ON THE STRATIFICATION OF NONCOMMUTATIVE PRIME SPECTRA

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ABSTRACT. We study rational actions of an algebraic torus G by automorphisms on an associative algebra R. The G-action on R induces a stratification of the prime spectrum $\operatorname{Spec} R$ which was introduced by Goodearl and Letzter. For a noetherian algebra R, Goodearl and Letzter showed that the strata of $\operatorname{Spec} R$ are isomorphic to the spectra of certain commutative Laurent polynomial algebras. The purpose of this note is to give a new proof of this result which works for arbitrary algebras R.

INTRODUCTION

Let G be an affine algebraic group and let R be an associative algebra on which G acts rationally by algebra automorphisms. The induced G-action on the set $\operatorname{Spec} R$ of all prime ideals of R leads to a stratification of $\operatorname{Spec} R$ which was pioneered by Goodearl and Letzter [3]. For the special case of an algebraic torus G and a noetherian algebra R, Goodearl and Letzter have given a description of the strata of $\operatorname{Spec} R$ in terms of the spectra of certain (commutative) Laurent polynomial algebras. Later, a different description of the strata was given in [6], for any algebra R and any connected affine algebraic group G. The purpose of this short note is to consolidate the description of [6], for the case of an algebraic torus G, with the earlier one due to Goodearl and Letzter, still working with a general algebra R.

Throughout, $\Bbbk$ will be an algebraically closed base field of arbitrary characteristic. The largest G-stable ideal of the algebra R that is contained in a given ideal I of R, called the *G-core* of I, will be denoted by $I{:}G$; so $I{:}G = \bigcap_{g \in G} g.I$. It is easy to see [5, Proposition 8(b)] that, for any rational action of an affine algebraic group G on R, the collection of all G-cores of prime ideals of R coincides with the set of all G-prime ideals of R; the latter set will be denoted by G-$\operatorname{Spec} R$. If the algebraic group G is connected, then G-$\operatorname{Spec} R$ is simply the set of all G-stable prime ideals of R [5, Proposition 19(a)]. The *Goodearl-Letzter stratification* of $\operatorname{Spec} R$ is the partition

$$\operatorname{Spec} R = \bigsqcup_{I \in G\text{-}\operatorname{Spec} R} \operatorname{Spec}_I R \qquad \text{with} \qquad \operatorname{Spec}_I R = \{P \in \operatorname{Spec} R \mid P{:}G = I\}\ .$$

For an algebraic torus G, we will give a description of each stratum $\operatorname{Spec}_I R$ in terms of the spectrum of a suitable affine *commutative* algebra Z_I. The construction of the algebra Z_I and the precise statement of the main result will be given in Section 1 while the proof will occupy Section 2.

1. THE STRATIFICATION THEOREM

Let G be a connected affine algebraic group over $\Bbbk$ and let R be an associative $\Bbbk$-algebra with a rational G-action by $\Bbbk$-algebra automorphism. For a given $I \in G$-$\operatorname{Spec} R$, let $\mathcal{C}(R/I)$ denote the extended centroid of the algebra R/I; this is a $\Bbbk$-field, called the *heart of* I, on which G acts via its action on R/I [5, 2.3]. We put

$$Z_I = \{c \in \mathcal{C}(R/I) \mid \text{the orbit } G.c \text{ spans a finite-dimensional } \Bbbk\text{-subspace of } \mathcal{C}(R/I)\}\ .$$

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Clearly, Z_I is a G-stable $\Bbbk$-subalgebra of $\mathcal{C}(R/I)$ and Z_I contains the subfield of G-invariants, $\mathcal{C}(R/I)^G$. Moreover, the G-action on Z_I is rational by [5, Lemma 18(b)].

We now focus on the case of an algebraic torus G. As usual, $X(G)$ will denote the lattice of rational characters of G. The following theorem, under the additional assumption that the algebra R is noetherian, is originally due to Goodearl and Letzter [3]; see also [2], [4] and [1, II.2.13].

Stratification Theorem. *Let G be an algebraic torus over $\Bbbk$ that acts rationally by algebra automorphisms on the $\Bbbk$-algebra R, and let $I \in G\text{-}\operatorname{Spec} R$. Then:*

(a) *There is an isomorphism*
$$Z_I \cong \mathcal{C}(R/I)^G \Gamma_I ,$$
the group algebra over the field $\mathcal{C}(R/I)^G$ of the sublattice $\Gamma_I = X(G/\operatorname{Ker}_G(Z_I)) \subseteq X(G)$.

