Multiplicative Invariant Theory

Séminaire AGATA Montpellier 06/08/2006
1st talk

Martin Lorenz

Temple University
Philadelphia
Multiplicative invariants: definitions, examples, historical roots, . . .
Overview

- **Multiplicative invariants**: definitions, examples, historical roots, . . .

- **Cohen-Macaulay rings**: a quick introduction and some recent results on multiplicative invariants
Overview

- **Multiplicative invariants**: definitions, examples, historical roots, . . .

- **Cohen-Macaulay rings**: a quick introduction and some recent results on multiplicative invariants

- **Some open problems**
Given: a group $G$ and a $G$-lattice $L \cong \mathbb{Z}^n$; so

$$G \rightarrow \text{GL}(L) \cong \text{GL}_n(\mathbb{Z})$$

an integral representation of $G$
Given: a group $G$ and a $G$-lattice $L \cong \mathbb{Z}^n$; so

$$G \rightarrow GL(L) \cong GL_n(\mathbb{Z})$$

Choose a base ring $\mathbb{k}$ and form the group algebra

$$\mathbb{k}[L] = \bigoplus_{m \in L} \mathbb{k}x^m \cong \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

$$x^m x^{m'} = x^{m+m'}$$

The $G$-action on $L$ extends uniquely to a “multiplicative” action by $\mathbb{k}$-algebra automorphisms on $\mathbb{k}[L]$:

$$g(x^m) = x^{g(m)} \quad (g \in G, m \in L)$$
Given: a group $G$ and a $G$-lattice $L \cong \mathbb{Z}^n$; so

$$G \to \text{GL}(L) \cong \text{GL}_n(\mathbb{Z})$$

Choose a base ring $k$ and form the group algebra

$$k[L] = \bigoplus_{m \in L} kx^m \cong k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], \quad x^m x^{m'} = x^{m+m'}$$

The $G$-action on $L$ extends uniquely to a “multiplicative” action by $k$-algebra automorphisms on $k[L]$.

The multiplicative invariant algebra is

$$k[L]^G = \{ f \in k[L] \mid g(f) = f \ \forall g \in G \}$$
Multiplicative invariants of the standard permutation lattice:

\( S_n \) is the symmetric group

\[ U_n = \bigoplus_1^n \mathbb{Z} e_i \cong \mathbb{Z}^n \]

action: \( \sigma(e_i) = e_{\sigma(i)} \) (\( \sigma \in S_n \))
Example #1

**Multiplicative invariants of the standard permutation lattice:**

$S_n$ is the symmetric group

$U_n = \bigoplus_1^n \mathbb{Z}e_i \cong \mathbb{Z}^n$

action: $\sigma(e_i) = e_{\sigma(i)}$ ($\sigma \in S_n$)

Put $x_i = x^{e_i} \in \mathbb{k}[U_n]$; so $\sigma(x_i) = x_{\sigma(i)}$ for $\sigma \in S_n$. Then

$$\mathbb{k}[U_n] = \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = \mathbb{k}[x_1, \ldots, x_n][s_n^{-1}]$$

where $s_n = \prod_1^n x_i$ is the $n^{th}$ elementary symmetric polynomial.
Example #1

Multiplicative invariants of the standard permutation lattice:

\[ S_n \text{ is the symmetric group} \]
\[ U_n = \bigoplus_1^n \mathbb{Z} e_i \cong \mathbb{Z}^n \]

action: \( \sigma(e_i) = e_{\sigma(i)} \) (\( \sigma \in S_n \))

\[ \therefore \quad k[U_n]^{S_n} = k[x_1, \ldots, x_n][s_{n-1}^{-1}]^{S_n} \]
\[ = k[x_1, \ldots, x_n]^{S_n}[s_{n-1}^{-1}] \]
\[ = k[s_1, \ldots, s_{n-1}, s_{n-1}^\pm] \]
\[ \cong k[\mathbb{Z}_+^{n-1} \oplus \mathbb{Z}] \]

elem. symmetric poly’s
Back to general multiplicative actions:

\[ L \quad \text{a } G\text{-lattice} \]
\[ k \quad \text{a commutative base ring} \]
\[ k[L] \quad \text{the group algebra} \]
$\mathbb{Z}$-structure:

$$k[L]^G = k \otimes \mathbb{Z}[L]^G$$
It suffices to consider **finite** groups:
Put

\[ L_{\text{fin}} = \{ m \in L \mid [G : G_m] < \infty \} . \]

