ON THE COHEN-MACAULAY PROPERTY OF MULTIPLICATIVE INVARIANTS

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ABSTRACT. We investigate the Cohen-Macaulay property for rings of invariants under multiplicative actions of a finite group $G$. By definition, these are $G$-actions on Laurent polynomial algebras $\mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ that stabilize the multiplicative group consisting of all monomials in the variables $x_i$. For the most part, we concentrate on the case where the base ring $\mathbb{k}$ is $\mathbb{Z}$. Our main result states that if $G$ acts non-trivially and the invariant ring $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^G$ is Cohen-Macaulay then the abelianized isotropy groups $G_m^{\text{ab}}$ of all monomials $m$ are generated by the bireflections in $G_m$ and at least one $G_m^{\text{ab}}$ is non-trivial. As an application, we prove the multiplicative version of Kemper’s 3-copies conjecture.

INTRODUCTION

This article is a sequel to [LPk]. Unlike in [LPk], however, our focus will be specifically on multiplicative invariants. In detail, let $L \cong \mathbb{Z}^n$ denote a lattice on which a finite group $G$ acts by automorphisms. The $G$-action on $L$ extends uniquely to an action by $\mathbb{k}$-algebra automorphisms on the group algebra $\mathbb{k}[L] \cong \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ over any commutative base ring $\mathbb{k}$. We are interested in the question when the subalgebra $\mathbb{k}[L]^G$ consisting of all $G$-invariant elements of $\mathbb{k}[L]$ has the Cohen-Macaulay property. The reader is assumed to have some familiarity with Cohen-Macaulay rings; a good reference on this subject is [BH].

It is a standard fact that $\mathbb{k}[L]$ is Cohen-Macaulay precisely if $\mathbb{k}$ is. On the other hand, while $\mathbb{k}[L]^G$ can only be Cohen-Macaulay when $\mathbb{k}$ is so, the latter condition is far from sufficient and rather stringent additional conditions on the action of $G$ on $L$ are required to ensure that $\mathbb{k}[L]^G$ is Cohen-Macaulay. Remarkably, the question whether or not $\mathbb{k}[L]^G$ is Cohen-Macaulay, for any given base ring $\mathbb{k}$, depends only on the rational isomorphism class
of the lattice $L$, that is, the isomorphism class of $L \otimes \mathbb{Z} \mathbb{Q}$ as $\mathbb{Q}[G]$-module; see Proposition 3.4 below. This is in striking contrast with most other ring theoretic properties of $\mathbb{k}[L]^G$ (e.g., regularity, structure of the class group) which tend to be sensitive to the $\mathbb{Z}$-type of $L$. For an overview, see [L1].

We will largely concentrate on the case where the base ring $\mathbb{k}$ is $\mathbb{Z}$. This is justified in part by the fact that if $\mathbb{Z}[L]^G$ is Cohen-Macaulay then likewise is $\mathbb{k}[L]^G$ for any Cohen-Macaulay base ring $\mathbb{k}$ (Lemma 3.2). Assuming $\mathbb{Z}[L]^G$ to be Cohen-Macaulay, we aim to derive group theoretical consequences for the isotropy groups $G_m = \{ g \in G \mid g(m) = m \}$ with $m \in L$. An element $g \in G$ will be called a $k$-reflection on $L$ if the sublattice $[g, L] = \{ g(m) - m \mid m \in L \}$ of $L$ has rank at most $k$ or, equivalently, if the $g$-fixed points of the $\mathbb{Q}$-space $L \otimes \mathbb{Z} \mathbb{Q}$ have codimension at most $k$. As usual, $k$-reflections with $k = 1$ and $k = 2$ will be referred to as reflections and bireflections. For any subgroup $H \leq G$, we let $H^{(2)}$ denote the subgroup generated by the elements of $H$ that act as bireflections on $L$. Our main result now reads as follows.

**Theorem.** Assume that $\mathbb{Z}[L]^G$ is Cohen-Macaulay. Then $G_m/G_m^{(2)}$ is a perfect group (i.e., equal to its commutator subgroup) for all $m \in L$. If $G$ acts non-trivially on $L$ then some $G_m$ is non-perfect.

It would be interesting to determine if the conclusion of the theorem can be strengthened to the effect that all isotropy groups $G_m$ are in fact generated by bireflections on $L$. I do not know if, for the latter to occur, it is sufficient that $G$ is generated by bireflections. The corresponding fact for reflection groups is known to be true: if $G$ is generated by reflections on $L$ (or, equivalently, on $L \otimes \mathbb{Z} \mathbb{Q}$) then so are all isotropy groups $G_m$; see [St, Theorem 1.5] or [Bou1, Exercise 8(a) on p. 139].

There is essentially a complete classification of finite linear groups generated by bireflections. In arbitrary characteristic, this is due to Guralnick and Saxl [GuS]; for the case of characteristic zero, see Huffman and Wales [HuW]. Bireflection groups have been of interest in connection with the problem of determining all finite linear groups whose algebra of polynomial invariants is a complete intersection. Specifically, suppose that $G \leq \text{GL}(V)$ for some finite-dimensional vector space $V$ and let $\mathcal{O}(V) = S(V^*)$ denote the algebra of polynomial functions on $V$. It was shown by Kac and Watanabe [KW] and independently by Gordeev [G1] that if the algebra $\mathcal{O}(V)^G$ of all $G$-invariant polynomial functions is a complete intersection then $G$ is generated by bireflections on $V$. The classification of all groups $G$ so that $\mathcal{O}(V)^G$ is a complete intersection has been achieved by Gordeev [G2] and by Nakajima [N].

The last assertion of the above Theorem implies in particular that if $\mathbb{Z}[L]^G$ is Cohen-Macaulay and $G$ acts non-trivially on $L$ then some element of
\( \mathcal{G} \) acts as a non-trivial bireflection on \( L \). Hence we obtain the following multiplicative version of Kemper’s 3-copies conjecture:

**Corollary.** If \( \mathcal{G} \) acts non-trivially on \( L \) and \( r \geq 3 \) then \( \mathbb{Z}[L^\oplus r]^\mathcal{G} \) is not Cohen-Macaulay.

The 3-copies conjecture was formulated by Kemper [K1, Vermutung 3.12] in the context of polynomial invariants. Using the above notation, the original conjecture states that if \( 1 \neq \mathcal{G} \leq {\text{GL}}(V) \) and the characteristic of the base field of \( V \) divides the order of \( \mathcal{G} \) (“modular case”) then the invariant algebra \( \mathcal{O}(V^{\oplus r})^\mathcal{G} \) will not be Cohen-Macaulay for any \( r \geq 3 \). This is still open. The main factors contributing to our success in the multiplicative case are the following:

- **Multiplicative actions are permutation actions:** \( \mathcal{G} \) permutes the \( k \)-basis of \( k[L] \) consisting of all “monomials”, corresponding to the elements of the lattice \( L \). Consequently, the cohomology \( H^*(\mathcal{G}, k[L]) \) is simply the direct sum of the various \( H^*(\mathcal{G}_m, k) \) with \( m \) running over a transversal for the \( \mathcal{G} \)-orbits in \( L \).
- **Up to conjugacy, there are only finitely many finite subgroups of \( \text{GL}_n(\mathbb{Z}) \) and these groups are explicitly known for small \( n \).** A crucial observation for our purposes is the following: if \( \mathcal{G} \) is a nontrivial finite perfect subgroup of \( \text{GL}_n(\mathbb{Z}) \) such that no \( 1 \neq g \in \mathcal{G} \) has eigenvalue 1 then \( \mathcal{G} \) is isomorphic to the binary icosahedral group and \( n \geq 8 \); see Lemma 2.3 below.

