GROUP ACTIONS AND RATIONAL IDEALS

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ABSTRACT. We develop the theory of rational ideals for arbitrary associative algebras R without assuming the standard finiteness conditions, noetherianness or the Goldie property. The Amitsur-Martindale ring of quotients replaces the classical ring of quotients which underlies the previous definition of rational ideals but is not available in a general setting.

Our main result concerns rational actions of an affine algebraic group G on R. Working over an algebraically closed base field, we prove an existence and uniqueness result for generic rational ideals in the sense of Dixmier: for every G-rational ideal I of R, the closed subset of the rational spectrum $\operatorname{Rat} R$ that is defined by I is the closure of a unique G-orbit in $\operatorname{Rat} R$. Under additional Goldie hypotheses, this was established earlier by Mœglin and Rentschler (in characteristic 0) and by Vonessen (in arbitrary characteristic), answering a question of Dixmier.

INTRODUCTION

0.1. Rational ideals have been rather thoroughly explored in various settings. In the simplest case, that of an affine commutative algebra R over an algebraically closed base field $\Bbbk$, rational ideals of R are the same as maximal ideals. More generally, this holds for any affine $\Bbbk$-algebra satisfying a polynomial identity [34]. For other classes of noncommutative algebras R, rational ideals are identical with primitive ideals, that is, annihilators of irreducible R-modules. Examples of such algebras include group algebras of polycyclic-by-finite groups over an algebraically closed base field $\Bbbk$ containing a non-root of unity [19] and enveloping algebras of finite-dimensional Lie algebras over an algebraically closed field $\Bbbk$ of characteristic 0 [24], [15]. Rational ideals of enveloping algebras have been the object of intense investigation by Dixmier, Joseph and many others from the late 1960s through the 80s; see §0.6 below. The fundamental results concerning algebraic group actions on rational ideal spectra, originally developed in the context of enveloping algebras, were later extended to general noetherian (or Goldie) algebras by Mœglin and Rentschler [25], [26], [28], [27] (for characteristic 0) and by Vonessen [39], [40] (for arbitrary characteristic). Currently, the description of rational ideal spectra in algebraic quantum groups is a thriving research topic; see the monograph [6] by Brown and Goodearl for an introduction. Again, rational ideals turn out to coincide with primitive ideals for numerous examples of quantum groups [6, II.8.5].

0.2. The aim of the present article is to liberate the theory of rational ideals of the standard finiteness conditions, noetherianness or the Goldie property, that are traditionally assumed in the literature. Thus, rational ideals are defined and explored here for an arbitrary associative algebra R (with 1) over some base field $\Bbbk$. The Amitsur-Martindale ring of quotients will play the role of the classical ring of quotients which underlies the usual definition of rational ideals but need not exist in general.

Specifically, for any prime ideal P of R, the center of the Amitsur-Martindale ring of quotients of R/P, denoted by $\mathcal{C}(R/P)$ and called the *extended centroid* of R/P, is an extension field of $\Bbbk$. The

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prime P will be called *rational* if

$$\mathcal{C}(R/P) = \Bbbk \, .$$

In the special case where R/P is right Goldie, $\mathcal{C}(R/P)$ coincides with the center of the classical ring of quotients of R/P; so our notion of rationality reduces to the familiar one in this case. Following common practise, we will denote the collection of all rational ideals of R by $\operatorname{Rat} R$; so

$$\operatorname{Rat} R \subseteq \operatorname{Spec} R \, ,$$

where $\operatorname{Spec} R$ is the collection of all prime ideals of R, as usual.

0.3. Besides always being available, the extended centroid turns out to lend itself rather nicely to our investigations. In fact, some of our arguments appear to be more straightforward than earlier proofs in more restrictive settings which were occasionally encumbered by the fractional calculus in classical rings of quotients and by the necessity to ensure the transfer of the Goldie property under various constructions. Section 1 is preliminary in nature and serves to deploy the definition and basic properties of extended centroids in a form suitable for our purposes. In particular, we show that all primitive ideals are rational under fairly general circumstances; see Proposition 6.

