Koszul algebras and the “master theorem”

Algebra Seminar
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• A combinatorial introduction: MacMahon’s “Master Theorem”
Overview

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- Generalized Koszul algebras: definition and a quick overview of some constructions of Manin
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• Application: Master Theorems via Koszul duality
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- Generalized Koszul algebras: definition and a quick overview of some constructions of Manin
- Application: Master Theorems via Koszul duality
- Recent work: superization and return to combinatorics
Part I: the “Master Theorem”
Generalized Matching Problem

At a reception, everybody checks their hats. There are \( n \) different types of hats, say

\[ m_i = \# \text{ hats of type } i \quad (\text{indistinguishable}) \]

If hats are returned randomly afterwards, what is the probability that nobody ends up with the type of hat they came with?
Generalized Matching Problem

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\[
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\]

If hats are returned randomly afterwards, what is the probability that nobody ends up with the type of hat they came with?

The original version had all \( m_i = 1 \); it was solved by Montmort (1713).
Reformulation: Suppose $I = \bigcup_{\ell=1}^{n} I_\ell$ with $|I_\ell| = m_\ell$. Determine the number of permutations $\sigma \in S_I$ so that

$$\forall \ell : \quad \sigma I_\ell \cap I_\ell = \emptyset$$

divided by $m_1!m_2! \cdots m_n!$
Reformulation: Suppose $I = \bigsqcup_{\ell=1}^{n} I_\ell$ with $|I_\ell| = m_\ell$. Determine the number of permutations $\sigma \in \mathfrak{S}_I$ so that

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Answer (after MacMahon, *Combinatory Analysis*, 1917)

It is the coefficient of $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ in the expansion of the following polynomial in commuting variables:

$$(x_2 + x_3 + \cdots + x_n)^{m_1}(x_1 + x_3 + \cdots + x_n)^{m_2} \cdots (x_1 + \cdots + x_{n-1})^{m_n}$$
To solve the generalized matching problem and other combinatorial problems, MacMahon proved the following “Master Theorem”
MacMahon’s Master Theorem (original version, 1917)

Given a matrix $A = (a_{ij})_{n \times n}$ over some commutative ring $R$ and commuting indeterminates $x_1, \ldots, x_n$ over $R$. For each $(m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n$, the $R$-coefficient of $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ in

$$\left( \sum_{j=1}^{n} a_{1j} x_j \right)^{m_1} \left( \sum_{j=1}^{n} a_{2j} x_j \right)^{m_2} \ldots \left( \sum_{j=1}^{n} a_{nj} x_j \right)^{m_n}$$

is identical to the corresponding coefficient in the power series

$$\text{det} \left( 1_{n \times n} - A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right)^{-1}$$
Applying this to the **Matching Problem**, we obtain that the desired number is given by the coefficient of $x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n}$ in

$$\det \left( 1_{n \times n} - A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right)^{-1}$$

with

$$A = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}$$
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Calculation of the determinant gives

$$\det \left( 1_{n \times n} - A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) = 1 - e_2 - 2e_3 - \cdots - (n - 1)e_n$$

 elem. symmetric poly’s
in $x_1, \ldots, x_n$
MacMahon presents the theorem as a result in the theory of permutations:
The standard proof of the MT uses Lagrange inversion, and the result is often viewed in the analytic context:

Percy Alexander MacMahon
1854 - 1929

Title page of MacMahon’s book containing the “Master Theorem”
(originally published at Cambridge, 1917)
Andrews’ Problem:


5. MacMahon’s Master Theorem and the Dyson Conjecture.

PROBLEM 5. Are there q-analogs of MacMahon’s Master Theorem and the Dyson Conjecture?