A brief outline of the contents of this article is as follows. The short preliminary Section 1 is devoted to general actions of a finite group \( \mathcal{G} \) on a commutative ring \( R \). This material relies rather heavily on [LPk]. We liberate a technical result from [LPk] from any a priori hypotheses on the characteristic; the new version (Proposition 1.4) states that if \( R \) and \( R^\mathcal{G} \) are both Cohen-Macaulay and \( H^i(\mathcal{G}, R) = 0 \) for \( 0 < i < k \) then \( H^k(\mathcal{G}, R) \) is detected by \( k + 1 \)-reflections. Section 2 then specializes to the case of multiplicative actions. We assemble the main tools required for the proof of the Theorem, which is presented in Section 3. The article concludes with a brief discussion of possible avenues for further investigation and some examples.

### 1. Finite Group Actions on Rings

1.1. In this section, \( R \) will be a commutative ring on which a finite group \( \mathcal{G} \) acts by ring automorphisms \( r \mapsto g(r) \ (r \in R, g \in \mathcal{G}) \). The subring of \( \mathcal{G} \)-invariant elements of \( R \) will be denoted by \( R^\mathcal{G} \).
1.2. Generalized reflections. Following [GK], we will say an element \( g \in \mathcal{G} \) acts as a \( k \)-reflection on \( R \) if \( g \) belongs to the inertia group
\[
I_G(\mathfrak{P}) = \{ g \in \mathcal{G} \mid g(r) - r \in \mathfrak{P} \forall r \in R \}
\]
of some prime ideal \( \mathfrak{P} \in \text{Spec} R \) with \( \text{height} \ \mathfrak{P} \leq k \). The cases \( k = 1 \) and \( k = 2 \) will be referred to as reflections and bireflections, respectively. Define the ideal \( I_R(g) \) of \( R \) by
\[
I_R(g) = \sum_{r \in R} (g(r) - r)R.
\]
Evidently, \( \mathfrak{P} \supseteq I_R(g) \) is equivalent to \( g \in I_G(\mathfrak{P}) \). Thus:
\[
g \text{ is a } k \text{-reflection on } R \text{ if and only if } \text{height } I_R(g) \leq k.
\]
For each subgroup \( \mathcal{H} \leq \mathcal{G} \), we put
\[
I_R(\mathcal{H}) = \sum_{g \in \mathcal{H}} I_R(g).
\]
It suffices to let \( g \) run over a set of generators of the group \( \mathcal{H} \) in this sum.

1.3. A height estimate. The cohomology \( H^*(\mathcal{G}, R) = \oplus_{n \geq 0} H^n(\mathcal{G}, R) \) has a canonical \( R^\mathcal{G} \)-module structure: for each \( r \in R^\mathcal{G} \), the map \( \rho: R \rightarrow R, s \mapsto rs \), is \( \mathcal{G} \)-equivariant and hence it induces a map on cohomology \( \rho_*: H^*(\mathcal{G}, R) \rightarrow H^*(\mathcal{G}, R) \). The element \( r \) acts on \( H^*(\mathcal{G}, R) \) via \( \rho_* \). Let \( \text{res}_\mathcal{H}^\mathcal{G}: H^*(\mathcal{G}, R) \rightarrow H^*(\mathcal{H}, R) \) denote the restriction map.

The following lemma extends [LPk, Proposition 1.4].

**Lemma.** For any \( x \in H^*(\mathcal{G}, R) \),
\[
\text{height } \text{ann}_{R^\mathcal{G}}(x) \geq \inf \{ \text{height } I_R(\mathcal{H}) \mid \mathcal{H} \leq \mathcal{G}, \text{res}_\mathcal{H}^\mathcal{G}(x) \neq 0 \}.
\]

**Proof.** Put \( \mathcal{X} = \{ \mathcal{H} \leq \mathcal{G} \mid \text{res}_\mathcal{H}^\mathcal{G}(x) = 0 \} \). For each \( \mathcal{H} \leq \mathcal{G} \), let \( R^\mathcal{H}_{R^\mathcal{G}} \) denote the image of the relative trace map \( R^\mathcal{H} \rightarrow R^\mathcal{G}, r \mapsto \sum_g g(r) \), where \( g \) runs over a transversal for the cosets \( \mathcal{g}\mathcal{H} \) of \( \mathcal{H} \) in \( \mathcal{G} \). By [LPk, Lemma 1.3],
\[
R^\mathcal{H}_{R^\mathcal{G}} \subseteq \text{ann}_{R^\mathcal{G}}(x) \quad \text{for all } \mathcal{H} \in \mathcal{X}.
\]
To prove the lemma, we may assume that \( \text{ann}_{R^\mathcal{G}}(x) \) is a proper ideal of \( R^\mathcal{G} \); for, otherwise \( \text{height } \text{ann}_{R^\mathcal{G}}(x) = \infty \). Choose a prime ideal \( \mathfrak{p} \) of \( R^\mathcal{G} \) with \( \mathfrak{p} \supseteq \text{ann}_{R^\mathcal{G}}(x) \) and \( \text{height } \mathfrak{p} = \text{height } \text{ann}_{R^\mathcal{G}}(x) \). If \( \mathfrak{P} \) is a prime of \( R \) that lies over \( \mathfrak{p} \) then
\[
R^\mathcal{H}_{R^\mathcal{G}} \subseteq \mathfrak{P} \quad \text{for all } \mathcal{H} \in \mathcal{X}
\]
and \( \text{height } \mathfrak{P} = \text{height } \mathfrak{p} \). By [LPk, Lemma 1.1], the above inclusion implies that
\[
[I_G(\mathfrak{P}) : I_R(\mathfrak{P})] \in \mathfrak{P} \quad \text{for all } \mathcal{H} \in \mathcal{X}.
\]
Put \( p = \text{char } R/\mathfrak{P} \) and let \( \mathcal{P} \leq I_G(\mathfrak{P}) \) be a Sylow \( p \)-subgroup of \( I_G(\mathfrak{P}) \) (so \( \mathcal{P} = 1 \) if \( p = 0 \)). Then \( I_R(\mathcal{P}) \subseteq \mathfrak{P} \) and \( [I_G(\mathfrak{P}) : \mathcal{P}] \notin \mathfrak{P} \). Hence,
\( \mathcal{P} \notin \mathcal{X} \) and \( \text{height } I_R(\mathcal{P}) \leq \text{height } \mathfrak{P} = \text{height } \text{ann}_{R^G}(x) \). This proves the lemma.

We remark that the lemma and its proof carry over verbatim to the more general situation where \( H^*(\mathcal{G}, R) \) is replaced by \( H^*(\mathcal{G}, M) \), where \( M \) is some module over the skew group ring of \( \mathcal{G} \) over \( R \); cf. [LPk]. However, we will not be concerned with this generalization here.