(b) *There is a G-equivariant order isomorphism*
$$\gamma\colon \operatorname{Spec}_I R \xrightarrow{\sim} \operatorname{Spec}(Z_I) .$$

2. Proof of the Stratification Theorem

2.1. **Reductions and preliminaries.** We begin with some remarks that hold for an arbitrary connected affine algebraic group G over $\Bbbk$. In order to describe the G-stratum $\operatorname{Spec}_I R$ for a given $I \in G\text{-}\operatorname{Spec} R$, we may replace R by R/I and thus assume that $I = 0$. In particular, R is a prime ring. We will write $\mathcal{C} = \mathcal{C}(R)$ and $Z = Z_0$ for brevity; so $\mathcal{C}$ is a commutative $\Bbbk$-field on which G acts by automorphisms and
$$Z = \{c \in \mathcal{C} \mid \text{the orbit } G.c \text{ spans a finite-dimensional } \Bbbk\text{-subspace of } \mathcal{C}\} .$$
Our goal is to give a description of Z and to establish a suitable order isomorphism
$$\gamma\colon \operatorname{Spec}_0 R = \{P \in \operatorname{Spec} R \mid P{:}G = 0\} \xrightarrow{\sim} \operatorname{Spec}(Z) .$$
We will need to consider various G-actions; they will usually be indicated by a simple dot as in the foregoing. When more precision is necessary, the G-action on $\mathcal{C}$ will be denoted by ρ. The group G also acts on $\Bbbk[G]$, the algebra of regular functions of G, via the right and left regular representations $\rho_r, \rho_\ell\colon G \to \operatorname{Aut}_{\Bbbk\text{-alg}}(\Bbbk[G])$; they are defined by $(\rho_r(x)f)(y) = f(yx)$ and $(\rho_\ell(x)f)(y) = f(x^{-1}y)$ for $x, y \in G$ and $f \in \Bbbk[G]$.

2.2. **The algebra Z.** Consider the Hopf $\mathcal{C}$-algebra
$$S = \mathcal{C} \otimes_\Bbbk \Bbbk[G] ;$$
this is an algebra of $\mathcal{C}$-valued functions on G via $(\sum_i c_i \otimes f_i)(g) = \sum_i c_i f_i(g)$. The group G acts on the ring S via $\rho \otimes \rho_r$. If $g \in G$ and $s = \sum_i c_i \otimes f_i \in S$ then $g.s \in S$ is the function $G \to \mathcal{C}$ that is given by
$$(g.s)(x) = \sum_i (g.c_i) f_i(xg) = g.\sum_i c_i f_i(xg) \quad (x \in G) .$$
Let S^G denote the subring of G-invariants in S. Thus, $s \in S^G$ if and only if $g^{-1}.s(1) = s(g)$ for all $g \in G$.

We claim that
$$S^G \subseteq Z \otimes_\Bbbk \Bbbk[G] . \tag{1}$$
To see this, let $s = \sum_1^r c_i \otimes f_i \in S^G$ with $\{f_i\}_1^r \subseteq \Bbbk[G]$ chosen $\Bbbk$-linearly independent. Choose $\{x_j\}_i^r \subseteq G$ such that the matrix $A_g = (f_i(x_j g))_{i,j}$ is invertible and put $B = (f_i(x_j))_{i,j}$. Then the equations $\sum_i c_i f_i(x) = \sum_i (g.c_i) f_i(xg)$ for all $x, g \in G$, with $x = x_j$ for $j = 1, \ldots, r$, can be written as the matrix equation $(c_1, \ldots, c_r)B = (g.c_1, \ldots, g.c_r)A_g$ or else $(g.c_1, \ldots, g.c_r) = (c_1, \ldots, c_r)BA_g^{-1}$. This shows that $G.c_i \subseteq \sum_{i=1}^r \Bbbk c_i$ and hence $c_i \in Z$, proving (1).

Next, we show that there is an isomorphism

$$\varepsilon\colon S^G \xrightarrow{\sim} Z \tag{2}$$

such that

$$\varepsilon \circ (1_Z \otimes \rho_l(g)) = \rho(g) \circ \varepsilon \tag{3}$$

holds for all $g \in G$. Indeed, by [5, Lemma 18(b)], the G-action on Z is rational: it arises from a map of $\Bbbk$-algebras $\Delta_Z : Z \to Z \otimes_{\Bbbk} \Bbbk[G]$, $c \mapsto \sum c_0 \otimes c_1$, via $g.c = \sum c_0 c_1(g)$. By [6, equations (17) and (18)], the $\Bbbk[G]$-linear extension of Δ_Z, which will also be denoted by Δ_Z, is an isomorphism of $\Bbbk[G]$-algebras

