RATIONAL GROUP ACTIONS ON AFFINE PI-ALGEBRAS

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To my friends Kenny and Toby

ABSTRACT. Let $R$ be an affine PI-algebra over an algebraically closed field $k$ and let $G$ be an affine algebraic $k$-group that acts rationally by algebra automorphisms on $R$. For $R$ prime and $G$ a torus, we show that $R$ has only finitely many $G$-prime ideals if and only if the action of $G$ on the center of $R$ is multiplicity free. This extends a standard result on affine algebraic $G$-varieties. Under suitable hypotheses on $R$ and $G$, we also prove a PI-version of a well-known result on spherical varieties and a version of Schelter’s catenarity theorem for $G$-primes.

1. INTRODUCTION

1.1. This article addresses the following general question:
Suppose a group $G$ acts by automorphisms on a ring $R$. When does $R$ have only finitely many $G$-prime ideals?

Recall that a proper $G$-stable (two-sided) ideal $I$ of $R$ is called $G$-prime if $AB \subseteq I$ for $G$-stable ideals $A$ and $B$ of $R$ implies that $A \subseteq I$ or $B \subseteq I$. The set of all $G$-prime ideals of $R$ will be denoted by $G$-$\text{Spec } R$.

Continuing our investigations in [21] and [22], our main focus will be on the case where $R$ is an algebra over an algebraically closed base field $k$ and $G$ is an affine algebraic $k$-group that acts rationally by $k$-algebra automorphisms on $R$; see 2.2 below for a brief reminder on rational actions. This setting will be assumed for the remainder of the Introduction. For noetherian $R$ and an algebraic torus $G$, the above question was stated as Problem II.10.6 in [7].

1.2. The question in 1.1 is motivated in part by the stratification of the prime spectrum $\text{Spec } R$ that is induced by the action of $G$. Namely, there is a surjection

$$\text{Spec } R \twoheadrightarrow G$-$\text{Spec } R$$

sending a given prime ideal $P$ to the largest $G$-stable ideal of $R$ that is contained in $P$. A precise description of the fibers of this map in terms of commutative algebras is given in [22, Theorem 9]. Hence, from a noncommutative perspective, the focus shifts to the description of $G$-$\text{Spec } R$, with finiteness being the optimal scenario. It turns out that, as long as the deformation parameters are sufficiently generic, $G$-$\text{Spec } R$ is indeed finite for all quantized coordinate algebras $R = \mathcal{O}_q(X)$ that have been analyzed in detail thus far, the acting group $G$ typically being a suitably chosen algebraic torus. Notable examples include the (generic) quantized coordinate rings of all semisimple algebraic groups (Joseph [14], Hodges, Levasseur and Toro [12]), quantum matrices and quantum Grassmannians (Cauchon, Lenagan and others; e.g. [8], [9] and [19]). Finiteness of $G$-$\text{Spec } R$ has also been observed for Leavitt path algebras $R$, again for the action of a suitable torus $G$ [1]. These finiteness results depend either on

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long calculations in \( R \) or else on finding a presentation of \( R \) as an iterated skew polynomial algebra, a class of algebras for which a finiteness result due to Goodearl and Letzter [10], [11] is available. A general finiteness criterion for \( G-\text{Spec} \, R \) is currently lacking.

1.3. In this article, we concentrate on the case of an affine \( k \)-algebra \( R \) satisfying a polynomial identity (PI). The class of PI-algebras, whose structural theory was pioneered by Kaplansky [15], Amitsur [2] and Procesi [27], combines aspects of noncommutativity with geometric properties that are familiar from commutative algebras. In order to give the finiteness problem in 1.1 a geometric perspective, we mention the following connection with \( \operatorname{G} \)-orbits of rational ideals. Here, a prime ideal \( P \) of \( R \) is called \textit{rational} if \( C(R/P) = k \), where \( C(\cdot) \) denotes the center of the classical ring of quotients of the ring in question. Rational primes are exactly the closed points of \( \text{Spec} \, R \); see 2.3.4 below for several equivalent characterizations of rationality. An ideal \( P \) is called \textit{G-rational} if the algebra of \( G \)-invariants \( C(R/P)^G \) coincides with \( k \). The subset of \( \text{Spec} \, R \) consisting of all rational primes of \( R \) will be denoted by \( \text{Rat} \, R \), and \( \text{G-Rat} \, R \subseteq \text{G-Spec} \, R \) will denote the set of all \( G \)-rational ideals. Since \( R \) satisfies the ascending chain condition for semiprime ideals (2.3.1), the Nullstellensatz (2.3.3) and the Dixmier-Mœglin equivalence (2.3.4), the following result is a special case of [22, Proposition 14].