1.4. A necessary condition. In this section, we assume that \( R \) is noetherian as \( R^G \)-module. This assumption is satisfied whenever \( R \) is an affine algebra over some noetherian subring \( k \subseteq R^G \); see [Bou2, Théorème 2 on p. 33]. Put

\[
\mathcal{X}_k = \{ \mathcal{H} \leq \mathcal{G} \mid \text{height } I_R(\mathcal{H}) \leq k \} .
\] (1.1)

Note that each \( \mathcal{H} \in \mathcal{X}_k \) consists of \( k \)-reflections on \( R \). The following proposition is a characteristic-free version of [LPk, Proposition 4.1].

Proposition. Assume that \( R \) and \( R^G \) are Cohen-Macaulay. If \( H^i(\mathcal{G}, R) = 0 \) \((0 < i < k)\) then the restriction map

\[
\text{res}^G_{\mathcal{X}_{k+1}} : \prod_{\mathcal{H} \in \mathcal{X}_{k+1}} H^k(\mathcal{H}, R) \to H^k(\mathcal{G}, R)
\]

is injective.

Proof. We may assume that \( H^k(\mathcal{G}, R) \neq 0 \). Let \( x \in H^k(\mathcal{G}, R) \) be nonzero and put \( a = \text{ann}_{R^G}(x) \). By [LPk, Proposition 3.3], \( \text{depth } a \leq k + 1 \). Since \( R^G \) is Cohen-Macaulay, \( \text{depth } a = \text{height } a \). Thus, Lemma 1.3 implies that \( k + 1 \geq \text{height } I_R(\mathcal{H}) \) for some \( \mathcal{H} \leq \mathcal{G} \) with \( \text{res}^G_{\mathcal{H}}(x) \neq 0 \). The proposition follows.

Note that the vanishing hypothesis on \( H^i(\mathcal{G}, R) \) is vacuous for \( k = 1 \). Thus, \( H^1(\mathcal{G}, R) \) is detected by bireflections whenever \( R \) and \( R^G \) are both Cohen-Macaulay.

2. Multiplicative Actions

2.1. For the remainder of this article, \( L \) will denote a lattice on which the finite group \( \mathcal{G} \) acts by automorphisms \( m \mapsto g(m) \) \((m \in L, g \in \mathcal{G})\). The group algebra of \( L \) over some commutative base ring \( k \) will be denoted by \( k[L] \). We will use additive notation in \( L \). The \( k \)-basis element of \( k[L] \) corresponding to the lattice element \( m \in L \) will be written as

\[
x^m .
\]
so \( x^0 = 1, x^{m+m'} = x^m x^{m'} \), and \( x^{-m} = (x^m)^{-1} \). The action of \( G \) on \( L \) extends uniquely to an action by \( \mathbb{k} \)-algebra automorphisms on \( \mathbb{k}[L] \):

\[
g(\sum_{m \in L} k_m x^m) = \sum_{m \in L} k_m x^{g(m)}.\]

The invariant algebra \( \mathbb{k}[L]^G \) is a free \( \mathbb{k} \)-module: a \( \mathbb{k} \)-basis is given by the \( G \)-orbit sums \( \sigma(m) = \sum_{m' \in G(m)} x^{m'} \), where \( G(m) \) denotes the \( G \) -orbit of \( m \in L \). Since all orbit sums are defined over \( \mathbb{Z} \), we have

\[
\mathbb{k}[L]^G = \mathbb{k} \otimes \mathbb{Z} [L]^G. \tag{2.1}
\]

2.2. Let \( \mathcal{H} \) be a subgroup of \( G \). We compute the height of the ideal \( I_{\mathbb{k}[L]}(\mathcal{H}) \) from §1.2. Let

\[
L^\mathcal{H} = \{ m \in L \mid g(m) = m \text{ for all } g \in \mathcal{H} \}
\]
denote the lattice of \( \mathcal{H} \)-invariants in \( L \) and define the sublattice \([\mathcal{H}, L]\) of \( L \) by

\[
[\mathcal{H}, L] = \sum_{g \in \mathcal{H}} [g, L],
\]

where \([g, L] = \{ g(m) - m \mid m \in L \}\). It suffices to let \( g \) run over a set of generators of the group \( \mathcal{H} \) in the above formulas.

**Lemma.** With the above notation, \( \mathbb{k}[L]/I_{\mathbb{k}[L]}(\mathcal{H}) \cong \mathbb{k}[L]/[\mathcal{H}, L] \) and

\[
\text{height } I_{\mathbb{k}[L]}(\mathcal{H}) = \text{rank } [\mathcal{H}, L] = \text{rank } L - \text{rank } L^\mathcal{H}.\]

**Proof.** Since the ideal \( I_{\mathbb{k}[L]}(\mathcal{H}) \) is generated by the elements \( x^{g(m)} m - 1 \) with \( m \in L \) and \( g \in \mathcal{H} \), the isomorphism \( \mathbb{k}[L]/I_{\mathbb{k}[L]}(\mathcal{H}) \cong \mathbb{k}[L]/[\mathcal{H}, L] \) is clear.

To prove the equality \( \text{rank } [\mathcal{H}, L] = \text{rank } L - \text{rank } L^\mathcal{H} \), note that the rational group algebra \( \mathbb{Q}[\mathcal{H}] \) is the direct sum of the ideals \( \mathbb{Q} \left( \sum_{g \in \mathcal{H}} g \right) \) and \( \sum_{g \in \mathcal{H}} \mathbb{Q}(g-1) \). This implies \( L \otimes \mathbb{Q} = (L^\mathcal{H} \otimes \mathbb{Q}) \oplus ([\mathcal{H}, L] \otimes \mathbb{Q}) \). Hence, \( \text{rank } L = \text{rank } L^\mathcal{H} + \text{rank } [\mathcal{H}, L] \).

To complete the proof, it suffices to show that

\[
\text{height } \mathfrak{p} = \text{rank } [\mathcal{H}, L]
\]

holds for any minimal covering prime \( \mathfrak{p} \) of \( I_{\mathbb{k}[L]}(\mathcal{H}) \). Put \( A = L/[\mathcal{H}, L] \) and \( \mathfrak{p} = \mathfrak{p}/I_{\mathbb{k}[L]}(\mathcal{H}) \), a minimal prime of \( \mathbb{k}[L]/I_{\mathbb{k}[L]}(\mathcal{H}) = \mathbb{k}[A] \). Further, put \( \mathfrak{p} = \mathfrak{p} \cap \mathbb{k} = \mathfrak{p} \cap k \). Since the extension \( k \hookrightarrow k[A] = \mathbb{k}[L]/I_{\mathbb{k}[L]}(\mathcal{H}) \) is free, \( \mathfrak{p} \) is a minimal prime of \( \mathbb{k} \); see [Bou3, Cor. on p. AC VIII.15]. Hence, descending chains of primes in \( \mathbb{k}[L] \) starting with \( \mathfrak{p} \) correspond in a 1-to-1 fashion to descending chains of primes of \( Q(\mathbb{k}/\mathfrak{p})[L] \) starting with the prime
that is generated by $\mathfrak{P}$. Thus, replacing $k$ by $Q(k/p)$, we may assume that $k$ is a field. But then
\[
\text{height } \mathfrak{P} = \dim k[L] - \dim k[L]/\mathfrak{P} = \text{rank } L - \dim k[L]/\mathfrak{P}.
\]
Let $\mathfrak{P}_0 = \mathfrak{P} \cap k[A_0]$, where $A_0$ denotes the torsion subgroup of $A$. Since $\mathfrak{P}$ is minimal, we have $\mathfrak{P} = \mathfrak{P}_0 k[A]$ and so $k[L]/\mathfrak{P} \cong k[A_0]/\mathfrak{P}_0$, where $k_0 = k[A_0]/\mathfrak{P}_0$ is a field. Thus, $\dim k[L]/\mathfrak{P} = \text{rank } A/A_0$. Finally, $\text{rank } A/A_0 = \text{rank } A = \text{rank } L - \text{rank } [H, L]$, which completes the proof. \hfill $\square$