First let us recall:

MacMahon’s Master Theorem (MacMahon (1894), (1915)). The coefficient of $X_1^{P_1}X_2^{P_2} \ldots X_n^{P_n}$ in
Noncommutative history

- First (somewhat) noncommutative version of MT in Cartier & Foata
  \[Problèmes combinatoires de commutation et réarrangements,\]
  SLN # 85 (1969)

- Garoufalidis, Lê and Zeilberger, The quantum MacMahon master theorem
  arXiv: math.QA/0303319

- Phùng Hô Hai and L.: qMT is a consequence of “Koszul duality” for quantum affine space
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- work of Foata & Han, Konvalinka & Pak, Etingof & Pak, …
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Koszul algebras and the “master theorem”
A considerable amount of research has been done by mathematical physicists on various quantum matrix identities; see, e.g.,

D. I. Gurevich, P. N. Pyatov, and P. A. Saponov:

*The Cayley-Hamilton theorem for quantum matrix algebras of GL(m|n) type*,


The techniques employed are quite different from ours.
Part II: Generalized Koszul algebras
Notation: $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$ graded $\mathbb{k}$-algebra
c connected: $\mathcal{A}_0 = \mathbb{k}$
$N$-Koszul algebras

**Notation:** $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$ graded $\mathbb{k}$-algebra

connected: $\mathcal{A}_0 = \mathbb{k}$

**Minimal presentation:** Write $\mathcal{A}_+ = \mathcal{A}_+^2 \oplus V$ for some graded subspace $V \subseteq \mathcal{A}_+ := \bigoplus_{n > 0} \mathcal{A}_n$ to get

$$T(V)/(R) \overset{\sim}{\rightarrow} \mathcal{A}$$

with a graded relation space $R \subseteq T(V) = \bigoplus_{n \geq 0} V^\otimes n$, chosen minimal: $(R) = R \oplus (V \otimes (R) + (R) \otimes V)$
**Notation:** $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$ graded $\mathbb{k}$-algebra connected: $\mathcal{A}_0 = \mathbb{k}$

For fixed $N \geq 2$ define the “jump function”

$$\nu_N(i) := \begin{cases} 
\frac{i}{2}N & \text{if } i \text{ is even} \\
\frac{i-1}{2}N + 1 & \text{if } i \text{ is odd}
\end{cases}$$

**Lemma:** The relations $R$ of $\mathcal{A}$ live in degrees $\geq N$ if and only if all $\text{Tor}^\mathcal{A}_i(\mathbb{k}, \mathbb{k})$ live in degrees $\geq \nu_N(i)$
**Notation:** \( \mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n \) graded \( k \)-algebra connected: \( \mathcal{A}_0 = k \)

**Definition:** The algebra \( \mathcal{A} \) is called \( N \)-Koszul if each \( \text{Tor}_i^\mathcal{A}(k, k) \) is concentrated in degree \( \nu_N(i) \)
\(N\)-Koszul algebras

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In this case, \( \mathcal{A} \) is \( N \)-homogeneous:

- \( V \) is concentrated in degree 1 \( (\text{b/c } \text{Tor}_1^\mathcal{A}(\mathbb{k}, \mathbb{k}) \cong V \text{ and } \nu_N(1) = 1) \)

- \( R \) is concentrated in degree \( N \) \( (\text{b/c } \text{Tor}_2^\mathcal{A}(\mathbb{k}, \mathbb{k}) \cong R \text{ and } \nu_N(2) = N) \)
Some background ($N = 2$)

- 2-Koszul ("Koszul") algebras were introduced by S. Priddy in connection with his investigation of Yoneda algebras $\text{Ext}_A(k, k)$.

Some background ($N = 2$)

- 2-Koszul (“Koszul”) algebras were introduced by S. Priddy in connection with his investigation of Yoneda algebras $\text{Ext}_A(k, k)$.


- Manin: 2-homogeneous (“quadratic”) algebras provide a convenient framework for the investigation of quantum group actions on noncommutative spaces.

Reference: Yu. I. Manin, *Quantum groups and noncommutative geometry*, Université de Montréal Centre de Recherches Mathématiques, Montreal, QC, 1988
• R. Berger: certain (Artin-Schelter) regular algebras are 3-Koszul, as are the “Yang-Mills algebras” defined by Connes and Dubois-Violette.

Some background (general $N \geq 2$)

- **R. Berger**: certain (Artin-Schelter) regular algebras are 3-Koszul, as are the “Yang-Mills algebras” defined by Connes and Dubois-Violette.