\textbf{Proposition 1.} Let \( R \) be an affine PI-algebra over the algebraically closed field \( k \) and let \( G \) be an affine algebraic \( k \)-group that acts rationally by \( k \)-algebra automorphisms on \( R \). Then the following are equivalent:

\begin{itemize}
  \item[(i)] \( \text{G-Spec} \, R \) is finite;
  \item[(ii)] \( \text{G-Rat} \, R \) is finite;
  \item[(iii)] \( G \) has finitely many orbits in \( \text{Rat} \, R \);
  \item[(iv)] \( \text{G-Rat} \, R = \text{G-Spec} \, R \).
\end{itemize}

Thus, the finiteness problem at hand amounts to determining when all \( G \)-primes of \( R \) are \( G \)-rational.

1.4. In studying the finiteness question 1.1 we may assume without loss that \( G \) is connected. In this case, all \( G \)-primes of \( R \) are actually prime, and hence \( \text{G-Spec} \, R \) is the set of all \( G \)-stable prime ideals of \( R \); see Lemma 4 below. The main result of this note concerns the special case where \( R \) is an affine PI-algebra and \( G \) is a torus; it generalizes a standard result on affine algebraic \( G \)-varieties [17, II.3.3 Satz 5].

\textbf{Theorem 2.} Let \( R \) be a prime affine PI-algebra over the algebraically closed field \( k \) and let \( G \) be an algebraic \( k \)-torus that acts rationally by \( k \)-algebra automorphisms on \( R \). Then \( \text{G-Spec} \, R \) is finite if and only if the action of \( G \) on the center \( Z(R) \) is multiplicity free: for each rational character \( \lambda : G \to k^\times \), the weight space \( Z(R)_\lambda = \{ r \in Z(R) \mid g.r = \lambda(g)r \text{ for all } g \in G \} \) has dimension at most 1.

The proof of Theorem 2 will be given in Section 3 after deploying some auxiliary results and a generous amount of background material in Section 2. We remark that if \( R \) is also assumed noetherian then Theorem 2 is quite a bit easier, being an immediate consequence of Proposition 7 and Lemma 8(b) below. We conclude, in Section 4, with two results for noetherian \( R \), namely a PI-version of a standard result on spherical varieties (Proposition 10) and a version of Schelter’s catenarity theorem for \( G \)-primes (Proposition 11).

\textbf{Notations and conventions.} All rings have a 1 which is inherited by subrings and preserved under homomorphisms. The action of the group \( G \) on the ring \( R \) will be written as \( G \times R \to R, (g.r) \mapsto g.r \). For any ideal \( I \) of \( R \), we will write \( I : G = \bigcap_{g \in G} g.I \); this is the largest \( G \)-stable ideal of \( R \) that is contained in \( I \). The symbol \( \subset \) denotes a proper inclusion.
2. Preliminaries

2.1. Finite centralizing ring extensions. A ring extension $R \subseteq S$ is called centralizing if $S = \text{RC}_S(R)$ where $\text{C}_S(R) = \{s \in S \mid sr = rs \text{ for all } r \in R\}$. In this case, for any prime ideal $P$ of $S$, the contraction $P \cap R$ is easily seen to be a prime ideal of $R$. A centralizing extension $R \subseteq S$ is called finite, if $S$ is finitely generated as left or, equivalently, right $R$-module. By results of G. Bergman [3, 4] (see also [29]), the classical relations of lying over and incomparability for prime ideals hold in any finite centralizing extension $R \subseteq S$:

- given $Q \subseteq \text{Spec } S$, there exists $P \subseteq \text{Spec } S$ such that $Q = P \cap R$ (Lying Over);
- if $P, P' \subseteq \text{Spec } S$ are such that $P \subseteq P'$ then $P \cap R \subseteq P' \cap R$ (Incomparability).

Lemma 3. Let $R \subseteq S$ be a finite centralizing extension of rings and let $G$ be a group acting by automorphisms on $S$ that stabilize $R$. Assume that every ideal $A$ of $S$ contains a finite product of primes each of which contains $A$. Then contraction yields a surjective map

$$G\text{-Spec } S \twoheadrightarrow G\text{-Spec } R, \quad I \mapsto I \cap R$$

of finite fibers. In particular, if one of $G\text{-Spec } S$ or $G\text{-Spec } R$ is finite then so is the other.

