walter Feit
1930 – 2004

$|\mathbf{O}_n(\mathbb{Z})| = 2^n n!$ is the largest order of any finite subgroup of $\mathrm{GL}_n(\mathbb{Q})$ for $n > 10$ and $n = 1, 3, 5$. Moreover, $\mathbf{O}_n(\mathbb{Z})$ is the unique subgroup of that order, up to conjugacy.

Large matrix groups (again)

Feit relies on an unfinished manuscript of **Weisfeiler** on the "Jordan bound" (based on the Enormous Theorem).

Boris Weisfeiler has been missing in Chile since January 4, 1985 (`http://boris.weisfeiler.com`)

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Weisfeiler's work has now been completed (and improved) by **Michael Collins** (preprints, Oxford University, 2005)