Specializing the lemma to the case where $H = \langle g \rangle$ for some $g \in G$, we see that $g$ acts as a $k$-reflection on $k[L]$ if and only if $g$ acts as a $k$-reflection on $L$, that is,
\[
\text{rank } [g, L] \leq k.
\]
Moreover, the collection of subgroups $\mathfrak{X}_k$ in equation (1.1) can now be written as
\[
\mathfrak{X}_k = \{ H \leq G \mid \text{rank } L/L^H \leq k \}.
\] (2.2)

2.3. Fixed-point-free lattices for perfect groups. The $G$-action on $L$ is called fixed-point-free if $g(m) \neq m$ holds for all $0 \neq m \in L$ and $1 \neq g \in G$. Recall also that the group $G$ is said to be perfect if $G^{ab} = G/[G, G] = 1$.

**Lemma.** Assume that $G$ is a nontrivial perfect group acting fixed-point-freely on the nonzero lattice $L$. Then $G$ is isomorphic to the binary icosahedral group $2.A_5 \cong \mathbb{SL}_2(\mathbb{F}_5)$ and rank $L$ is a multiple of $8$.

**Proof.** Put $V = L \otimes \mathbb{C}$, a nonzero fixed-point-free $\mathbb{C}[G]$-module. By a well-known theorem of Zassenhaus (see [Wo, Theorem 6.2.1]), $G$ is isomorphic to the binary icosahedral group $2.A_5$ and the irreducible constituents of $V$ are 2-dimensional. The binary icosahedral group has two irreducible complex representations of degree 2; they are Galois conjugates of each other and both have Frobenius-Schur indicator $-1$. We denote the corresponding $\mathbb{C}[G]$-modules by $V_1$ and $V_2$. Both $V_i$ occur with the same multiplicity in $V$, since $V$ is defined over $\mathbb{Q}$. Thus, $V \cong (V_1 \oplus V_2)^m$ for some $m$ and rank $L = 4m$. We have to show that $m$ is even. Since both $V_i$ have indicator $-1$, it follows that $V_1 \oplus V_2$ is not defined over $\mathbb{R}$, whereas each $V_i^2$ is defined over $\mathbb{R}$; see [I, (9.21)]. Thus, $V_1 \oplus V_2$ represents an element $x$ of order 2 in the cokernel of the scalar extension map $G_0(\mathbb{R}[G]) \to G_0(\mathbb{C}[G])$, and $mx = 0$. Therefore, $m$ must be even, as desired. \hfill $\square$

We remark that the binary icosahedral group $2.A_5$ is isomorphic to the subgroup of the nonzero quaternions $\mathbb{H}^*$ that is generated by $(a + i + ja^*)/2$ and $(a + j + ka^*)/2$, where $a = (1 + \sqrt{5})/2$ and $a^* = (1 - \sqrt{5})/2$ and $\{1, i, j, k\}$ is the standard $\mathbb{R}$-basis of $\mathbb{H}$. Thus, letting $2.A_5$ act on $\mathbb{H}$ via left
multiplication, \( \mathbb{H} \) becomes a 2-dimensional fixed-point-free complex representation of \( 2.A_5 \). It is easy to see that this representation can be realized over \( K = \mathbb{Q}(i, \sqrt{5}) \); so \( \mathbb{H} = V \otimes_K \mathbb{C} \) with \( \dim_{\mathbb{Q}} V = 2[K : \mathbb{Q}] = 8 \). Any \( 2.A_5 \)-lattice for \( V \) will be fixed-point-free and have rank 8.

2.4. **Isotropy groups.** The isotropy group of an element \( m \in L \) in \( G \) will be denoted by \( G_m \); so

\[
G_m = \{ g \in G \mid g(m) = m \}.
\]

The \( G \)-lattice \( L \) is called *faithful* if \( \ker_G(L) = \bigcap_{m \in L} G_m = 1 \). The following lemma, at least part (a), is well-known. We include the proof for the reader’s convenience.

**Lemma.**

(a) The set of isotropy groups \( \{ G_m \mid m \in L \} \) is closed under conjugation and under taking intersections.

(b) Assume that the \( G \)-lattice \( L \) is faithful. If \( G_m (m \in L) \) is a minimal non-identity isotropy group then \( G_m \) acts fixed-point-freely on \( L/L^{G_m} \neq 0 \).

**Proof.** Consider the \( \mathbb{Q}[G] \)-module \( V = L \otimes_{\mathbb{Z}} \mathbb{Q} \). The collection of isotropy groups \( G_m \) remains unchanged when allowing \( m \in V \). Moreover, for any subgroup \( H \leq G \), \( L/L^H \) is an \( H \)-lattice with \( L/L^H \otimes_{\mathbb{Z}} \mathbb{Q} \cong V/V^H \).

(a) The first assertion is clear, since \( ^g G_m = G_{g(m)} \) holds for all \( g \in G, m \in V \). For the second assertion, let \( M \) be a non-empty subset of \( V \) and put \( G_M = \bigcap_{m \in M} G_m \). We must show that \( G_M = G_m \) for some \( m \in V \). Put \( W = V^{G_M} \). If \( g \in G \setminus G_M \) then \( W^g = \{ w \in W \mid g(w) = w \} \) is a proper subspace of \( W \), since some element of \( M \) does not belong to \( W^g \). Any \( m \in W \setminus \bigcup_{g \in G \setminus G_M} W^g \) satisfies \( G_m = G_M \).

(b) Let \( H = G_m \) be a minimal non-identity member of \( \{ G_m \mid m \in V \} \). As \( \mathbb{Q}[H] \)-modules, we may identify \( V^H \oplus V/V^H \). If \( 0 \neq v \in V/V^H \) then \( H_v = H \cap G_v \subsetneq H \). In view of (a), our minimality assumption on \( H \) forces \( H_v = 1 \). Thus, \( H \) acts fixed-point-freely on \( V/V^H \), and hence on \( L/L^H \). \( \Box \)

**Proposition.** Assume that \( L \) is a faithful \( G \)-lattice such that all minimal isotropy groups \( 1 \neq G_m (m \in L) \) are perfect. Then \( \text{rank} L/L^H \geq 8 \) holds for every nonidentity subgroup \( H \leq G \).