Proof. First, we note that the $G$-primes of $S$ are exactly the ideals of the form $P;G$ with $P \subseteq \text{Spec } S$. Indeed, it is straightforward to check that $P;G$ is $G$-prime. Conversely, for any given $I \subseteq G\text{-Spec } S$, there are finitely many $P_i \subseteq \text{Spec } S$ (not necessarily distinct) with $I \subseteq P_i$ and $\prod_i P_i \subseteq I$. But then $I \subseteq P;G$ for each $i$ and $\prod_i P_i;G \subseteq I$, whence $I = P;G$ for some $i$. In particular, each $I \subseteq G\text{-Spec } S$ is semiprime. The group $G$ permutes the finitely many primes of $S$ that are minimal over $I$ and $G$-primeness forces these primes to form a single $G$-orbit. Therefore, we may write $I = P;G$ with $P \subseteq \text{Spec } S$ having a finite $G$-orbit. Similar remarks apply to the ring $R$, because every ideal $B$ of $R$ also contains a finite product of primes each of which contains $B$; this follows from the fact that $B$ contains some finite power of $BS \cap R$ by [20, Corollary 1.4].

Now let $I \subseteq G\text{-Spec } S$ be given and let $A, B$ be $G$-stable ideals of $R$ such that $AB \subseteq I \cap R$. Then $AS = SA$ is a $G$-stable ideal of $S$ and similarly for $B$. Since $(AS)(BS) = ABS \subseteq I$, we must have $AS \subseteq I$ or $BS \subseteq I$ and hence $A \subseteq I \cap R$ or $B \subseteq I \cap R$. Thus contraction yields a well-defined map $G\text{-Spec } S \twoheadrightarrow G\text{-Spec } R$.

For surjectivity of the contraction map, let $J \subseteq G\text{-Spec } R$ be given and write $J = Q;G$ with $Q \subseteq \text{Spec } R$. By Lying Over we may choose $P \subseteq \text{Spec } S$ with $Q = P \cap R$. Putting $I = P;G$ we obtain a $G$-prime of $S$ such that $J = I \cap R$.

Finally, assume that $I \subseteq G\text{-Spec } S$ contracts to a given $J \subseteq G\text{-Spec } R$. Write $I = P;G$ with $P \subseteq \text{Spec } S$ having a finite $G$-orbit. We claim that $P$ must be minimal over the ideal $JS$. Indeed, suppose that $JS \subseteq P' \subseteq P$ for some $P' \subseteq \text{Spec } S$. Then Incomparability gives $P \cap R \supset P' \cap R \supset J = \bigcap_{g \in G} g.(P \cap R)$. Since the last intersection is finite and $P' \cap R$ is prime, we conclude that $g.(P \cap R) \subseteq P' \cap R$ for some $g \in G$. Hence, $g.(P \cap R) \subset P \cap R$ which is impossible. This proves minimality of $P$ over $JS$. It follows that there are finitely many possibilities for $P$, and hence there are finitely many possibilities for $I$. This completes the proof of the lemma.

The hypothesis that every ideal of $S$ contains a finite product of prime divisors is of course satisfied, by Noether’s classical argument, if $S$ satisfies the ascending chain condition for ideals. More importantly for our purposes, the hypothesis also holds for any affine PI-algebra $S$ over some commutative noetherian ring by Braun’s Theorem [30, 6.3.39].

2.2. Rational group actions. Let $G$ be an affine algebraic $k$-group, where $k$ is an algebraically closed field, and let $k[G]$ denote the Hopf algebra of regular functions on $G$. A $k$-vector space $M$ is called a $G$-module if $M$ is a $k[G]$-comodule; see Jantzen [13, 2.7-2.8] or Waterhouse [32, 3.1-3.2]. Writing
the comodule structure map $\Delta_M : M \to M \otimes k[G]$ as $\Delta_M(m) = \sum m_0 \otimes m_1$, the group $G$ acts by $k$-linear transformations on $M$ via

$$g.m = \sum m_0 m_1(g) \quad (g \in G, m \in M).$$

Such $G$-actions, called rational $G$-actions, are in particular locally finite: the $G$-orbit of any $m \in M$ is contained in the finite-dimensional $k$-subspace of $M$ that is generated by $\{m_0\}$. If $G$ acts rationally on $M$ then it does so on all $G$-subquotients of $M$. Moreover, every irreducible $G$-submodule of $M$ is finite-dimensional, and the sum of all irreducible $G$-submodules is an essential $G$-submodule of $M$; it is called the socle of $M$ and denoted by $soc_G M$. In the following, we will denote the set of isomorphism classes of irreducible $G$-modules by $\text{irr} G$ and, for each $E \in \text{irr} G$, we let

$$[R : E] \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

denote the multiplicity of $E$ as a composition factor of $R$; see [13, I.2.14].