\textbf{stabilizer of } m \in L

\( G \) acts on \( L_{\text{fin}} \) through the finite quotient \( G = G/\text{Ker}_G(L_{\text{fin}}) \),

and

\[ \mathbb{k}[L]^G = \mathbb{k}[L_{\text{fin}}]^G \]
In particular, $k[L]^G$ is always affine over $k$ (Hilbert # 14 ok).
In general, $k[L]$ has no grading (connected) that is preserved by the action of $G$.

\[ \Rightarrow \] computational theory not yet highly developed

\[ \exists \text{ some GAP & MAGMA-programs (L., Marc Renault)} \]
Example #2

The “weight lattice” \( A_{n-1}^* = U_n / \mathbb{Z}(e_1 + \cdots + e_n) \cong \mathbb{Z}^{n-1} \)

\((S_n \text{ and } U_n = \bigoplus_1^n \mathbb{Z}e_i \text{ as in Example #1})\)
Example #2

The “weight lattice” $A^*_{n-1} = U_n / \mathbb{Z}(e_1 + \cdots + e_n) \cong \mathbb{Z}^{n-1}$

($S_n$ and $U_n = \bigoplus_1^n \mathbb{Z}e_i$ as in Example #1)

So

$U_n \rightarrow A^*_{n-1} \quad \leadsto \quad \mathbb{k}[U_n] \rightarrow \mathbb{k}[A^*_{n-1}]$

$\leadsto \quad \mathbb{k}[U_n]S_n \rightarrow \mathbb{k}[A^*_{n-1}]S_n$
Example #2

The “weight lattice” 

\[ A^*_n = U_n / \mathbb{Z}(e_1 + \cdots + e_n) \cong \mathbb{Z}^{n-1} \]

\[(S_n \text{ and } U_n = \bigoplus_1^n \mathbb{Z} e_i \text{ as in Example #1})\]

So

\[ U_n \rightarrow A^*_n \rightarrow \mathbb{k}[U_n] \rightarrow \mathbb{k}[A^*_n] \]
\[ \cong \mathbb{k}[U_n] S_n \rightarrow \mathbb{k}[A^*_n] S_n \]

Under the last map, \( x^{e_1 + \cdots + e_n} = s_n \rightarrow 1 \).
The “weight lattice” $A^*_{n-1} = U_n / \mathbb{Z}(e_1 + \cdots + e_n) \cong \mathbb{Z}^{n-1}$

(S_n and $U_n = \bigoplus_1^n \mathbb{Z}e_i$ as in Example #1)

So

$$U_n \to A^*_{n-1} \rightsquigarrow \mathbb{k}[U_n] \to \mathbb{k}[A^*_{n-1}]$$

$$\rightsquigarrow \mathbb{k}[U_n]S_n \to \mathbb{k}[A^*_{n-1}]S_n$$

Under the last map, $x^{e_1 + \cdots + e_n} = s_n \mapsto 1$.

$$\therefore \quad \mathbb{k}[A^*_{n-1}]S_n \cong \mathbb{k}[U_n]S_n/(s_n - 1)$$

$$= \mathbb{k}[s_1, \ldots, s_{n-1}, s_n^{\pm 1}]/(s_n - 1)$$

$$\cong \mathbb{k}[s_1, \ldots, s_{n-1}] \cong \mathbb{k}[\mathbb{Z}^{n-1}]$$
The root lattice $A_{n-1} = \{ \sum_i z_i e_i \in U_n \mid \sum_i z_i = 0 \} \cong \mathbb{Z}^{n-1}$ (notation as before)
Example #3

The root lattice \( A_{n-1} = \{ \sum_i z_i e_i \in U_n \mid \sum_i z_i = 0 \} \simeq \mathbb{Z}^{n-1} \)
(notation as before)