After sending out the first version of this article, we learned that much of the material in this section was previously known, partly even for nonassociative rings. For the convenience of the reader, we have opted to leave our proofs intact while also indicating, to the best of our knowledge, the original source of each result.

0.4. In Section 2, we consider actions of a group G by $\Bbbk$-algebra automorphisms on R. Such an action induces G-actions on the extended centroid $\mathcal{C}(R)$ and on the set of ideals of R. Recall that a proper G-stable ideal I of R is said to be *G-prime* if $AB \subseteq I$ for G-stable ideals A and B of R implies that $A \subseteq I$ or $B \subseteq I$. In this case, the subring $\mathcal{C}(R/I)^G$ of G-invariants in $\mathcal{C}(R/I)$ is an extension field of $\Bbbk$. The G-prime I is called *G-rational* if

$$\mathcal{C}(R/I)^G = \Bbbk \, .$$

We will denote the collections of all G-prime and all G-rational ideals of R by G-$\operatorname{Spec} R$ and G-$\operatorname{Rat} R$, respectively; so

$$G\text{-}\operatorname{Rat} R \subseteq G\text{-}\operatorname{Spec} R \, .$$

The action of G on the set of ideals of R preserves both $\operatorname{Spec} R$ and $\operatorname{Rat} R$. Writing the corresponding sets of G-orbits as $G\backslash \operatorname{Spec} R$ and $G\backslash \operatorname{Rat} R$, the assignment $P \mapsto \bigcap_{g\in G} g.P$ always yields a map

$$G\backslash \operatorname{Spec} R \longrightarrow G\text{-}\operatorname{Spec} R \, . \tag{1}$$

Under fairly mild hypotheses, (1) is surjective: this certainly holds whenever every G-orbit in R generates a finitely generated ideal of R; see Proposition 8(b). In Proposition 12 we show that (1) always restricts to a map

$$G\backslash \operatorname{Rat} R \longrightarrow G\text{-}\operatorname{Rat} R \, . \tag{2}$$

More stringent conditions are required for (2) to be surjective. If the group G is finite then (1) is easily seen to be a bijection, and it follows from Lemma 10 that (2) is bijective as well.

0.5. Section 3 focuses on rational actions of an affine algebraic $\Bbbk$-group G on R; the basic definitions will be recalled at the beginning of the section. Working over an algebraically closed base field $\Bbbk$, we show that (2) is then a bijection:

Theorem 1. *Let R be an associative algebra over the algebraically closed field $\Bbbk$ and let G be an affine algebraic group over $\Bbbk$ acting rationally by $\Bbbk$-algebra automorphisms on R. Then the map $P \mapsto \bigcap_{g\in G} g.P$ yields a surjection $\operatorname{Rat} R \twoheadrightarrow G\text{-}\operatorname{Rat} R$ whose fibres are the G-orbits in $\operatorname{Rat} R$.*

The theorem quickly reduces to the situation where G is connected. Theorem 22 gives a description of the fibre of the map $\operatorname{Rat} R \to G\text{-}\operatorname{Rat} R$ over any given G-rational ideal of R for connected G. This description allows us to prove transitivity of the G-action on the fibres by simply invoking an earlier result of Vonessen [40, Theorem 4.7] on subfields of the rational function field $\Bbbk(G)$ that are stable under the regular G-action. Under suitable Goldie hypotheses, Theorem 1 is due to Mœglin and Rentschler [27, Théorème 2] in characteristic 0 and to Vonessen [40, Theorem 2.10] in arbitrary characteristic. The basic outline of our proof of Theorem 1 via the description of the fibres as in Theorem 22 is adapted from the groundbreaking work of Mœglin, Rentschler and Vonessen. However, the generality of our setting necessitates a complete reworking of the material and our presentation contains numerous simplifications over the original arguments.