We will be primarily concerned with the situation where $G$ acts rationally by algebra automorphisms on a $k$-algebra $R$. This is equivalent to $R$ being a right $k[G]$-comodule algebra in the sense of [26, 4.1.2]. In the special case where $G \cong (k^\times)^d$ is an algebraic torus, we have $\text{irr} G = X(G) \cong \mathbb{Z}^d$, the lattice of rational characters $\chi : G \to k^\times$. We will usually write $\Lambda = X(G)$. A rational $G$-action on $R$ is equivalent to a $\mathbb{Z}^d$-grading $R = \bigoplus_{d \in \mathbb{Z}^d} R_{\lambda}$ of the algebra $R$. This follows from the fact that $k[G]$ is the group algebra $k\Lambda$ of the lattice $\Lambda \cong \mathbb{Z}^d$, and $k\Lambda$-comodule algebras are the same as $\Lambda$-graded algebras; see [26, 4.1.7]. The homogeneous component of $R$ of degree $\lambda$ is the weight space

$$R_{\lambda} = \{r \in R \mid g.r = \lambda(g)r \text{ for all } g \in G\},$$

and $[R : \lambda] = \dim_k R_{\lambda}$.

The following lemma was referred to in the Introduction.

**Lemma 4.** Let $R$ be a $k$-algebra, where $k$ is an algebraically closed field, and let $G$ be an affine algebraic $k$-group that acts rationally by $k$-algebra automorphisms on $R$. If $G^0 \subseteq G$ denotes the connected component of the identity, then $G^0$-Spec $R$ consists of all ordinary prime ideals of $R$ that are $G^0$-stable. Moreover, $G$-Spec $R$ is finite if and only if $G^0$-Spec $R$ is finite.

**Proof.** For the assertion that all $G^0$-primes are prime, see [21, Proposition 19(a)].

The second assertion, that $G$-Spec $R$ is finite if and only if $G^0$-Spec $R$ is so, actually holds for any (normal) subgroup $N \subseteq G$ having finite index in $G$ in place of $G^0$. Putting $G = G/N$, we first note that the $G$-primes of $R$ are exactly the ideals of the form $P = \bigcap_{x \in G} x.Q$ with $Q \in N$-Spec $R$. Indeed, $\bigcap_{x \in G} x.Q$ is easily seen to be $G$-prime. Conversely, any $P \in G$-Spec $R$ has the form $P = P':G$ with $P' \in$ Spec $R$ by [21, Proposition 8], and hence we may take $Q = P':N$. Moreover, the intersection $\bigcap_{x \in G} x.Q$ determines the $N$-prime $Q$ to within $G$-conjugacy, because all $x.Q$ are $N$-prime ideals of $R$ and $G$ is finite. Therefore, finiteness of $N$-Spec $R$ is equivalent to finiteness of $G$-Spec $R$. \qed

### 2.3. Some ring theoretic background on affine PI-algebras.

Let $R$ be an affine PI-algebra over a commutative noetherian ring $k$. The following facts are well-known.

#### 2.3.1. Semiprime ideals.

The ring $R$ satisfies the ascending chain condition for semiprime ideals and, for each ideal $I$ of $R$, there are only finitely many primes of $R$ that are minimal over $I$. If $I$ is semiprime then $R/I$ is a right and left Goldie ring and the extended centroid of $R/I$, in the sense of Martindale [24], is given by $\mathcal{C}(R/I) = Z(Q(R/I))$, the center of the classical ring of quotients of $R/I$. If $I$ is prime then $\mathcal{C}(R/I)$ is identical to the field of fractions of $Z(R/I)$ by Posner’s Theorem. See [30, 6.1.30, 6.3.36*], [25, 13.6.9], [21, 1.4.2] for all this.
2.3.2. $G$-prime ideals. By Braun’s Theorem [30, 6.3.39], every ideal $I$ of $R$ contains a finite product of primes that contain $I$. As in the proof of Lemma 3 it follows that, for any group $G$ acting by ring automorphisms on $R$, the $G$-primes of $R$ are exactly the ideals of the form $P:G$ with $P \in \text{Spec } R$. Moreover, $P$ can be chosen to have a finite $G$-orbit. In particular, every $I \in G-\text{Spec } R$ is semiprime. The ring of $G$-invariants $C(R/I)^G$ is a field for every $I \in G-\text{Spec } R$; see [21, Prop. 9].

2.3.3. Nullstellensatz. If $k$ is a Jacobson ring then so is $R$: every prime ideal of $R$ is an intersection of primitive ideals. Moreover, if $P$ is a primitive ideal of $R$ then $P$ is maximal; in fact, $k/P \cap k$ is a field and $R/P$ is a finite-dimensional algebra over this field; see [30, 6.3.3].

2.3.4. Rational ideals and the Dixmier-Mœglin equivalence. Now assume that $k$ is an algebraically closed field. Recall that a prime ideal $P$ of $R$ is said to be rational if $C(R/P) = k$ or, equivalently, $Z(R/P) = k$. By Posner’s Theorem [30, 6.1.30], this forces $P$ to have finite $k$-codimension in $R$. In fact, for any prime ideal of $R$, the following properties coincide (Dixmier-Mœglin equivalence), the implications $\Rightarrow$ being either trivial or immediate from the Nullstellensatz:

finite codimensional $\equiv$ maximal $\equiv$ locally closed in $\text{Spec } R \equiv$ primitive $\equiv$ rational .