Here,

\[ \mathbb{k}[A_{n-1}] = \mathbb{k}[U_n]_0, \]

the degree 0-component for the \((S_n\text{-stable})\) “total degree” grading of \( \mathbb{k}[U_n] = \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \).
Example #3

The root lattice $A_{n-1} = \{ \sum_i z_i e_i \in U_n \mid \sum_i z_i = 0 \} \cong \mathbb{Z}^{n-1}$ (notation as before)

Get

$$\mathbb{k}[A_{n-1}]^{S_n} = \mathbb{k}[U_n]^{0} \cong \mathbb{k}[M]$$

with

$$M = \{ (t_1, \ldots, t_{n-1}) \in \mathbb{Z}_+^{n-1} \mid \sum i t_i \in n\mathbb{Z} \} ,$$

a submonoid of $\mathbb{Z}_+^{n-1}$. 
Example #3

The root lattice $A_{n-1} = \{ \sum_{i} z_i e_i \in U_n \mid \sum_{i} z_i = 0 \} \cong \mathbb{Z}^{n-1}$

(notation as before)

$\mathbb{C}[A_{n-1}]^{S_n}$ is not regular:

($n > 2$; picture for $n = 3$)
Two $G$-lattices $L$ and $L'$ are called \textbf{rationally isomorphic} if $L \otimes_{\mathbb{Z}} \mathbb{Q} \cong L' \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Q}[G]$-modules.
Two $G$-lattices $L$ and $L'$ are called rationally isomorphic if $L \otimes \mathbb{Z} \mathbb{Q} \cong L' \otimes \mathbb{Z} \mathbb{Q}$ as $\mathbb{Q}[G]$-modules.

Rationally isomorphic lattices can have very different multiplicative invariant algebras . . .
Two $G$-lattices $L$ and $L'$ are called rationally isomorphic if $L \otimes \mathbb{Z} \mathbb{Q} \cong L' \otimes \mathbb{Z} \mathbb{Q}$ as $\mathbb{Q}[G]$-modules.

**Example:** The $S_n$-lattices $A^*_{n-1}$ and $A_{n-1}$ are rationally isomorphic:

$$A^*_{n-1} \otimes \mathbb{Z} \mathbb{Q} \cong A_{n-1} \otimes \mathbb{Z} \mathbb{Q} \cong S^{(n-1,1)}$$

the Specht module for the partition $(n - 1, 1)$ of $n$. Yet,

- $\mathbb{Z}[A^*_{n-1}]^{S_n}$ is a polynomial ring over $\mathbb{Z}$
- $\mathbb{Z}[A_{n-1}]^{S_n}$ is not regular and not a UFD for $n > 2$. 

Montpellier 06/08/2006: Multiplicative Invariant Theory – p. 8/19

\[ R(\mathfrak{g}) \cong \mathbb{Z}[\Lambda]^W \cong \mathbb{Z}[x_1, \ldots, x_{\text{rank } \mathfrak{g}}] \]

where \( R(\mathfrak{g}) = \) representation ring of a semisimple Lie algebra \( \mathfrak{g} \), \( \Lambda = \) weight lattice of \( \mathfrak{g} \), and \( W = \) Weyl group.

\[ R(g) \cong \mathbb{Z}[\Lambda]^W \cong \mathbb{Z}[x_1, \ldots, x_{\text{rank } g}] \]

where \( R(g) \) = representation ring of a semisimple Lie algebra \( g \), \( \Lambda \) = weight lattice of \( g \), and \( W \) = Weyl group.

Steinberg, Richardson (1970s)

\[
R(\mathfrak{g}) \cong \mathbb{Z}[\Lambda]^W \cong \mathbb{Z}[x_1, \ldots, x_{\text{rank } \mathfrak{g}}]
\]

where \( R(\mathfrak{g}) \) = representation ring of a semisimple Lie algebra \( \mathfrak{g} \), \( \Lambda \) = weight lattice of \( \mathfrak{g} \), and \( W \) = Weyl group.