In the notation of equation (2.2), the conclusion of the proposition can be stated as follows:

\[
\mathcal{X}_k = \{ 1 \} \text{ for all } k < 8.
\]

**Proof of the Proposition.** Put \( \overline{H} = \bigcap_{m \in L^H} G_m \). Then \( \overline{H} \geq H \) and \( L^{\overline{H}} = L^H \). Lemma 2.4(a) further implies that \( \overline{H} = G_m \) for some \( m \). Replacing \( H \) by \( \overline{H} \), we may assume that \( H \) is a nonidentity isotropy group. If
is not minimal then replace \( \mathcal{H} \) by a smaller nonidentity isotropy group; this does not increase the value of rank \( L/L^\mathcal{H} \). Thus, we may assume that \( \mathcal{H} \) is a minimal nonidentity isotropy group, and hence \( \mathcal{H} \) is perfect. By Lemma 2.4(b), \( \mathcal{H} \) acts fixed-point-freely on \( L/L^\mathcal{H} \neq 0 \) and Lemma 2.3 implies that rank \( L/L^\mathcal{H} \geq 8 \), proving the proposition. \( \square \)

2.5. Cohomology. Let \( \mathcal{X} \) denote any collection of subgroups of \( \mathcal{G} \) that is closed under conjugation and under taking subgroups. We will investigate injectivity of the restriction map

\[
\text{res}^G_{\mathcal{X}} : H^k(\mathcal{G}, k[L]) \to \prod_{\mathcal{H} \in \mathcal{X}} H^k(\mathcal{H}, k[L]) .
\]

This map was considered in Proposition 1.4 for \( \mathcal{X} = \mathcal{X}_{k+1} \).

**Lemma.** The map \( \text{res}^G_{\mathcal{X}} : H^k(\mathcal{G}, k[L]) \to \prod_{\mathcal{H} \in \mathcal{X}} H^k(\mathcal{H}, k[L]) \) is injective if and only if the restriction maps

\[
H^k(\mathcal{G}_m, k) \to \prod_{\mathcal{H} \leq \mathcal{G}_m} \prod_{\mathcal{H} \in \mathcal{X}} H^k(\mathcal{H}, k)
\]

are injective for all \( m \in L \).

**Proof.** As \( k[\mathcal{G}] \)-module, \( k[L] \) is a permutation module:

\[
k[L] \cong \bigoplus_{m \in \mathcal{G} \setminus L} k[\mathcal{G}/\mathcal{G}_m] ,
\]

where \( k[\mathcal{G}/\mathcal{G}_m] = k[\mathcal{G}] \otimes k[\mathcal{G}_m] k \) and \( \mathcal{G} \setminus L \) is a transversal for the \( \mathcal{G} \)-orbits in \( L \). For each subgroup \( \mathcal{H} \leq \mathcal{G} \),

\[
k[\mathcal{G}/\mathcal{G}_m]|_{\mathcal{H}} \cong \bigoplus_{g \in \mathcal{H} \setminus \mathcal{G}/\mathcal{G}_m} k[\mathcal{H}/\mathcal{G}_m \cap \mathcal{H}] ;
\]

see [CR, 10.13]. Therefore, \( \text{res}^G_{\mathcal{H}} \) is the direct sum of the restriction maps

\[
H^k(\mathcal{G}, k[\mathcal{G}/\mathcal{G}_m]) \to H^k(\mathcal{H}, k[\mathcal{G}/\mathcal{G}_m]) = \bigoplus_{g \in \mathcal{H} \setminus \mathcal{G}/\mathcal{G}_m} H^k(\mathcal{H}, k[\mathcal{H}/\mathcal{G}_m \cap \mathcal{H}]) .
\]

By the Eckmann-Shapiro Lemma [Br, III(5.2),(6.2)], \( H^k(\mathcal{G}, k[\mathcal{G}/\mathcal{G}_m]) \cong H^k(\mathcal{G}_m, k) \) and \( H^k(\mathcal{H}, k[\mathcal{H}/\mathcal{G}_m \cap \mathcal{H}]) \cong H^k(\mathcal{G}_m \cap \mathcal{H}, k) \). In terms of these isomorphisms, the above restriction map becomes

\[
\rho_{\mathcal{H},m} : H^k(\mathcal{G}_m, k) \to \bigoplus_{g \in \mathcal{H} \setminus \mathcal{G}/\mathcal{G}_m} H^k(\mathcal{G}_m \cap \mathcal{H}, k)
\]

\[
[f] \mapsto \bigl( [h \mapsto f(g^{-1} h g)] \bigr)_g
\]

\]
where \([\ldots]\) denotes the cohomology class of a \(k\)-cocycle and \(h\) stands for a \(k\)-tuple of elements of \(g\mathbb{G}_m \cap \mathcal{H}\). Therefore,

\[
\ker \rho_{\mathcal{H}, m} = \bigcap_{g \in \mathcal{H} \setminus \mathbb{G} / \mathbb{G}_m} \ker \left( \text{res}_{\mathcal{G}_m \cap \mathcal{H}^g} : H^k(\mathcal{G}_m, \mathbb{k}) \to H^k(\mathcal{G}_m \cap \mathcal{H}^g, \mathbb{k}) \right).
\]

Thus, \(\ker \text{res}_{\mathcal{X}}\) is isomorphic to the direct sum of the kernels of the restriction maps

\[
H^k(\mathcal{G}_m, \mathbb{k}) \to \prod_{\mathcal{H} \in \mathcal{X}} H^k(\mathcal{G}_m \cap \mathcal{H}^g, \mathbb{k})
\]

with \(m \in \mathcal{G} \setminus \mathcal{L}\). Finally, by hypothesis on \(\mathcal{X}\), the groups \(\mathcal{G}_m \cap \mathcal{H}^g\) with \(\mathcal{H} \in \mathcal{X}\) are exactly the groups \(\mathcal{H} \in \mathcal{X}\) with \(\mathcal{H} \leq \mathcal{G}_m\). The lemma follows. \(\square\)

**Corollary.** Let \(\mathbb{k} = \mathbb{Z}/(|\mathcal{G}|)\) and \(k = 1\). Then \(\text{res}_{\mathcal{X}}\) injective if and only if all \(\mathcal{G}_m^{ab}\) \((m \in \mathcal{L})\) are generated by the images of the subgroups \(\mathcal{H} \leq \mathcal{G}_m\) with \(\mathcal{H} \in \mathcal{X}\).