0.6. The systematic investigation of rational ideals in the enveloping algebra $U(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ over an algebraically closed field $\Bbbk$ of characteristic 0 was initiated by Gabriel [31], [14]. As mentioned in §0.1, it was eventually established that "rational" is tantamount to "primitive" for ideals of $U(\mathfrak{g})$; over an uncountable base field $\Bbbk$, this is due to Dixmier [9]. The reader is referred to the standard reference [11] for a detailed account of the theory of primitive ideals in enveloping algebras; for an updated survey, see [37]. Here we just mention that the original motivation behind Theorem 1 and its predecessors was a question of Dixmier [10] (see also [11, Problem 11]) concerning primitive ideals of $U(\mathfrak{g})$. Specifically, if G is the adjoint algebraic group of $\mathfrak{g}$ then, for any ideal $\mathfrak{k}$ of $\mathfrak{g}$ and any primitive ideal Q of $U(\mathfrak{g})$, the ideal $I = Q \cap U(\mathfrak{k})$ of $U(\mathfrak{k})$ is G-rational [9]. Dixmier asked if the following are true for I:

(a) $I = \bigcap_{g \in G} g.P$ for some primitive ideal P of $U(\mathfrak{k})$, and
(b) any two such primitive ideals belong to the same G-orbit.

The earlier version of Theorem 1, due to Mœglin and Rentschler, settled both (a) and (b) in the affirmative. Letting $\operatorname{Prim} U(\mathfrak{k})$ denote the collection of all primitive ideals of $U(\mathfrak{k})$ endowed with the Jacobson-Zariski topology, (a) says that the set $\{J \in \operatorname{Prim} U(\mathfrak{k}) \mid J \supseteq I\}$ is the closure of the orbit $G.P$ in $\operatorname{Prim} U(\mathfrak{k})$. Following Dixmier [10] such P are called *generic* for I. The uniqueness of generic orbits as in (b) was proved for solvable $\mathfrak{g}$ in [4] and generally (over uncountable $\Bbbk$) in [36]; this fact was instrumental for the proof that the Dixmier and Duflo maps are injective in the solvable and algebraic case, respectively (Rentschler [35], Duflo [12]).

0.7. In future work, we hope to address some topological aspects of $\operatorname{Rat} R$ endowed with the Jacobson-Zariski topology from $\operatorname{Spec} R$. Finally, it remains to bring the machinery developed herein to bear on new classes of algebras that lack the traditional finiteness conditions.

1. THE EXTENDED CENTROID

Throughout this section, R will denote an associative ring. It is understood that all rings have a 1 which is inherited by subrings and preserved under homomorphisms.

1.1. The Amitsur-Martindale ring of quotients. Let $\mathscr{E} = \mathscr{E}(R)$ denote the filter consisting of all (two-sided) ideals I of R such that

$$\operatorname{l.ann}_R I = \{r \in R \mid rI = 0\} = 0 \ .$$

The right Amitsur-Martindale ring of quotients, introduced for prime rings R by Martidale [22] and in general by Amitsur [1], is defined by

$$\mathrm{Q_r}(R) = \varinjlim_{I \in \mathscr{E}} \operatorname{Hom}(I_R, R_R) \ .$$

Explicitly, the elements of $\mathrm{Q_r}(R)$ are equivalence classes of right R-module maps $f \colon I_R \to R_R$ with $I \in \mathscr{E}$; the map f is defined to be equivalent to $f' \colon I'_R \to R_R$ ($I' \in \mathscr{E}$) if f and f' agree on some ideal $J \subseteq I \cap I'$, $J \in \mathscr{E}$. In this case, f and f' actually agree on $I \cap I'$; see [1, Lemma 1]. The sum of two elements $q, q' \in \mathrm{Q_r}(R)$, represented by $f\colon I_R \to R_R$ $(I \in \mathscr{E})$ and $f'\colon I'_R \to R_R$ $(I' \in \mathscr{E})$, respectively, is defined to be the class of $f + f'\colon I \cap I' \to R$. Similarly, the product $qq' \in \mathrm{Q_r}(R)$ is the class of the composite $f \circ f'\colon I'I \to R$. This makes $\mathrm{Q_r}(R)$ into a ring; the identity element is the class of the identity map Id_R on R. Sending an element $r \in R$ to the equivalence class of the map $\lambda_r\colon R \to R$, $x \mapsto rx$, yields an embedding of R as a subring of $\mathrm{Q_r}(R)$. Suppose the element $q \in \mathrm{Q_r}(R)$ is represented by $f\colon I_R \to R_R$ $(I \in \mathscr{E})$. Then the equality $f \circ \lambda_r = \lambda_{f(r)}$ $(r \in I)$ shows that $qI \subseteq R$.