Steinberg, Richardson (1970s)

"\( \Delta \)-methods" for group rings: Passman, Zalesskii, Roseblade, Dan Farkas \( \rightsquigarrow \) “multiplicative invariants” (mid 1980s)
Jordan (1880): $\text{GL}_n(\mathbb{Z})$ has only finitely many finite subgroups up to conjugacy.
Jordan (1880): $\text{GL}_n(\mathbb{Z})$ has only finitely many finite subgroups up to conjugacy.

\[ \therefore \text{there are only finitely many multiplicative invariant algebras } \mathbb{k}[L]^G \text{ (up to } \cong) \text{ with rank } L \text{ bounded} \]
## Finite Linear Groups

<table>
<thead>
<tr>
<th>$n$</th>
<th># fin. $G \leq \text{GL}_n(\mathbb{Z})$ (up to conj.)</th>
<th># max' $G$ (up to conj.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>13</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>73</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>710</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>6079</td>
<td>17</td>
</tr>
<tr>
<td>6</td>
<td>85311</td>
<td>39</td>
</tr>
</tbody>
</table>
More on finite linear groups in the second talk . . .
Next: Cohen-Macaulay (CM) rings
Hypotheses:

- $R$ a comm. noetherian ring
- $\alpha$ an ideal of $R$
Cohen-Macaulay Rings

- **Hypotheses:**
  - $R$ a comm. noetherian ring
  - $\mathfrak{a}$ an ideal of $R$

- **Always:**

  \[
  \text{height } \mathfrak{a} \geq \text{depth } \mathfrak{a} = \inf \{i \mid H^i_{\mathfrak{a}}(R) \neq 0\}
  \]
Cohen-Macaulay Rings

- **Hypotheses:**
  - $R$ a comm. noetherian ring
  - $\alpha$ an ideal of $R$

- **Always:**
  - $\text{height} \, \alpha \geq \text{depth} \, \alpha = \inf \{i \mid H^i_{\alpha}(R) \neq 0\}$

- **Def:** $R$ is **Cohen-Macaulay** iff equality holds for all (maximal) ideals $\alpha$.

(Zariski) topology
dimension theory

(homological) algebra
Some Examples of CM Rings

- **Standard example:** $R$ an affine domain/PID $\mathbb{k}$, finite / some polynomial subalgebra $P = \mathbb{k}[x_1, \ldots, x_n]$. Then:

$$R \text{ CM } \iff R \text{ is free over } P$$
Some Examples of CM Rings

- **Standard example:** \( R \) an affine domain/PID \( \mathbb{k} \), finite / some polynomial subalgebra \( P = \mathbb{k}[x_1, \ldots, x_n] \). Then:

\[
R \text{ CM } \iff R \text{ is free over } P
\]

- **Hierarchy:**

```
catenary
regular \rightarrow complete \cap \rightarrow Gorenstein
\rightarrow CM
```

- dim 0
- dim 1 reduced
- dim 2 normal
Hypotheses: $R$ a CM ring
$G$ a finite group acting on $R$
Invariant Rings

Hypotheses: \( R \) a CM ring
\( G \) a finite group acting on \( R \)

If the \textbf{trace map} \( R \to R^G, r \mapsto \sum_G g(r) \), is epi ("non-modular case") then \( R^G \) is CM; otherwise usually not.
Invariant Rings

Hypotheses: \( R \) a CM ring
\( G \) a finite group acting on \( R \)

If the trace map \( R \rightarrow R^G, r \mapsto \sum_{g} g(r) \), is epi ("non-modular case") then \( R^G \) is CM; otherwise usually not.

Here is a necessary condition . . .
Invariant Rings

Hypotheses:

- $R$ a CM ring
- $G$ a finite group acting on $R$
- $R_k = \{ k\text{-reflections on } R \}$
- Assume $R$ noetherian

Automorphisms belonging to the inertia group of some prime of height $\leq k$

Theorem 1 (L. Pathak)

If $R^G$ CM & $H^i(G, R) = 0 \ (0 < i < k)$ then

$$\text{res: } H^k(G, R) \rightarrow \prod_{\mathcal{H} \subseteq R_{k+1}} H^k(\mathcal{H}, R)$$
Invariant Rings

Hypotheses:  