**Proof.** By the lemma with \(k = 1\), the hypothesis on the restriction map says that all restrictions

\[
H^1(\mathcal{G}_m, \mathbb{k}) \to \prod_{\mathcal{H} \in \mathcal{X}} H^1(\mathcal{H}, \mathbb{k})
\]

are injective. Now, for each \(\mathcal{H} \leq \mathcal{G}\), \(H^1(\mathcal{H}, \mathbb{k}) = \text{Hom}(\mathcal{H}^{ab}, \mathbb{k}) \cong \mathcal{H}^{ab}\), where the last isomorphism holds by our choice of \(\mathbb{k}\). Therefore, injectivity of the above map is equivalent to \(\mathcal{G}_m^{ab}\) being generated by the images of all \(\mathcal{H} \leq \mathcal{G}_m\) with \(\mathcal{H} \in \mathcal{X}\). \(\square\)

### 3. The Cohen-Macaulay Property

**3.1.** Continuing with the notation of §2.1, we now turn to the question when the invariant algebra \(\mathbb{k}[L]^{\mathcal{G}}\) is Cohen-Macaulay. Our principal tool will be Proposition 1.4. We remark that the Cohen-Macaulay hypothesis of Proposition 1.4 simplifies slightly in the setting of multiplicative actions: it suffices to assume that \(\mathbb{k}[L]^{\mathcal{G}}\) is Cohen-Macaulay. Indeed, in this case the base ring \(\mathbb{k}\) is also Cohen-Macaulay, because \(\mathbb{k}[L]^{\mathcal{G}}\) is free over \(\mathbb{k}\), and then \(\mathbb{k}[L]\) is Cohen-Macaulay as well; see [BH, Exercise 2.1.23 and Theorems 2.1.9, 2.1.3(b)].

**3.2. Base rings.** Our main interest is in the case where \(\mathbb{k} = \mathbb{Z}\). As the following lemma shows, if \(\mathbb{Z}[L]^{\mathcal{G}}\) is Cohen-Macaulay then so is \(\mathbb{k}[L]^{\mathcal{G}}\) for any Cohen-Macaulay base ring \(\mathbb{k}\).

**Lemma.** The following are equivalent:

(a) \(\mathbb{Z}[L]^{\mathcal{G}}\) is Cohen-Macaulay;

(b) \(\mathbb{k}[L]^{\mathcal{G}}\) is Cohen-Macaulay whenever \(\mathbb{k}\) is;
(c) \( \mathbb{k}[L]^G \) is Cohen-Macaulay for \( \mathbb{k} = \mathbb{Z}/(|G|) \);
(d) \( \mathbb{F}_p[L]^G \) is Cohen-Macaulay for all primes \( p \) dividing \( |G| \).

**Proof.** (a) \( \Rightarrow \) (b): Put \( S = \mathbb{k}[L]^G \) and consider the extension of rings \( \mathbb{k} \hookrightarrow S \). This extension is free; see §2.1. By [BH, Exercise 2.1.23], \( S \) is Cohen-Macaulay if (and only if) \( Q \) [BH, Theorem 2.1.3(b)], it suffices to show that localization of \( S \) is Cohen-Macaulay, where \( p = P \cap \mathbb{k} \). But \( S_{\mathfrak{q}}/pS_{\mathfrak{q}} \) is a localization of \( (S/pS)_{\mathfrak{q},0} \cong Q(k/p)[L]^G \); see equation (2.1). Therefore, by [BH, Theorem 2.1.3(b)], it suffices to show that \( Q(k/p)[L]^G \) is Cohen-Macaulay. In other words, we may assume that \( k \) is a field. By [BH, Theorem 2.1.10], we may further assume that \( k = \mathbb{Q} \) or \( k = \mathbb{F}_p \). But equation (2.1) implies that \( Q[L]^G \cong \mathbb{Z}[L]_{\mathfrak{q},0}^G \) and \( \mathbb{F}_p[L]^G \cong \mathbb{Z}[L]^G/(p) \). Since \( \mathbb{Z}[L]^G \) is assumed Cohen-Macaulay, [BH, Theorem 2.1.3] implies that \( Q[L]^G \) and \( \mathbb{F}_p[L]^G \) are Cohen-Macaulay, as desired.

(b) \( \Rightarrow \) (c) is clear.

(c) \( \Rightarrow \) (d): Write \( |G| = \prod_p p^{n_p} \). Then \( k[L] \cong \prod_p \mathbb{Z}/(p^{n_p})[L]^G \) and \( \mathbb{Z}/(p^{n_p})[L]^G \) is a localization of \( k[L]^G \). Therefore, \( \mathbb{Z}/(p^{n_p})[L]^G \) is Cohen-Macaulay, by [BH, Theorem 2.1.3(b)]. If \( n_p \neq 0 \) then it follows from [BH, Theorem 2.1.3(a)] that \( \mathbb{Z}(p)[L]^G \) and \( \mathbb{F}_p[L]^G \cong \mathbb{Z}(p)[L]^G/(p) \) are Cohen-Macaulay.

(d) \( \Rightarrow \) (a): First, (d) implies that \( \mathbb{F}_p[L]^G \) is Cohen-Macaulay for all primes \( p \). For, if \( p \) does not divide \( |G| \) then \( \mathbb{F}_p[L]^G \) is always Cohen-Macaulay; see [BH, Corollary 6.4.6]. Now let \( \mathfrak{p} \) be a maximal ideal of \( \mathbb{Z}[L] \). Then \( \mathfrak{p} \cap \mathbb{Z} = (p) \) for some prime \( p \) and \( \mathbb{Z}[L]_{\mathfrak{p},0}^G \) is a localization of \( \mathbb{Z}[L]^G/(p) = \mathbb{F}_p[L]^G \). Thus, \( \mathbb{Z}[L]_{\mathfrak{p},0}^G/(p) \) is Cohen-Macaulay and [BH, Theorem 2.1.3(a)] further implies that \( \mathbb{Z}[L]_{\mathfrak{p}}^G \) is Cohen-Macaulay. Since, \( \mathfrak{p} \) was arbitrary, (a) is proved.

Since normal rings of (Krull) dimension at most 2 are Cohen-Macaulay, the implication (d) \( \Rightarrow \) (b) of the lemma shows that \( \mathbb{k}[L]^G \) is certainly Cohen-Macaulay whenever \( \mathbb{k} \) is Cohen-Macaulay and \( L \) has rank at most 2.

3.3. **Proof of the Theorem.** We are now ready to prove the Theorem stated in the Introduction. Recall that, for any subgroup \( \mathcal{H} \leq \mathcal{G} \), \( \mathcal{G}^{(2)} \) denotes the subgroup generated by the elements of \( \mathcal{H} \) that act as bireflections on \( L \) or, equivalently, by the subgroups of \( \mathcal{H} \) that belong to \( \mathcal{X}_2 \); see (2.2). Throughout, we assume that \( \mathbb{Z}[L]^G \) is Cohen-Macaulay.

We first show that \( \mathcal{G}_m/\mathcal{G}_m^{(2)} \) is a perfect group for all \( m \in L \). Put \( \mathbb{k} = \mathbb{Z}/(|G|) \). Then \( \mathbb{k}[L]^G \) is Cohen-Macaulay, by Lemma 3.2. Therefore, the restriction \( H^1(\mathcal{G}, \mathbb{k}[L]) \rightarrow \prod_{\mathcal{H} \in \mathcal{X}_2} H^1(\mathcal{H}, \mathbb{k}[L]) \) is injective, by Proposition 1.4; see the remark in §3.1. Corollary 2.5 yields that all \( \mathcal{G}_m^{ab} \) are generated by the images of the subgroups \( \mathcal{H} \leq \mathcal{G}_m \) with \( \mathcal{H} \in \mathcal{X}_2 \). In other words,
each $G_m^{ab}$ is generated by the images of the bireflections in $G_m$. Therefore,
\[
\left( G_m/G_m^{(2)} \right)^{ab} = 1,
\]
as desired.