We summarize the foregoing and some easy consequences thereof in the following proposition. Complete details can be found in [1] and in [32, Proposition 10.2], for example.

Proposition 2. *The ring* $\mathrm{Q_r}(R)$ *has the following properties:*

i. *There is a ring embedding* $R \hookrightarrow \mathrm{Q_r}(R)$*;*
ii. *for each* $q \in \mathrm{Q_r}(R)$*, there exits* $I \in \mathscr{E}$ *with* $qI \subseteq R$*;*
iii. *if* $qI = 0$ *for* $q \in \mathrm{Q_r}(R)$ *and* $I \in \mathscr{E}$ *then* $q = 0$*;*
iv. *given* $f\colon I_R \to R_R$ *with* $I \in \mathscr{E}$*, there exists* $q \in \mathrm{Q_r}(R)$ *with* $qr = f(r)$ *for all* $r \in I$*.*

Furthermore, $\mathrm{Q_r}(R)$ *is characterized by these properties: any other ring satisfying* (i) – (iv) *is* R*-isomorphic to* $\mathrm{Q_r}(R)$*.*

1.2. The extended centroid. The extended centroid of R is defined to be the center of $\mathrm{Q_r}(R)$; it will be denoted by $\mathcal{C}(R)$:

$$\mathcal{C}(R) = \mathcal{Z}(\mathrm{Q_r}(R)) \ .$$

It is easy to see from Proposition 2 that $\mathcal{C}(R)$ coincides with the centralizer of R in $\mathrm{Q_r}(R)$:

$$\mathcal{C}(R) = C_{\mathrm{Q_r}(R)}(R) = \{q \in \mathrm{Q_r}(R) \mid qr = rq \ \forall r \in R\} \ .$$

In particular, the center $\mathcal{Z}(R)$ of R is contained in $\mathcal{C}(R)$. Moreover, an element $q \in \mathrm{Q_r}(R)$ belongs to $\mathcal{C}(R)$ if and only if q is represented by an (R,R)-*bimodule* map $f\colon I \to R$ with $I \in \mathscr{E}$; in this case, every representative $f'\colon I'_R \to R_R$ $(I' \in \mathscr{E})$ of q is an (R,R)-bimodule map; see [1, Theorem 3].

1.2.1. By reversing sides, one can define the left ring of quotients $\mathrm{Q}_\ell(R)$ and its center $\mathcal{C}_\ell(R) = \mathcal{Z}(\mathrm{Q}_\ell(R))$ as above. However, we will mainly be concerned with semiprime rings, that is, rings R having no nonzero ideals of square 0. In that case, $\mathrm{l.ann}_R\, I = \mathrm{r.ann}_R\, I$ holds for every ideal I of R; so the definition of $\mathscr{E}(R)$ is symmetric. Moreover, any $q \in \mathcal{C}(R)$ is represented by an (R,R)-bimodule map $f\colon I \to R$ with $I \in \mathscr{E}$. The class of f in $\mathrm{Q}_\ell(R)$ is an element $q' \in \mathcal{C}_\ell(R)$, and the map $q \mapsto q'$ yields an isomorphism $\mathcal{C}(R) \xrightarrow{\sim} \mathcal{C}_\ell(R)$. In the following, we shall always work with $\mathrm{Q_r}(R)$ and $\mathcal{C}(R)$.

1.2.2. Let R be semiprime. Then one knows that $\mathcal{C}(R)$ is a von Neumann regular ring. Moreover, R is prime if and only if $\mathcal{C}(R)$ is a field; see [1, Theorem 5].

1.3. Central closure. Rings R such that $\mathcal{C}(R) \subseteq R$ are called *centrally closed*. In this case, $\mathcal{C}(R) = \mathcal{Z}(R)$. For every semiprime ring R, the subring $R\mathcal{C}(R)$ of $\mathrm{Q_r}(R)$ is a semiprime centrally closed ring called the *central closure* of R; see [2, Theorem 3.2]. If R is prime then so is the central closure $R\mathcal{C}(R)$ by Proposition 2(ii).