2.4. The trace ring of a prime PI-ring. Let $R$ be a prime PI-ring with center $C = Z(R)$. By Posner’s Theorem [30, 6.1.30], the central localization $Q(R) = R_{C\setminus\{0\}}$ is a central simple algebra over the field of fractions $F = Q(C) = C(R)$. For each $q \in Q(R)$ we can consider the reduced characteristic polynomial $c_q(X) \in F[X]$. In detail, letting $F^{\text{alg}}$ denote an algebraic closure of $F$, we have an isomorphism of $F^{\text{alg}}$-algebras

\[ \varphi : Q(R) \otimes_F F^{\text{alg}} \cong M_n(F^{\text{alg}}) \]  

for some $n$. This isomorphism allows us to define $c_q(X)$ as the characteristic polynomial of the matrix $\varphi(q \otimes 1) \in M_n(F^{\text{alg}})$. One can show that $c_q(X)$ has coefficients in $F$ and is independent of the choice of the isomorphism $\varphi$; see [28, §9a] or [6, §12.3].

The commutative trace ring of $R$, by definition, is the $C$-subalgebra of $F$ that is generated by the coefficients of all polynomials $c_r(X)$ with $r \in R$; this algebra will be denoted by $T$. The trace ring of $R$, denoted by $TR$, is the $C$-subalgebra of $Q(R)$ that is generated by $R$ and $T$. The following result is standard; see [25, 13.9.11] or [31, 3.2].

Lemma 5. Let $R$ be a prime PI-ring that is an affine algebra over some commutative noetherian ring $k$. Then $T$ is an affine commutative $k$-algebra and $TR$ is a finitely generated $T$-module. Furthermore, $TR$ is finitely generated as $R$-module if and only if $R$ is noetherian.

Now suppose that a group $G$ acts by ring automorphisms on $R$. The action of $G$ extends uniquely to an action on the trace ring $TR$, and this action stabilizes $T$. To see this, note that the $G$-action on $R$ extends uniquely to an action on the ring of fractions $Q(R)$. Each $g \in G$ stabilizes $F = Z(Q(R))$, and hence $g$ yields an automorphism of $F[X]$ via its action on the coefficients of polynomials. The reduced characteristic polynomials of $q \in Q(R)$ and of $g.q$ are related by

\[ c_{g.q}(X) = g.c_q(X) . \]  

Indeed, extending $g$ to a field automorphism of $F^{\text{alg}}$, we obtain automorphisms $M_n(g) \in \text{Aut } M_n(F^{\text{alg}})$ and $\alpha_g \in \text{Aut } Q(R) \otimes_F F^{\text{alg}}$, the latter being defined by $\alpha_g(q \otimes f) = g.q \otimes g.f$. Fixing $\varphi$ as in (1) we obtain an isomorphism of $F^{\text{alg}}$-algebras $M_n(g)^{-1} \circ \varphi \circ \alpha_g : Q(R) \otimes_F F^{\text{alg}} \cong M_n(F^{\text{alg}})$. Using this isomorphism to compute reduced characteristic polynomials, we see that $c_{g.q}(X) = g^{-1}.c_q(X)$, proving (2). Since $g.r \in R$ for $r \in R$, equation (2) shows that the commutative trace ring $T$ is stable under the action of $G$ on $Q(R)$, and hence so is the trace ring $TR$. For rational actions, we have the following result of Vonessen [31, Proposition 3.4].
Lemma 6 (Vonessen). Let \( R \) be a prime PI-algebra over an algebraically closed field \( \mathbb{k} \) and let \( G \) be an affine algebraic \( \mathbb{k} \)-group that acts rationally by \( \mathbb{k} \)-algebra automorphisms on \( R \). Then the induced \( G \)-actions on \( TR \) and on \( T \) are rational as well.

In general, the finiteness problem 1.1 transfers nicely to trace rings.

Proposition 7. Let \( R \) be a prime PI-ring that is an affine algebra over some commutative noetherian ring. Let \( G \) be a group acting by ring automorphism on \( R \) and consider the induced \( G \)-actions on \( T \) and on \( TR \). Then \( G \)-Spec \( T \) is finite if and only if \( G \)-Spec \( TR \) is finite. If \( R \) is noetherian, then this is also equivalent to \( G \)-Spec \( R \) being finite.

Proof. Lemma 3, applied to the finite centralizing extension \( T \subseteq TR \) (Lemma 5), tells us that finiteness of \( G \)-Spec \( TR \) is equivalent to finiteness of \( G \)-Spec \( T \). If \( R \) is noetherian then we may argue in the same way for the finite centralizing extension \( R \subseteq TR \).