\( R \) a CM ring  
\( G \) a finite group acting on \( R \)  
\( \mathcal{R}_k = \{ k\text{-reflections on } R \} \)  
Assume \( R \) noetherian \( /R^G \)

**Theorem 1**  
(L. - Pathak)

If \( R^G \) CM & \( H^i(G, R) = 0 \) \( (0 < i < k) \) then

\[ \text{res}: H^k(G, R) \twoheadrightarrow \prod_{\mathcal{H} \subseteq \mathcal{R}_{k+1}} H^k(\mathcal{H}, R) \]

Note:  
The \((H^i = 0)\)-cond\(^n\) is vacuous for \( k = 1 \)  
\( \leadsto \) bi-reflections.
Main ingredients of proof:

- spectral sequences by Ellingsrud & Skjelbred:

\[
E_2^{p,q} = H^p_\alpha(H^q(G, M)) \quad \Rightarrow \quad H^{p+q}_\alpha(G, M)
\]

\[
E_2^{p,q} = H^p(G, H^q_\alpha(M)) \quad \Rightarrow \quad H^{p+q}_\alpha(G, M)
\]

- calculation of the closed set in \( \text{Spec} \, R^G \) determined by the image of transfer \( R^\mathcal{H} \to R^G \) for \( \mathcal{H} \leq G \).
Notations: \( G \) is a finite group \( \neq 1 \)
\( L \) a \( G \)-lattice, WLOG faithful
Notations: \( G \) is a finite group \( \neq 1 \)
\( L \) a \( G \)-lattice, WLOG faithful

So \( G \hookrightarrow \text{GL}(L), g \mapsto g_L \). In this setting,

\[
g \in G \text{ is a } k\text{-reflection on } k[L] \iff \text{rank}(g_L - \text{Id}_L) \leq k
\]

"\( g \) is a \( k \)-reflection on \( L \)" (or on \( L \otimes_\mathbb{Z} \mathbb{Q} \))
Multiplicative Invariants: CM-property

**Notations:**

\[ G \] is a finite group \( \neq 1 \)

\[ L \] a \( G \)-lattice, WLOG faithful

Decoding Theorem 1 and mixing it with some representation theory, notably a result of **Zassenhaus** on fixed-point-free complex representations of perfect groups, we obtain . . .
Notations: \( G \) is a finite group \( \neq 1 \)
\( L \) a \( G \)-lattice, WLOG faithful

**Theorem 2**

(L, Trans AMS ’06)

Assume that \( \mathbb{Z}[L]^G \) is CM. Then all \( G_m/\mathcal{R}^2(G_m) \) are perfect groups, but not all \( G_m \) are.

stabilizer of \( m \in L \)

subgroup gen. by bireflections on \( L \)
Notations: \( G \) is a finite group \( \neq 1 \)
\( L \) a \( G \)-lattice, WLOG faithful

**Theorem 2**

Assume that \( \mathbb{Z}[L]^G \) is CM. Then all \( G_m/R^2(G_m) \)
are perfect groups, but not all \( G_m \) are.

**Corollary** ("3-copies conjecture") \( \mathbb{Z}[L^r]^G \) is never CM
for \( r \geq 3 \).
Notations: \( G \) is a finite group \( \neq 1 \)
\( L \) a \( G \)-lattice, WLOG faithful

Theorem 2  
(L, Trans AMS ’06)

Assume that \( \mathbb{Z}[L]^G \) is CM. Then all \( G_m/R^2(G_m) \) are perfect groups, but not all \( G_m \) are.