The following lemma goes back to Martindale [22].

Lemma 3. *Let* R *be a prime centrally closed ring and let* S *be an algebra over the field* $C = \mathcal{C}(R)$*. Then:*

(a) *Every nonzero ideal* I *of* $R \otimes_C S$ *contains an element* $0 \neq r \otimes s$ *with* $r \in R$*,* $s \in S$*.*
(b) *If* S *is simple then every nonzero ideal* I *of* $R \otimes_C S$ *intersects* R *nontrivially. Consequently,* $R \otimes_C S$ *is prime.*
(c) *If* I *is a prime ideal of* $R \otimes_C S$ *such that* $I \cap R = 0$ *then* $I = R \otimes_C (I \cap S)$*.*

Proof. (a) Fix a C-basis $\{s_i\}$ of S. Consider an element $0 \neq t = \sum_i r_i \otimes s_i \in I$ with a minimal number of nonzero R-coefficients r_i among all nonzero elements of I and choose i_0 with $r = r_{i_0} \neq 0$. Then the element $rxt - txr = \sum_{i \neq i_0} (rxr_i - r_i xr) \otimes s_i$ must be zero for all $x \in R$. Hence $rxr_i = r_i xr$ holds for all i, and by [22, Theorem 1], there are $c_i \in C$ such that $r_i = rc_i$. Therefore, $t = r \otimes s$ with $s = \sum_i c_i s_i \in S$.

(b) If S is simple then we can make $s = 1$ in (a), and so $0 \neq r \in I \cap R$. Since R is prime, it follows that $R \otimes_C S$ is prime as well.

(c) Suppose for a contradiction that $I \supsetneq R \otimes_C (I \cap S)$. Replacing S by $S/(I \cap S)$, we may assume that $I \neq 0$ but $I \cap R = 0$ and $I \cap S = 0$. Choosing $r \otimes s \in I$ as in (a), we obtain that $I \supseteq S(r \otimes s)R = rR \otimes_C Ss$. Since I is prime, we must have $r \in I$ or $s \in I$, whence the desired contradiction. $\square$

1.4. Examples.

1.4.1. If R is a simple ring, or a finite product of simple rings, then $\mathscr{E}(R) = \{R\}$, and hence $\mathrm{Q_r}(R) = R$ by Proposition 2(i)(ii). Thus, R is certainly centrally closed in this case. Less trivial examples of centrally closed rings include crossed products $R * F$ with R a simple ring and F a free semigroup on at least two generators ([32, Theorem 13.4]) and Laurent power series rings $R((x))$ over centrally closed rings R ([20]).

1.4.2. If R is semiprime right Goldie then $\mathcal{C}(R) = \mathcal{Z}(\mathrm{Q_{cl}}(R))$, the center of the classical ring of quotients of R. Indeed, $\mathrm{Q_{cl}}(R)$ coincides with the maximal ring of quotients $\mathrm{Q_{max}}(R)$ in this case; see, e.g., Lambek [17, Prop. 4.6.2]. Furthermore, the Amitsur-Martindale ring of quotients $\mathrm{Q_r}(R)$ is R-isomorphic to the subring of $\mathrm{Q_{max}}(R)$ consisting of all $q \in \mathrm{Q_{max}}(R)$ such that $qI \subseteq R$ for some $I \in \mathscr{E}(R)$; see, e.g., [33, Chap. 24] or [29, Chap. 3]. This isomorphism yields an isomorphism $\mathcal{C}(R) \cong \mathcal{Z}(\mathrm{Q_{max}}(R))$.