- Berger, Dubois-Violette and Wambst extend Manin’s theory to general $N$-homogeneous algebras

Manin’s constructions

... as extended to $N$-homogeneous algebras by Berger, Dubois-Violette and Wambst
Manin’s constructions

Notation:

\[ \mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n \] is \( N \)-homogeneous:

- generators \( V = \mathcal{A}_1 \)
- relations \( R \subseteq V \otimes N \)

\[ \mathcal{A} = A(V, R) \]
$\mathcal{A} = A(V, R) \leadsto \text{new } N\text{-homogeneous algebras:}$

- **dual algebra:** $\mathcal{A}^! = A(V^*, R^\perp)$

- **endomorphism bialgebra:**

$$\text{end } \mathcal{A} = \mathcal{A}^! \bullet \mathcal{A} = A(V^* \otimes V, \pi_N(R^\perp \otimes R))$$

$\text{End}(V): \text{matrices}$
Example: Quantum affine $n$-space ($N = 2$)

For fixed scalars $0 \neq q_{ij} \in k$ ($1 \leq i < j \leq n$), define

$$A_{q}^{n|0} := k \langle x_1, \ldots, x_n \rangle / (x_j x_i - q_{ij} x_i x_j \mid 1 \leq i < j \leq n)$$

Get $\left( A_{q}^{n|0} \right)! = A_{q}^{0|n}$ with generators $x^1, \ldots, x^n$ and relations

$$x^\ell x^\ell = 0, \ x^k x^\ell + q_{k\ell} x^\ell x^k = 0 \quad (k < \ell)$$
Example: Quantum affine $n$-space ($N = 2$)

The endomorphism “semigroup” $\text{end} A_q^{n|0}$ has generators $z^j_i := x^j \otimes x_i$ and relations

**column relations:** $z^\ell_j z^\ell_i = q_{ij} z^\ell_i z^\ell_j$ (all $\ell, i < j$)

**cross relations:** $q_{ij} z^k_i z^\ell_j - q_{k\ell} z^k_j z^k_i = z^k_j z^\ell_i - q_{ij} q_{k\ell} z^\ell_i z^k_j$ (for $i < j, k < \ell$)
For any $N$-homogeneous $\mathcal{A}$, there is an $N$-complex

$$\ldots \xrightarrow{d} \mathcal{A} \otimes \mathcal{A}_{i+1}^! \xrightarrow{d} \mathcal{A} \otimes \mathcal{A}_i^! \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{A} \rightarrow 0$$

The **Koszul complex** $K(\mathcal{A}) = \bigoplus_{n \geq 0} K(\mathcal{A})^n$ is the following contraction of this complex

$$\ldots \xrightarrow{d^{N-1}} \mathcal{A} \otimes \mathcal{A}_{N+1}^! \xrightarrow{d} \mathcal{A} \otimes \mathcal{A}_N^! \xrightarrow{d^{N-1}} \mathcal{A} \otimes \mathcal{A}_1^! \xrightarrow{d} \mathcal{A} \rightarrow 0$$
Theorem: (a) $A$ is $N$-Koszul
\[ \iff K(A) \text{ is exact in degrees } > 0 \]
\[ \iff \text{all } K(A)^n (n > 0) \text{ are exact} \]

(b) $K(A)$ and all $K(A)^n$ are complexes of end $A$-comodules

Part (a) is due to R. Berger, part (b) to Phùng Hồ Hai, Benoît Kriegk & L.
Part III: MT from Koszul duality
The remainder of this talk is based on


**Notation:**

- $\mathcal{B}$: some bialgebra over $\mathbb{k}$ (later: $\mathcal{B} = \text{end} \ A$)
- $R_{\mathcal{B}}$: Grothendieck ring of all $\mathcal{B}$-comodules that are finite-dimensional over $\mathbb{k}$
Notation:

- $B$ some bialgebra over $\mathbb{k}$ (later: $B = \text{end}_A$)
- $R_B$ Grothendieck ring of all $B$-comodules that are finite-dimensional/$\mathbb{k}$

In more detail:

- $B$-comodule $V \rightsquigarrow [V] \in R_B$
- $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ exact \(\rightsquigarrow [V] = [U] + [W] \) in $R_B$
- Multiplication in $R_B$ is given by the tensor product of $B$-comodules
Characters