$$\Delta_Z\colon Z \otimes_{\Bbbk} \Bbbk[G] \xrightarrow{\sim} Z \otimes_{\Bbbk} \Bbbk[G]$$

that satisfies the "intertwining formula" $\Delta_Z \circ (\rho \otimes \rho_r)(g) = (1_Z \otimes \rho_r)(g) \circ \Delta_Z$ for each $g \in G$. It follows that Δ_Z yields an isomorphism of $S^G = (Z \otimes_{\Bbbk} \Bbbk[G])^G$ with the sub algebra of $(1_Z \otimes \rho_r)(G)$-invariants in $Z \otimes_{\Bbbk} \Bbbk[G]$. Since the latter algebra is clearly Z, the isomorphism (2) follows. It is easy to see that the isomorphism (2) is just the restriction to S^G of the Hopf counit $S \to \mathcal{C}$, $s \mapsto s(1)$. In particular, for any $s \in S^G$, we have

$$(\varepsilon \circ (1_Z \otimes \rho_l(g)))(s) = s(g^{-1}) = g.s(1) = (\rho(g) \circ \varepsilon)(s) ,$$

proving (3).

2.3. **The case of an algebraic torus.** Now let $G \cong (\Bbbk^\times)^d$ be an algebraic torus over $\Bbbk$ and let $\Lambda = X(G) \cong \mathbb{Z}^d$ be its lattice of rational characters. Then $\Bbbk[G] = \Bbbk\Lambda$, the group algebra of Λ over $\Bbbk$. As it is customary to use additive notation for the lattice Λ, we will write the standard $\Bbbk$-basis of $\Bbbk[G]$ as $\{\mathbf{x}^\lambda \mid \lambda \in \Lambda\}$; so $\mathbf{x}^\lambda \mathbf{x}^{\lambda'} = \mathbf{x}^{\lambda+\lambda'}$ and $\mathbf{x}^\lambda(g) = \langle \lambda, g\rangle \in \Bbbk^\times$ for $g \in G$. Then $\rho_r(g)\mathbf{x}^\lambda = \langle \lambda, g\rangle \mathbf{x}^\lambda$ and

$$S = \bigoplus_{\lambda \in \Lambda} \mathcal{C} \otimes_{\Bbbk} \Bbbk\mathbf{x}^\lambda \cong \mathcal{C}\Lambda ,$$

the group algebra of Λ over the field $\mathcal{C}$. Consider an element $s = \sum_\lambda s_\lambda \otimes \mathbf{x}^\lambda \in S$ with $s_\lambda \in \mathcal{C}$. Then $g.s = \sum_\lambda g.s_\lambda \otimes \langle \lambda, g\rangle \mathbf{x}^\lambda$ for $g \in G$. Hence, $s \in S^G$ if and only if $g.s_\lambda = \langle -\lambda, g\rangle s_\lambda$ for all g, λ. Putting $\mathcal{C}_\lambda = \{c \in \mathcal{C} \mid g.c = \langle \lambda, g\rangle c \text{ for all } g \in G\}$ and noting that each nonzero $\mathcal{C}_\lambda$ is 1-dimensional over the fixed field $\mathcal{C}^G$, we have

$$S^G = \bigoplus_{\lambda \in \Lambda} \mathcal{C}_{-\lambda} \otimes_{\Bbbk} \Bbbk\mathbf{x}^\lambda \cong \mathcal{C}^G\Gamma , \tag{4}$$

the group algebra of the sublattice $\Gamma = \{\lambda \in \Lambda \mid \mathcal{C}_\lambda \neq 0\}$ over $\mathcal{C}^G$. We remark that $\Gamma = X(G/N)$ is the character lattice of the torus G/N, where N is the kernel of the action of G on Z; so $Z = \bigoplus_{\lambda \in \Gamma} \mathcal{C}_\lambda$. The isomorphisms (4) and (2) prove part (a) of the Theorem. Note also the S is free over S^G.