1.4.3. Let R be a semiprime homomorphic image of the enveloping algebra $U(\mathfrak{g})$ of a finite-dimensional Lie algebra $\mathfrak{g}$ over some base field $\Bbbk$. Answering a question of Rentschler, we show here that

$$\mathrm{Q_r}(R) \textit{ consists of all } \operatorname{ad}\mathfrak{g}\textit{-finite elements of } \mathrm{Q_{cl}}(R).$$

Here, $\operatorname{ad}: U(\mathfrak{g}) \to \operatorname{End}_{\Bbbk} \mathrm{Q_{cl}}(R)$ is the standard adjoint action, given by $\operatorname{ad} x(q) = xq - qx$ for $x \in \mathfrak{g}$ and $q \in \mathrm{Q_{cl}}(R)$, and q is called $\operatorname{ad}\mathfrak{g}$-finite if the $\Bbbk$-subspace $\operatorname{ad} U(\mathfrak{g})(q)$ of $\mathrm{Q_{cl}}(R)$ is a finite dimensional. To prove the claim, recall from §1.4.2 that $\mathrm{Q_r}(R) = \{q \in \mathrm{Q_{cl}}(R) \mid qI \subseteq R \text{ for some } I \in \mathscr{E}(R)\}$. First consider $q \in \mathrm{Q_r}(R)$. Letting R_n and $I_n = I \cap R_n$ $(n \geq 0)$ denote the filtrations of R and I, respectively, that are induced by the canonical filtration of $U(\mathfrak{g})$ ([11, 2.3.1]), we have $I = I_s R$ and $qI_s \subseteq R_t$ for suitable $s, t \geq 0$. Since both I_s and R_t are $\operatorname{ad}(\mathfrak{g})$-stable, it follows that $\operatorname{ad} U(\mathfrak{g})(q) I_s \subseteq R_t$. Furthermore, $\operatorname{l.ann}_{\mathrm{Q_{cl}}(R)} I_s = \operatorname{l.ann}_{\mathrm{Q_{cl}}(R)} I = 0$; so $\operatorname{ad} U(\mathfrak{g})(q)$ embeds into $\operatorname{Hom}_{\Bbbk}(I_s, R_t)$ proving that q is $\operatorname{ad}\mathfrak{g}$-finite. Conversely, suppose that $q \in \mathrm{Q_{cl}}(R)$ is $\operatorname{ad}\mathfrak{g}$-finite and let $\{q_i\}_1^m$ be a $\Bbbk$-basis of $\operatorname{ad} U(\mathfrak{g})(q)$. Each $D_i = \{r \in R \mid q_i r \in R\}$ is an essential right ideal of R, and hence $I = \bigcap_1^m D_i = \{r \in R \mid \operatorname{ad} U(\mathfrak{g})(q) r \subseteq R\}$ is an essential right ideal of R which is also $\operatorname{ad}(\mathfrak{g})$-stable, since this holds for $\operatorname{ad} U(\mathfrak{g})(q)$ and R. Therefore, $I \in \mathscr{E}(R)$ which shows that $q \in \mathrm{Q_r}(R)$.

1.5. Centralizing homomorphisms. A ring homomorphism $\varphi\colon R \to S$ is called *centralizing* if the ring S is generated by $\varphi(R)$ and the centralizer $C_S(\varphi(R)) = \{s \in S \mid s\varphi(r) = \varphi(r)s \ \forall r \in R\}$. Surjective ring homomorphisms are clearly centralizing, and composites of centralizing homomorphisms are again centralizing. Note also that any centralizing homomorphism $\varphi\colon R \to S$ sends the center $\mathcal{Z}(R)$ of R to $\mathcal{Z}(S)$. Finally, φ induces a map $\operatorname{Spec} S \to \operatorname{Spec} R$, $P \mapsto \varphi^{-1}(P)$.

For any $q \in \mathrm{Q_r}(R)$, we define the ideal D_q of R by

$$D_q = \{r \in R \mid qRr \subseteq R\} . \tag{3}$$

By Proposition 2(ii), $D_q \in \mathscr{E}(R)$. If $q \in \mathcal{C}(R)$ then the description of the ideal D_q simplifies to $D_q = \{r \in R \mid qr \subseteq R\}$.

Lemma 4. *Let $\varphi\colon R \to S$ be a centralizing homomorphism of rings. Put*

$$\mathcal{C}_\varphi = \{q \in \mathcal{C}(R) \mid \operatorname{l.ann}_S \varphi(D_q) = 0\} .$$

Then $R\mathcal{C}_\varphi$ is a subring of $\mathrm{Q_r}(R)$ containing R. The map φ extends uniquely to a centralizing ring homomorphism $\widetilde{\varphi}\colon R\mathcal{C}_\varphi \to S\mathcal{C}(S)$. In particular, $\widetilde{\varphi}(\mathcal{C}_\varphi) \subseteq \mathcal{C}(S)$.