**Defn:** Let $V$ be a $\mathcal{B}$-comodule; so have $\delta_V : V \to V \otimes \mathcal{B}$ . The **character** of $\chi_V$ is the image of $\delta_V$ under the map

$$\text{Hom}_k(V, V \otimes \mathcal{B}) \sim \text{End}_k(V) \otimes \mathcal{B} \xrightarrow{\text{trace} \otimes \text{Id}} k \otimes \mathcal{B} = \mathcal{B}$$
**Def**: Let $V$ be a $B$-comodule; so have $\delta_V : V \to V \otimes B$. The **character** of $\chi_V$ is the image of $\delta_V$ under the map

$$\text{Hom}_k(V, V \otimes B) \xrightarrow{\sim} \text{End}_k(V) \otimes B \xrightarrow{\text{trace} \otimes \text{Id}} k \otimes B = B$$

**Lemma**: There is a commutative diagram of ring maps
For any $N$-homogeneous algebra $\mathcal{A}$, define the following Poincaré series in $R_{\text{end}}\mathcal{A}[t]$

$$P_{\mathcal{A}}(t) = \sum_i [\mathcal{A}_i] t^i$$

and

$$P_{\mathcal{A}^\ast,N}(-t) = \sum_i (-1)^i [\mathcal{A}_N^\ast(i)] t^{\nu_N(i)}$$
For any $N$-homogeneous algebra $\mathcal{A}$, define the following Poincaré series in $R_{\text{end}} \mathcal{A}[t]$

$$P_{\mathcal{A}}(t) = \sum_i [A_i] t^i \quad \text{and} \quad P_{\mathcal{A}^*,N}(-t) = \sum_i (-1)^i [A_{\nu N}(i)] t^{\nu N}(i)$$

**Theorem:** If $\mathcal{A}$ is $N$-Koszul then, in $R_{\text{end}} \mathcal{A}[t]$,

$$P_{\mathcal{A}}(t) \cdot P_{\mathcal{A}^*,N}(-t) = 1$$
Transport the identity in the Theorem from $R_{\text{end}}A[t]$ to $\text{end} A[t]$ via characters

$\leadsto$ a MT for any $N$-Koszul algebra $A$
Transport the identity in the Theorem from $R_{\text{end}} \mathcal{A}[t]$ to $\text{end} \mathcal{A}[t]$ via characters

$\rightsquigarrow$ a MT for any $N$-Koszul algebra $\mathcal{A}$

This can be specialized to “easier” $\mathcal{B}[t]$ via algebra maps $\text{end} \mathcal{A} \to \mathcal{B}$
$N$-Koszul algebra $\mathcal{A}$ \rightarrow Koszul complex $K(\mathcal{A})$

MT-equation in $\text{end } \mathcal{A}[t]$ \leftarrow equation in $R_{\text{end } \mathcal{A}[t]}$

characters
Examples

- $\mathcal{A} = S(V)$
  affine $n$-space
  $\leadsto$ the original MT
  (MacMahon)
Examples

- $\mathcal{A} = S(V)$ affine $n$-space $\leadsto$ the original MT (MacMahon)
- $\mathcal{A} = A_q^{n|0}$ quantum $n$-space $\leadsto$ qMT (Garoufalidis, Lê and Zeilberger)
Examples

- \( \mathcal{A} = S(V) \)
  affine \( n \)-space \( \leadsto \) the original MT (MacMahon)

- \( \mathcal{A} = A_q^n \)
  quantum \( n \)-space \( \leadsto \) qMT (Garoufalidis, Lê and Zeilberger)

- \( \mathcal{A} = A(V, Y_N(V^\otimes N)) \)
  \( N \)-symmetric algebra (Berger) \( \leadsto \) \( N \)-MT (Etingof and Pak)

antisymmetrizer \( \in \mathbb{k}[\mathfrak{S}_N] \)
Part IV: Superization
This amounts to putting a \( \mathbb{Z}/2 \)-grading ("parity": \( \hat{v} \)) on the generating space \( V \) of \( \mathcal{A} = A(V, R) \) and requiring the relations \( R \subseteq V \otimes N \) to be parity-graded.