Note that the conclusions of Theorem 2 only refer to the \textit{rational} type of \( L \). In fact . . .
**Notations:**

- $\mathcal{G}$ is a finite group $\neq 1$
- $L$ a $\mathcal{G}$-lattice, WLOG faithful

---

**Theorem 2**  
*(L, Trans AMS ’06)*

*Assume that $\mathbb{Z}[L]^\mathcal{G}$ is CM. Then all $\mathcal{G}_m/\mathcal{R}^2(\mathcal{G}_m)$ are perfect groups, but not all $\mathcal{G}_m$ are.*

---

**Proposition**

*If $k[L]^\mathcal{G}$ is CM then so is $k[L']^\mathcal{G}$ for any $\mathcal{G}$-lattice $L'$ rationally isomorphic to $L$.*
Example: $S_n$-lattices

What are the $S_n$-lattices $L$ such that $\mathbb{Z}[L]^{S_n}$ is CM?
Example: \(S_n\)-lattices

We know:

- only the structure of \(L_\mathbb{Q} = L \otimes_{\mathbb{Z}} \mathbb{Q}\) matters (Proposition)

- \(S_n\) must act as a bireflection group on \(L\) (Theorem 2), and hence on all simple constituents of \(L_\mathbb{Q}\)
Classification results of irreducible finite linear groups containing a bireflection (Huffman and Wales, 70s) imply, for \( n \geq 7 \):

\[
L_\mathbb{Q} \cong \mathbb{Q}^r \oplus (\mathbb{Q}^-)^s \oplus (A_{n-1})^t_{\mathbb{Q}} \quad (s + t \leq 2)
\]

sign representation of \( S_n \)
Example: $S_n$-lattices

In all cases, $\mathbb{Z}[L]^{S_n}$ is indeed CM, with the possible exception of

$$L = A_{n-1}^2$$

This case reduces to

<table>
<thead>
<tr>
<th>Problem</th>
<th>Are the &quot;vector invariants&quot; $\mathbb{F}_p[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n}$ CM?</th>
</tr>
</thead>
<tbody>
<tr>
<td>(open for $p \leq n/2$)</td>
<td></td>
</tr>
</tbody>
</table>
Let $L$ be a $G$-lattice, where $G$ is a finite group.
Example #4

Multiplicative inversion in rank 2: 
($k = \mathbb{Z}$)

$C_2 = \langle g \mid g^2 = 1 \rangle$
$L_2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$
action: $g(e_i) = -e_i$
Example #4

**Multiplicative inversion in rank 2:**

\((k = \mathbb{Z})\)

\[C_2 = \langle g \mid g^2 = 1 \rangle\]

\[L_2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2\]

**action:** \(g(e_i) = -e_i\)

So:

\[\mathbb{Z}[L_2] = \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}] \text{ with } g(x_i) = x_i^{-1}\]
Example #4

Multiplicative inversion in rank 2: \( (k = \mathbb{Z}) \)

\( C_2 = \langle g \mid g^2 = 1 \rangle \)

\( L_2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \)

action: \( g(e_i) = -e_i \)

Straightforward calculation gives

\[
\mathbb{Z}[L_2]^{C_2} = \mathbb{Z}[\xi_1, \xi_2] \oplus \eta \mathbb{Z}[\xi_1, \xi_2]
\]

where \( \xi_i = x_i + x_i^{-1} \) and \( \eta = x_1x_2 + x_1^{-1}x_2^{-1} \)
Multiplicative inversion in rank 2: \( (k = \mathbb{Z}) \)

\[ C_2 = \langle g \mid g^2 = 1 \rangle \]

\[ L_2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \]

Action: \( g(e_i) = -e_i \)

One observes \( \eta \xi_1 \xi_2 = \eta^2 + \xi_1^2 + \xi_2^2 - 4 \). Hence,

\[ \mathbb{Z}[L_2]^C_2 \cong \mathbb{Z}[x, y, z]/(x^2 + y^2 + z^2 - xyz - 4) \].
Example #4

Multiplicative inversion in rank 2: $(k = \mathbb{Z})$

\[ C_2 = \langle g \mid g^2 = 1 \rangle \]
\[ L_2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \]

action: $g(e_i) = -e_i$

\[ \mathbb{Z}[L_2]^C_2 \] is not a semi-group algebra:
One more problem . . .

Linear groups

\[ G \subseteq GL(V) \text{ a finite linear group (} V \text{ some vector space).} \]

**Question**

If \( G \) is generated by bireflections, is this also true for all \( G_v \ (v \in V) \)?

The answer is “NO” for general \( k \)-reflections (Zalesskiï), but it is “YES” for reflections (\( k = 1 \); Steinberg, Serre).
One more problem ...