3. MAIN RESULT

Throughout this section, \( R \) denotes an affine PI-algebra over an algebraically closed field \( \mathbb{k} \) and \( G \) will be an affine algebraic \( \mathbb{k} \)-group that acts rationally by \( \mathbb{k} \)-algebra automorphisms on \( R \).

3.1. Sufficient criteria for \( G \)-rationality. By Proposition 1 we know that \( G \)-Spec \( R \) is finite if and only if all \( G \)-primes of \( R \) are \( G \)-rational. Therefore, \( G \)-rationality criteria are essential. As usual, the algebra \( R \) will be called \( G \)-prime if the zero ideal of \( R \) is \( G \)-prime; similarly for \( G \)-rationality.

Lemma 8. Assume that \( R \) is \( G \)-prime.

(a) If there is an \( N \in \mathbb{Z} \) such that \( [\text{soc}_G Z(R) : E] \leq N \) for all \( E \in \text{irr} \) \( G \) then \( R \) is \( G \)-rational.

(b) If \( G \) is connected solvable then \( R \) is \( G \)-rational if and only if \( [\text{soc}_G Z(R) : E] \leq 1 \) for all \( E \in \text{irr} \) \( G \).

Proof. (a) For a given \( q \in C(R)^G \) put \( I = \{ r \in R \mid qr = r \} \); this is a nonzero \( G \)-stable ideal of \( R \). Therefore, \( J = I^N \cap Z(R) \) is a nonzero \( G \)-stable ideal of \( Z(R) \); see [30, 6.1.28]. Note that \( q^iJ \subseteq Z(R) \) for \( 0 \leq i \leq N \). We have \( E \mapsto J \) for some \( E \in \text{irr} \) \( G \) and multiplication with \( q^i \) yields a \( G \)-equivariant map \( E \mapsto J \to Z(R) \). Since \( \dim_{\mathbb{k}G}(E, Z(R)) = [\text{soc}_G Z(R) : E] \leq N \), there are \( k_i \in \mathbb{k} \), not all 0, such that \( c = \sum_{i=0}^N k_i q^i \) annihilates \( E \). But nonzero elements of \( C(R)^G \) are invertible; so we must have \( c = 0 \). Thus, \( q \) is algebraic over \( \mathbb{k} \) and so \( q \in \mathbb{k} \).

(b) The condition is sufficient by part (a). For the converse, assume that \( E_1 \oplus E_2 \subseteq Z(R) \) for isomorphic \( E_1 \in \text{irr} \) \( G \). By the Lie-Kolchin Theorem [5, III.10.5], \( E_i = \mathbb{k} x_i \) for suitable \( x_i \). Since \( x_i \) generates a \( G \)-stable two-sided ideal, \( x_i \) is regular in \( R \). The quotient \( x_1 x_2^{-1} \in C(R) \) is a non-scalar \( G \)-invariant; so \( R \) is not \( G \)-rational.

Remark. A simplified version of the argument in the proof of (a), without recourse to [30, 6.1.28], establishes the following general fact: Let \( A \) be an arbitrary (associative) \( \mathbb{k} \)-algebra and let \( G \) be a group that acts on \( A \) by locally finite \( \mathbb{k} \)-algebra automorphisms. If there is an \( N \in \mathbb{Z} \) such that \( [A : E] \leq N \) holds for all finite-dimensional irreducible \( \mathbb{k}G \)-modules \( E \), where \( [A : E] \) denotes the multiplicity of \( E \) as composition factor of \( A \) as in [13, I.2.14], then \( G \)-Spec \( A = G \)-Rat \( A \).

3.2. Regular primes. Recall from (1) that if \( R \) is prime, then the classical ring of quotients \( Q(R) \) is a central simple algebra over the field of fractions \( F = Q(Z(R)) \). The \( PI \) degree of \( R \), by definition, is the degree of this central simple algebra: \( \Pi \deg R = \sqrt{\dim_F Q(R)} \). For any \( P \in \text{Spec} \) \( R \), one has \( \Pi \deg R/P \leq \Pi \deg R \). The prime \( P \) is called regular if equality holds here. The regular primes form an open subset of \( \text{Spec} \) \( R \). See [30, p. 104] or [25, 13.7.2] for all this.

Now let \( G \) be an algebraic \( \mathbb{k} \)-torus. In particular, \( G \) is connected and so \( G \)-Spec \( R \) consists of the \( G \)-stable prime ideals of \( R \) by Lemma 4.
Lemma 9. Let $G$ be an algebraic $k$-torus and assume that $R$ is prime. Then, for every regular $P \in G\text{-Spec } R$, we have $\text{tr deg}_R C(R/P)G \leq \text{tr deg}_R C(R)G$. Consequently, if $R$ is $G$-rational then all regular primes in $G\text{-Spec } R$ belong to $G\text{-Rat } R$.