We claim that each G-stable ideal $\mathfrak{a}$ of S is generated by its intersection with S^G:

$$\mathfrak{a} = (\mathfrak{a} \cap S^G)S . \tag{5}$$

For the nontrivial inclusion $\subseteq$, let $s = \sum_\lambda s_\lambda \otimes \mathbf{x}^\lambda \in \mathfrak{a}$ be given. In order to show that $s \in (\mathfrak{a} \cap S^G)S$, we argue by induction on the size of $\operatorname{Supp}(s) = \{\lambda \in \Lambda \mid s_\lambda \neq 0\}$, the length of s. Our claim being clear for $s = 0$, assume that $s \neq 0$. Suppose there exists an element $0 \neq t \in \mathfrak{a}$ with $\operatorname{Supp}(t) \subsetneqq \operatorname{Supp}(s)$. Multiplying t and s with suitable units of the form $c \otimes \mathbf{x}^\mu$, we may assume that $0 \in \operatorname{Supp}(t)$ and $t_0 = s_0 = 1$. Since t and $s - t$ are shorter than s, they both belong to $(\mathfrak{a} \cap S^G)S$ and hence $s \in (\mathfrak{a} \cap S^G)S$ as well. Therefore, we may assume that if $t \in \mathfrak{a}$ satisfies $\operatorname{Supp}(t) \subsetneqq \operatorname{Supp}(s)$ then $t = 0$. Continuing to assume that $s_0 = 1$, this holds in particular for $t = s - g.s = \sum_{0 \neq \lambda} (s_\lambda - \langle \lambda, g\rangle g.s_\lambda) \otimes \mathbf{x}^\lambda$ for each $g \in G$. Therefore, we must have $s \in S^G$ and (5) is proved.

Now let $\mathfrak{b}$ be an ideal of S^G and let $\mathfrak{a}$ denote sum of all ideals of S that contract to $\mathfrak{b}$. Since S is free over S^G, we have $\mathfrak{a} \cap S^G = \mathfrak{b}$. Moreover, $\mathfrak{a}$ is clearly G-stable and so (5) gives that $\mathfrak{a} = \mathfrak{b}S$. Thus, $\mathfrak{b}S$ is the unique largest ideal of S that contracts to $\mathfrak{b}$.

2.4. **The prime correspondence.** We start with some reminders from [6]. For now, let G again be an arbitrary connected affine algebraic group over $\Bbbk$ and let $\Bbbk(G) = \operatorname{Fract} \Bbbk[G]$ be the field of rational functions of G. The G-action on S via $\rho \otimes \rho_r$ extends uniquely to an action of G on the following localization of S:

$$T = \mathcal{C} \otimes_{\Bbbk} \Bbbk(G) \; .$$

Let $\operatorname{Spec}^G(T)$ denote the collection of all G-stable prime ideals of T. Then [6, Theorem 9] establishes an order isomorphism

$$c\colon \operatorname{Spec}_0 R \xrightarrow{\sim} \operatorname{Spec}^G(T) \tag{6}$$

with the following G-equivariance property, for $P \in \operatorname{Spec}_0 R$ and $g \in G$:

$$c(g.P) = (1_{\mathcal{C}} \otimes \rho_\ell(g))(c(P)) \, . \tag{7}$$

Since T is the localization S at the nonzero elements of $\Bbbk[G]$, contraction and extension yields a G-equivariant order isomorphism $\operatorname{Spec} T \xrightarrow{\sim} \{\mathfrak{p} \in \operatorname{Spec} S \mid \mathfrak{p} \cap \Bbbk[G] = 0\}$. Note that $\Bbbk[G]$ is a G-simple ring, because G acts transitively on itself by right multiplication. Therefore, each $\mathfrak{p} \in \operatorname{Spec}^G(S)$ satisfies $\mathfrak{p} \cap \Bbbk[G] = 0$, and hence the above bijection restricts to a bijection

$$\operatorname{Spec}^G(T) \xrightarrow{\sim} \operatorname{Spec}^G(S) \; , \tag{8}$$

given by contraction and extension.

We now return to the case of an algebraic torus G. Then the map $\mathfrak{p} \mapsto \mathfrak{p} \cap S^G$ injects $\operatorname{Spec}^G(S)$ into $\operatorname{Spec}(S^G)$ by (5). Moreover, for any $\mathfrak{q} \in \operatorname{Spec}(S^G)$, we know that $\mathfrak{p} = \mathfrak{q}S$ is the unique largest ideal of S that contracts to $\mathfrak{q}$, which implies that $\mathfrak{p} \in \operatorname{Spec}^G(S)$. Thus we obtain a bijection

$$\operatorname{Spec}^G(S) \xrightarrow{\sim} \operatorname{Spec}(S^G) \tag{9}$$

that is again given by contraction and extension. From (6) - (9) in conjunction with the isomorphism (2) we obtain the desired order isomorphism $\gamma\colon \operatorname{Spec}_0 R \xrightarrow{\sim} \operatorname{Spec}(Z)$. The fact that γ is G-equivariant is immediate from the G-equivariance property (7) of the bijection (6) and the intertwining formula (3). This completes the proof of the Stratification Theorem.

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