Now assume that $G$ acts non-trivially on $L$. Our goal is to show that some isotropy group $G_m$ is non-perfect. Suppose otherwise. Replacing $G$ by $G/\ker G(L)$ we may assume that $1 \neq G$ acts faithfully on $L$. Then $\mathcal{X}_k = \{1\}$ for all $k < 8$, by Proposition 2.4. It follows that
\[
k = \inf \{i > 0 \mid H^i(G, \mathbb{k}[L]) \neq 0\} \geq 7.
\]
Indeed, if $k < 7$ then Proposition 1.4 implies that $0 \neq H^k(G, \mathbb{k}[L])$ embeds into $\prod_{i \in \mathbb{X}_{k+1}} H^k(H, \mathbb{k}[L])$ which is trivial, because $\mathcal{X}_{k+1} = \{1\}$. By Lemma 2.5 with $\mathcal{X} = \{1\}$, we conclude that
\[H^i(G_m, \mathbb{k}) = 0 \text{ for all } m \in L \text{ and all } 0 < i < 7.\]

On the other hand, choosing $G_m$ minimal with $G_m \neq 1$, we know by Lemmas 2.3 and 2.4(b) that $G_m$ is isomorphic to the binary icosahedral group $2.A_5$. The cohomology of $2.A_5$ is 4-periodic (see [Br, p. 155]). Hence, $H^3(G_m, \mathbb{k}) \cong H^{-1}(G_m, \mathbb{k}) = \text{ann}_{\mathbb{k}}(\sum_{G_m} \varphi) \cong \mathbb{Z}/(|G_m|) \neq 0.$ This contradiction completes the proof of the Theorem. \hfill \square

3.4. Rational invariance. We now show that the Cohen-Macaulay property of $\mathbb{k}[L]^G$ depends only on the rational isomorphism class of the $G$-lattice $L$. Recall that $G$-lattices $L$ and $L'$ are said to be **rationally isomorphic** if $L \otimes \mathbb{Q} \cong L' \otimes \mathbb{Q}$ as $\mathbb{Q}[G]$-modules. In this section, $\mathbb{k}$ denotes any commutative base ring.

**Proposition.** If $\mathbb{k}[L]^G$ is Cohen-Macaulay then so is $\mathbb{k}[L']^G$ for any $G$-lattice $L'$ that is rationally isomorphic to $L$.

**Proof.** Assume that $L \otimes \mathbb{Q} \cong L' \otimes \mathbb{Q}$. Replacing $L'$ by an isomorphic copy inside $L \otimes \mathbb{Q}$, we may assume that $L \supseteq L'$ and $L/L'$ is finite. Then $\mathbb{k}[L]$ is finite over $\mathbb{k}[L']$ which in turn is integral over $\mathbb{k}[L]^G$. Therefore, $\mathbb{k}[L]$ is integral over $\mathbb{k}[L']^G$, and hence so is $\mathbb{k}[L]^G$.

We now invoke a ring-theoretic result of Hochster and Eagon [HE] (or see [BH, Theorem 6.4.5]): Let $R \supseteq S$ be an integral extension of commutative rings having a Reynolds operator, that is, an $S$-linear map $R \to S$ that restricts to the identity on $S$. If $R$ is Cohen-Macaulay then $S$ is Cohen-Macaulay as well.

To construct the requisite Reynolds operator, consider the truncation map
\[
\pi: \mathbb{k}[L] \to \mathbb{k}[L'], \quad \sum_{m \in L} k_m x^m \mapsto \sum_{m \in L'} k_m x^m.
\]
This is a Reynolds operator for the extension $k[L] \supseteq k[L']$ that satisfies $\pi(g(f)) = g(\pi(f))$ for all $g \in G$, $f \in k[L]$. Therefore, $\pi$ restricts to a Reynolds operator $k[L]^G \to k[L']^G$ and the proposition follows. \hfill \Box

The proposition in particular allows to reduce the general case of the Cohen-Macaulay problem for multiplicative invariants to the case of effective $G$-lattices. Recall that the $G$-lattice $L$ is effective if $L^G = 0$. For any $G$-lattice $L$, the quotient $L/L^G$ is an effective $G$-lattice; this follows, for example, from the fact that $L$ is rationally isomorphic to the $G$-lattice $L^G \oplus L/L^G$.

**Corollary.** $k[L]^G$ is Cohen-Macaulay if and only if this holds for $k[L/L^G]^G$.

**Proof.** By the proposition, we may replace $L$ by $L' = L^G \oplus L/L^G$. But $k[L']^G \cong k[L/L^G]^G \otimes_k k[L]^G$, a Laurent polynomial algebra over $k[L/L^G]^G$. Thus, by [BH, Theorems 2.1.3 and 2.1.9], $k[L]^G$ is Cohen-Macaulay if and only if $k[L/L^G]^G$ is Cohen-Macaulay. The corollary follows. \hfill \Box

3.5. **Remarks and examples.**

3.5.1. **Abelian bireflection groups.** It is not hard to show that if $G$ is a finite abelian group acting as a bireflection group on the lattice $L$ then $\mathbb{Z}[L]^G$ is Cohen-Macaulay. Using Corollary 3.4 and an induction on rank $L$, the proof reduces to the verification that $\mathbb{Z}[L]^G$ is Cohen-Macaulay for $L = \mathbb{Z}^n$ and $G = \text{diag}(\pm 1, \ldots, \pm 1) \cap \text{SL}_n(\mathbb{Z})$. Direct computation shows that, for $n \geq 2$, 

$$\mathbb{Z}[L]^G = \mathbb{Z}[\xi_1, \ldots, \xi_n] \oplus \eta \mathbb{Z}[\xi_1, \ldots, \xi_n]$$

where $\xi_i = x^{e_i} + x^{-e_i}$ is the $G$-orbit sum of the standard basis element $e_i \in \mathbb{Z}^n$ and $\eta$ is the orbit sum of $\sum_i e_i = (1, \ldots, 1)$.

It would be worthwhile to try and extend this result to larger classes of bireflection groups. The aforementioned classification of bireflection groups in [GuS] will presumably be helpful in this endeavor.

3.5.2. **Subgroups of reflection groups.** Assume that $G$ acts as a reflection group on the lattice $L$ and let $H$ be a subgroup of $G$ with $[G : H] = 2$. Then $H$ acts as a bireflection group. (More generally, if $G$ acts as a $k$-reflection group and $[G : H] = m$ then $H$ acts as a $km$-reflection group; see [L1].) Presumably $\mathbb{Z}[L]^H$ will always be Cohen-Macaulay, but I have no proof. For an explicit example, let $G = S_n$ be the symmetric group on $\{1, \ldots, n\}$ and let $L = U_n$ be the standard permutation lattice for $S_n$; so $U_n = \bigoplus_{i=1}^n \mathbb{Z} e_i$ with $s(e_i) = e_{s(i)}$ for $s \in S_n$. Transpositions act as reflections on $U_n$ and 3-cycles as bireflections. Let $A_n \leq S_n$ denote the alternating group. To compute $\mathbb{Z}[U_n]^{A_n}$, put $x_i = x^{e_i} \in \mathbb{Z}[U_n]$. Then $\mathbb{Z}[U_n] = \mathbb{Z}[x_1, \ldots, x_n][s_n^{-1}]$, where $s_n = x^{\sum_i e_i} = \prod_i x_i$ is the $n$th elementary symmetric function, and $S_n$ acts via $s(x_i) = x_{s(i)}$ ($s \in S_n$).
Therefore, $\mathbb{Z}[U_n]^{A_n} = \mathbb{Z}[x_1, \ldots, x_n]^{A_n}[s_n^{-1}]$. The ring $\mathbb{Z}[x_1, \ldots, x_n]^{A_n}$ of polynomial $A_n$-invariants has the following form; see [S, Theorem 1.3.5]:

$$\mathbb{Z}[x_1, \ldots, x_n]^{A_n} = \mathbb{Z}[s_1, \ldots, s_n] \oplus d\mathbb{Z}[s_1, \ldots, s_n],$$

where $s_i$ is the $i$th elementary symmetric function and

$$d = \frac{1}{2} (\Delta + \Delta_+)$$

with $\Delta_+ = \prod_{i<j}(x_i + x_j)$ and $\Delta = \prod_{i<j}(x_i - x_j)$, the Vandermonde determinant. Thus,

$$\mathbb{Z}[U_n]^{A_n} = \mathbb{Z}[s_1, \ldots, s_{n-1}, s_n^{\pm 1}] \oplus d\mathbb{Z}[s_1, \ldots, s_{n-1}, s_n^{\pm 1}]$$

This is Cohen-Macaulay, being free over $\mathbb{Z}[s_1, \ldots, s_{n-1}, s_n^{\pm 1}]$.

### 3.5.3. $S_n$-lattices

If $L$ is a lattice for the symmetric group $S_n$ such that $\mathbb{Z}[L]^{S_n}$ is Cohen-Macaulay then the Theorem implies that $S_n$ acts as a bireflection group on $L$, and hence on all simple constituents of $L \otimes_{\mathbb{Z}} \mathbb{Q}$. The simple $\mathbb{Q}[S_n]$-modules are the Specht modules $S^\lambda$ for partitions $\lambda$ of $n$. If $n \geq 7$ then the only partitions $\lambda$ so that $S_n$ acts as a bireflection group on $S^\lambda$ are $(n)$, $(1^n)$ and $(n-1, 1)$; this follows from the lists in [Hu] and [W].

The corresponding Specht modules are trivial module, $\mathbb{Q}$, the sign module $\mathbb{Q}^-$, and the rational root module $A_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q}$, where $A_{n-1} = \{ \sum_i z_i e_i \in U_n \mid \sum_i z_i = 0 \}$ and $U_n$ is as in §3.5.2. Thus, if $n \geq 7$ and $\mathbb{Z}[L]^{S_n}$ is Cohen-Macaulay then we must have

$$L \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^r \oplus (\mathbb{Q}^-)^s \oplus (A_{n-1} \otimes_{\mathbb{Z}} \mathbb{Q})^t$$

with $s + t \leq 2$. In most cases, $\mathbb{Z}[L]^{S_n}$ is easily seen to be Cohen-Macaulay. Indeed, we may assume $r = 0$ by Corollary 3.4. If $s + t \leq 1$ then $S_n$ acts as a reflection group on $L$ and so $\mathbb{Z}[L]^{S_n}$ is Cohen-Macaulay by [L2]. For $t = 0$ we may quote the last remark in §3.2. This leaves the cases $s = t = 1$ and $s = 0, t = 2$ to consider.

If $s = t = 1$ then add a copy of $\mathbb{Q}$ so that $L$ is rationally isomorphic to $U_n \oplus \mathbb{Z}^-$. Using the notation of §3.5.2 and putting $t = x^{(0_{n,1})} \in \mathbb{Z}[U_n \oplus \mathbb{Z}^-]$ the invariants are:

$$\mathbb{Z}[U_n \oplus \mathbb{Z}^-]^{S_n} = R \oplus R\varphi$$

with $R = \mathbb{Z}[s_1, \ldots, s_{n-1}, s_n^{\pm 1}, t, t^{-1}]$ and $\varphi = \frac{1}{2} (\Delta_+ + \Delta) t + \frac{1}{2} (\Delta_+ - \Delta) t^{-1}$.

If $s = 0$ and $t = 2$ then we may replace $L$ by the lattice $U_n^2 = U_n \oplus U_n$. By Lemma 3.2 $\mathbb{Z}[U_n^2]^{S_n}$ is Cohen-Macaulay precisely if $\mathbb{F}_p[U_n^2]^{S_n}$ is Cohen-Macaulay for all primes $p \leq n$. As in §3.5.2, one sees that $\mathbb{F}_p[U_n^2]^{S_n}$ is a localization of the algebra “vector invariants” $\mathbb{F}_p[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n}$. By [K2, Corollary 3.5], this algebra is known to be Cohen-Macaulay for $n/2 < p \leq n$, but the primes $p \leq n/2$ apparently remain to be dealt with.
3.5.4. Ranks \( \leq 4 \). As was pointed out in §3.2, \( \mathbb{Z}[L]^G \) is always Cohen-Macaulay when rank \( L \leq 2 \).

For \( L = \mathbb{Z}^3 \), there are 32 \( \mathbb{Q} \)-classes of finite subgroups \( G \leq \text{GL}_3(\mathbb{Z}) \). All \( G \) are solvable; in fact, their orders divide 48. The Sylow 3-subgroup \( H \leq G \), if nontrivial, is generated by a bireflection of order 3. Thus, \( \mathbb{F}_3[L]^H \) is Cohen-Macaulay, and hence so is \( \mathbb{F}_3[L]^G \). Therefore, by Lemma 3.2, \( \mathbb{Z}[L]^G \) is Cohen-Macaulay if and only if \( \mathbb{F}_2[L]^G \) is Cohen-Macaulay, and for this to occur, \( G \) must be generated by bireflections. It turns out that 3 of the 32 \( \mathbb{Q} \)-classes consist of non-bireflection groups; these classes are represented by the cyclic groups
\[
\left\langle \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \right\rangle
\]
of orders 2, 4 and 6 (the latter two classes each split into 2 \( \mathbb{Z} \)-classes). For the \( \mathbb{Q} \)-classes consisting of bireflection groups, Pathak [P] has checked explicitly that \( \mathbb{F}_2[L]^G \) is indeed Cohen-Macaulay.

In rank 4, there are 227 \( \mathbb{Q} \)-classes of finite subgroups \( G \leq \text{GL}_4(\mathbb{Z}) \). All but 5 of them consist of solvable groups and 4 of the non-solvable classes are bireflection groups, the one exception being represented by \( S_5 \) acting on the signed root lattice \( \mathbb{Z}^- \otimes_{\mathbb{Z}} A_4 \). Thus, if the group \( G/G^{(2)} \) is perfect then it is actually trivial, that is, \( G \) is a bireflection group. It also turns out that, in this case, all isotropy groups \( G_m \) are bireflection groups. There are exactly 71 \( \mathbb{Q} \)-classes that do not consist of bireflection groups. By the foregoing, they lead to non-Cohen-Macaulay multiplicative invariant algebras. The \( \mathbb{Q} \)-classes consisting of bireflection groups have not been systematically investigated yet. The searches in rank 4 were done with [GAP].

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