Proof. Let $P \in G\text{-Spec } R$ be regular. Put $n = \text{PI deg } R$ and let $g_n(R)^+$ denote the Formanek center of $R$; this is a $G$-stable ideal of $Z(R)$ such that $g_n(R)^+ \subseteq P$ (cf. [30, 6.1.37] or [25, 13.7.2(i)]).

Therefore, we may choose a semi-invariant $c \in g_n(R)^+ \lambda$ with $c \notin P$. The group $G$ acts rationally on the localization $R_c = R[1/c]$ and $R_c$ is Azumaya by the Artin-Procesi Theorem [25, 13.7.14]. Therefore, $Z(R_c)$ maps onto $Z(R_c/P R_c)$ and $Z(R_c)\lambda$ maps onto $Z(R_c/P R_c)\lambda$ for all $\lambda \in X(G)$. The map $Z(R_c) \rightarrow Z(R_c/P R_c)$ extends to a $G$-equivariant epimorphism $Z(R_p) \rightarrow C(R/P) = Q(Z(R_c/P R_c))$, where $p = P \cap Z(R)$. Since $Z(R_p)^G \subseteq C(R)^G$, it suffices to show that $Z(R_p)^G$ maps onto $C(R/P)^G$. But, given $q \in C(R/P)^G$, we can find a semi-invariant $0 \neq x \in Z(R_c/P R_c)\lambda$ such that $qx \in Z(R_c/P R_c)$, and we can further find $y, z \in Z(R_c)\lambda$ with $y \mapsto x$ and $z \mapsto qx$. Then $zy^{-1} \in Z(R_p)^G$ maps to $q$. This proves the lemma.

3.3. Proof of Theorem 2. Let $G$ be an algebraic $k$-torus and assume that $R$ is prime. We need to show that $G\text{-Spec } R$ is finite if and only if the action of $G$ on $Z(R)$ is multiplicity free. By Lemma 8(b), the latter property is equivalent to $G$-rationality of $R$, and this is certainly necessary for $G\text{-Spec } R$ to be finite by Proposition 1.

Now assume that $R$ is $G$-rational. By Proposition 1 we must show that all $G$-primes of $R$ are $G$-rational. Lemma 9 ensures this for the regular $G$-primes. In particular, we may assume that $n := \text{PI deg } R > 1$. Now consider $P \in G\text{-Spec } R$ with $\text{PI deg } R/P < n$. Then $P$ contains the ideal $\alpha = g_n(R)R \subseteq R$; this is a nonzero $G$-stable common ideal of $R$ and of the trace ring $R' := TR$ of $R$ (cf. [30, 6.1.37 and 6.3.28]). All primes of $R$ that are minimal over $\alpha$ are $G$-stable. Let $Q$ be one of these primes such that $Q \subseteq P$. It suffices to show that $Q$ is $G$-rational. For, then we may replace $R$ by $R/Q$, and since $\text{PI deg } R/Q < n$, we may argue by induction that $P$ is $G$-rational.

First, we claim that there exists $Q' \in G\text{-Spec } R'$ with $Q' \cap R = Q$. Indeed, choosing $Q'$ to be a $G$-stable ideal of $R'$ that is maximal subject to the requirement that $Q' \cap R \subseteq Q$, it is straightforward to see that $Q'$ is $G$-prime. If $Q' \cap R \neq Q$ then $Q' \nsubseteq \alpha$ by minimality of $Q$ over $\alpha$. Thus, $Q' + \alpha$ is a $G$-stable ideal of $R'$ which properly contains $Q'$ and yet also satisfies $(Q' + \alpha) \cap R = (Q' \cap R) + \alpha \subseteq Q$. Since this contradicts our maximal choice of $Q'$, we must have $Q' \cap R = Q$ as claimed.

Next, we show that $Q'$ is $G$-rational. To see this, recall from Lemma 6 that $G$ acts rationally on the trace rings $T$ and $R'$. Moreover, $T$ is an affine commutative $k$-algebra that is $G$-rational, because $Q(T)^G = C(T)^G = k$. Therefore, by the case $n = 1$, we know that $G\text{-Spec } T$ is finite. By Proposition 7, $G\text{-Spec } R'$ is finite as well, and in view of Proposition 1, this forces $Q'$ to be $G$-rational.

Finally, we show that $Q$ is $G$-rational; this will finish the proof. But $C(R/Q) \subseteq C(R'/Q')$ and $C(R'/Q')^G = k$ by the foregoing. Therefore, $C(R/Q)^G = k$ as desired.

4. Related results

In this section, $R$ and $G$ are as in the previous section and $R$ is also assumed noetherian.

4.1. Actions of reductive groups. Recall from Lemma 6 that the induced $G$-action on the commutative trace ring $T$ is rational. This fact allows us to invoke results from algebraic geometry.