Manin’s constructions have to be carried out in the category of vector superspaces using the "rule of signs"

\[
c_{U,V} : U \otimes V \rightsquigarrow V \otimes U \ , \quad u \otimes v \mapsto (-1)^{\hat{u}\hat{v}} v \otimes u
\]
Superization

\[ A = A(V, R) \] is a superalgebra

\[ \text{end } A \] is a super bialgebra

\[ K(A) \] is a complex of super comodules/\text{end } A

have additional ring maps \( s\dim : R_{\text{end } A} \to \mathbb{Z} \) and \( \chi^s : R_{\text{end } A} \to \text{end } A : \)

\[
\begin{align*}
\text{sdim } V &= \dim V^0_0 - \dim V^1_1 \\
\text{strace } (F_{i}^{j}) &= \sum (-1)^{i} F_{i}^{j}
\end{align*}
\]

13.4. The relations for variables in our super-analogue are somewhat different from those studied in the literature (see e.g. [M3]). Note also that our super-determinant is different from the Berezinian [B] (see also [GGRW, M1]). We are somewhat puzzled by this and hope to obtain the "real" super-analogue in the future.

13.5. The relations studied in this paper always lead to quadratic algebras. While the deep reason lies in the Koszul duality, the fact that Koszulity can be extended to non-quadratic algebras is suggestive [Be]. The first such effort is made in [EP] where...
The superalgebra $S_N(V)$

Notation: $V \cong \mathbb{k}^{p|q}$: basis $x_1, \ldots, x_p$, $x_{p+1}, \ldots, x_{p+q}$

- even: $\hat{i} = \overline{0}$
- odd: $\hat{i} = \overline{1}$

$d = p + q$

assume $\text{char } \mathbb{k} = 0$
The superalgebra $S_N(V)$

**Defn:** The $N$-symmetric superalgebra $S_N(V)$ is generated by $x_1, \ldots, x_d$ subject to the relations

$$
\sum_{\sigma \in S_N} (\text{sgn } \sigma) (-1)^{\sum_{(r,s) \in \text{inv } \sigma} \hat{i}_r \hat{i}_s} x_{i_{\sigma^{-1}(1)}} \cdots x_{i_{\sigma^{-1}(N)}} = 0
$$

with $\text{inv } \sigma = \{(r, s) \mid r < s \text{ but } \sigma(r) > \sigma(s)\}$
**Defn:** The $N$-symmetric superalgebra $S_N(V)$ is generated by $x_1, \ldots, x_d$ subject to the relations

$$
\sum_{\sigma \in \mathfrak{S}_N} (\text{sgn } \sigma)(-1)^{\sum_{(r,s) \in \text{inv } \sigma} \hat{i}_r \hat{i}_s} x_{i_{\sigma^{-1}(1)}} \cdots x_{i_{\sigma^{-1}(N)}} = 0
$$

with $\text{inv } \sigma = \{(r, s) \mid r < s \text{ but } \sigma(r) > \sigma(s)\}$

**Example** $S_2(V) = S(V) \cong S(V_0) \otimes \Lambda(V_1)$ is supercommutative: $[v, v'] = vv' - (-1)^{\hat{v} \hat{v}'} v'v = 0$
MT for $A = S_N(V)$

- $A = S_N(V)$ is indeed $N$-Koszul
- $A_\ell$ has $k$-basis the monomials $x_i = x_{i_1}x_{i_2}\ldots x_{i_\ell}$ for sequences $i = (i_1,\ldots,i_\ell) \in \{1,\ldots,d\}^\ell$ satisfying:

  $i$ has no connected subsequence $(j_1,\ldots,j_N)$ with
  
  $j_1 < j_2 < \cdots < j_m \leq p < j_{m+1} \leq \cdots \leq j_N$ for some $m$

Denote this collection of sequences $i$ by

$$\Lambda(p|q, N)_\ell$$
MT for $\mathcal{A} = S_N(V)$

- end $\mathcal{A}$ is highly noncommutative (even for $N = 2$ and $V = V_0$).

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MT for $\mathcal{A} = S_N(V)$

- $\text{end} \mathcal{A}$ is highly noncommutative (even for $N = 2$ and $V = V_0$).