Proposition 10. Let $R$ be an affine noetherian PI-algebra over an algebraically closed field $k$ and let $G$ be an affine algebraic $k$-group that acts rationally by $k$-algebra automorphisms on $R$. Assume that $R$ is prime and that $G$ is connected reductive. Let $F = Q(Z(R))$ denote the field of fractions of the center of $R$, and let $F^B \subseteq F$ denote the invariant subfield of a Borel subgroup $B \leq G$. If $F^B = k$ then $B\text{-Spec } R$ is finite (and hence $G\text{-Spec } R$ is finite as well).
Proof. By Proposition 7, $B$-$\text{Spec} R$ is finite if and only if $B$-$\text{Spec} T$ is finite. Now, $T$ is an affine commutative domain over $k$ and the field of fractions of $T$ is $F$. By a standard result on spherical varieties [16, Corollary 2.6], the condition $FB = k$ implies that there are only finitely many $B$-orbits in $\text{Rat} T$. The latter fact is equivalent to finiteness of $B$-$\text{Spec} T$ by Proposition 1, which proves the proposition.

4.2. Catenary. A partially ordered set $(P, \leq)$ is said to be catenary if, given any two $x < x'$ in $P$, all saturated chains $x = x_0 < x_1 < \cdots < x_r = x'$ have the same finite length $r = r(x, x')$.

In the commutative case, the following observation goes back to conversations that I had with R. Rentschler a long time ago; cf. [23, §3]. As usual, $\text{GK dim}$ denotes Gelfand-Kirillov dimension.

Proposition 11. Let $R$ be an affine noetherian PI-algebra over an algebraically closed field $k$ and let $G$ be an affine algebraic $k$-group that acts rationally by $k$-algebra automorphisms on $R$. If the connected component of the identity of $G$ is solvable then the poset $(G$-$\text{Spec} R, \subseteq)$ is catenary. In fact, every saturated chain $P = P_0 \subset P_1 \subset \cdots \subset P_r = P'$ in $G$-$\text{Spec} R$ has length $r = \text{GK dim} R/P - \text{GK dim} R/P'$.

Proof. First assume that $G$ is connected; so $G$-$\text{Spec} R$ consists of the $G$-stable primes of $R$. In view of Schelter’s catenary theorem for $\text{Spec} R$ [30, 6.3.43], we need to show that any two neighbors $Q \subset P$ in $G$-$\text{Spec} R$ are also neighbors when viewed in $\text{Spec} R$. Passing to $R/Q$ we may assume that the algebra $R$ is prime and $P$ is a minimal nonzero member of $G$-$\text{Spec} R$, and we need to show that $P$ has height 1 in $\text{Spec} R$. But $P \cap Z(R)$ is a nonzero $G$-stable ideal of $Z(R)$ and hence the Lie-Kolchin Theorem provides us with a $G$-eigenvector $0 \neq z \in P \cap Z(R)$. The ideal $P$ is a minimal prime over $(z)$. For, if $(z) \subseteq P' \subset P$ for some $P' \in \text{Spec} R$ then $(z) \subseteq P'/G \subset P$ and $P'/G \in G$-$\text{Spec} R$, contradicting the fact that $P$ is a minimal nonzero member of $G$-$\text{Spec} R$. Thus $P$ is minimal over $(z)$ as claimed, and the principal ideal theorem [25, 4.1.11] gives that $P$ has height 1 as desired.

In general, let $G^0$ denote the connected component of the identity of $G$ and put $\overline{G} = G/G^0$. Write $Q = \bigcap_{x \in G} x.\overline{Q}$ and $P = \bigcap_{x \in G} x.\overline{P}$ for suitable $\overline{Q}, \overline{P} \in G^0$-$\text{Spec} R$ as in the proof of Lemma 4. Since these intersections are finite intersections of $G^0$-primes of $R$, we can arrange that $\overline{Q} \subset \overline{P}$. The ideals $Q$ and $P$ are neighbors in $G^0$-$\text{Spec} R$. For, if $\overline{Q} \subset T \subset \overline{P}$ for some $T \in G^0$-$\text{Spec} R$ then $Q \subset \bigcap_{x \in G} x.T \subset P$ since $G$ is finite, which contradicts the fact that $Q$ and $P$ are neighbors in $G$-$\text{Spec} R$. By the first paragraph of the proof, $\overline{Q}$ and $\overline{P}$ are also neighbors in $\text{Spec} R$, and hence $\text{GK dim} R/\overline{Q} = \text{GK dim} R/\overline{P} + 1$ by Schelter’s theorem. Since $\text{GK dim} R/\overline{Q} = \text{GK dim} R/\overline{P}$ and $\text{GK dim} R/P = \text{GK dim} R/P$ by [18, Corollary 3.3], the proof is complete.

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