But: $\exists$ canonical

$$\text{end} \mathcal{A} \rightarrow \mathcal{O}(E(V)) = S(V^* \otimes V) = \mathbb{k}[x_{ij}^i \mid 1 \leq i, j \leq d]$$

$x^i \otimes x^j$; parity $\hat{i} + \hat{j}$ supercommute
MT for $A = S_N(V)$

- end $A$ is highly noncommutative (even for $N = 2$ and $V = V_{\overline{0}}$).

But: $\exists$ canonical

$$\text{end } A \to \mathcal{O}(E(V)) = S(V^* \otimes V) = \mathbb{k}[x_{ij}^i | 1 \leq i, j \leq d]$$

\[ \downarrow \]

generic supermatrix of type $p|q$:

$$X = (x_{ij}^i) = \begin{pmatrix} \text{even}_{p \times p} & \text{odd} \\ \text{odd} & \text{even}_{q \times q} \end{pmatrix}$$
Berezinian: Given a supermatrix

\[ \Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{(even, odd)} \]

over some supercommutative \( \mathbb{k}\)-superalgebra, one has:

- \( \Phi \) is invertible \( \iff \) \( A \) and \( D \) are invertible
- In this case, one defines

\[
\text{ber } \Phi = \det(A) \det(D - CA^{-1}B)^{-1} = \det(D)^{-1} \det(A - BD^{-1}C)
\]
Notation: Put

\[ y_i = \sum_j x_j \otimes x_i^j \in \mathcal{A} \otimes \mathcal{O}(E(V)) \]

In \( \mathcal{A}_\ell \otimes \mathcal{O}(E(V)) = \bigoplus_{i \in \Lambda(p|q,N)_\ell} x_i \otimes \mathcal{O}(E(V)) \) consider the elements

\[ y_i = x_i \otimes X(i) + \ldots \]

with

\[ X(i) \in \mathcal{O}(E(V))_\ell \]

“the coefficient”
Theorem: Let $X = (x^i_j)_{d \times d}$ be the generic supermatrix of type $p|q$. Then, in the power series ring $\mathbb{k}[x^i_j \mid \text{all } i, j]_{\mathbb{Z}[t]}$,\[
\sum_{\ell} \sum_{i \in \Lambda(p|q,N)_{\ell}} (-1)^i X(i) t^\ell \cdot \sum_{m \equiv 0,1 \mod N} (-1)^{m \mod N} e_m t^m = 1
\]

Here, $e_m$ is the $m^{th}$ elementary supersymmetric function:\[
\text{ber}(1 + tX) = \sum_{n \geq 0} e_n t^n
\]
MT for $A = S_N(V)$

- For $N = 2$ and $V$ pure even ($q = 0$), this is the original MacMahon MT.
- For general $N$ and $V$ pure even, we obtain the $N$-MT of Etingof and Pak.
In general, applying the counit to the MT for $\mathcal{A}$ yields the super Hilbert series

$$H^s_A(t) = \sum_{\ell \geq 0} \text{sdim } \mathcal{A}_\ell t^\ell$$
In general, applying the counit to the MT for $\mathcal{A}$ yields the super Hilbert series

$$H^s_{\mathcal{A}}(t) = \sum_{\ell \geq 0} \text{sdim } \mathcal{A}_\ell \, t^\ell$$

In our MT for $\mathcal{A} = S_N(V)$, we have $X \mapsto 1_{d \times d}$, all coefficients $X(i) \mapsto 1$ and

$$\text{ber}(1 + tX) \mapsto \text{ber}(1 + t1_{d \times d}) = (1 + t)^{p-q}$$

This gives …
Back to combinatorics: a binomial identity

\[
\sum_{\ell \geq 0} \left( \sum_{i \in \Lambda(p|q,N)_{\ell}} (-1)^{\hat{i}} \right) t^{\ell}
\]

\[
= \begin{cases} 
\sum_{m \equiv 0,1 \mod N} (-1)^m \mod N \left( \binom{p-q}{m} t^m \right)^{-1} & \text{if } p \geq q \\
\sum_{m \equiv 0,1 \mod N} (-1)^{\alpha_N(m)} \left( \binom{m+q-p-1}{q-p-1} t^m \right)^{-1} & \text{if } p < q
\end{cases}
\]

\[
= m - (m \mod N)
\]