Proof. Put

$$R_\varphi = \{q \in \mathrm{Q_r}(R) \mid \operatorname{l.ann}_S \varphi(D_q) = 0\} .$$

Since $R = \{q \in \mathrm{Q_r}(R) \mid 1 \in D_q\}$, we certainly have $R \subseteq R_\varphi$. For $q, q' \in \mathrm{Q_r}(R)$, one easily checks that $D_{q'}D_q \subseteq D_q \cap D_{q'} \subseteq D_{q+q'}$ and $D_{q'}D_q \subseteq D_{qq'}$. Moreover, if $\varphi(D_q)$ and $\varphi(D_{q'})$ both have zero left annihilator in S then so does $\varphi(D_{q'}D_q) = \varphi(D_{q'})\varphi(D_q)$. This shows that $q + q' \in R_\varphi$ and $qq' \in R_\varphi$ for $q, q' \in R_\varphi$; so R_φ is a subring of $\mathrm{Q_r}(R)$ containing R. Since $\mathcal{C}_\varphi = \mathcal{Z}(R_\varphi)$, it follows that $R\mathcal{C}_\varphi$ is also a subring of $\mathrm{Q_r}(R)$ containing R.

Now let $q \in \mathcal{C}_\varphi$ be given. Then $\varphi(D_q)S = \varphi(D_q)C_S(\varphi(R)) \in \mathscr{E}(S)$. Define $\overline{q}\colon \varphi(D_q)S \to S$ by

$$\overline{q}(\sum_i \varphi(x_i)c_i) = \sum_i \varphi(qx_i)c_i$$

for $x_i \in D_q$, $c_i \in C_S(\varphi(R))$. To see that $\overline{q}$ is well-defined, note that, for each $d \in D_q$, we have

$$\begin{aligned}\sum_i \varphi(x_i)c_i\varphi(qd) &= \sum_i \varphi(x_i)\varphi(qd)c_i = \sum_i \varphi(x_iqd)c_i \\ &= \sum_i \varphi(qx_id)c_i = \sum_i \varphi(qx_i)\varphi(d)c_i \\ &= \sum_i \varphi(qx_i)c_i\varphi(d) .\end{aligned}$$

Thus, if $\sum_i \varphi(x_i)c_i = \sum_j \varphi(y_j)e_j$ with $x_i, y_j \in D_q$ and $c_i, e_j \in C_S(\varphi(R))$ then the above computation gives

$$0 = \left(\sum_i \varphi(x_i)c_i - \sum_j \varphi(y_j)e_j\right)\varphi(qD_q) = \left(\sum_i \varphi(qx_i)c_i - \sum_j \varphi(qy_j)e_j\right)\varphi(D_q) ,$$

and so $0 = \sum_i \varphi(qx_i)c_i - \sum_j \varphi(qy_j)e_j$. Therefore, $\overline{q}$ is well-defined.

It is straightforward to check that $\overline{q}$ is an (S,S)-bimodule map. Hence, the class of $\overline{q}$ in $\mathrm{Q_r}(R)$ is an element $\widetilde{\varphi}(q) \in \mathcal{C}(S)$. The map $q \mapsto \widetilde{\varphi}(q)$ is a ring homomorphism $\mathcal{C}_\varphi \to \mathcal{C}(S)$ which yields the desired extension $\widetilde{\varphi}\colon R\mathcal{C}_\varphi \to S\mathcal{C}(S)$: $\widetilde{\varphi}(\sum_i r_iq_i) = \sum_i \varphi(r_i)\widetilde{\varphi}(q_i)$ for $r_i \in R$, $q_i \in \mathcal{C}_\varphi$. Well-definedness and uniqueness of $\widetilde{\varphi}$ follow easily from the fact that, given finitely many $x_i \in R_\varphi$, there is an ideal D of R with $\operatorname{l.ann}_S \varphi(D) = 0$ and $x_iD \subseteq R$ for all i. □

In the special case where both R and S are commutative domains in Lemma 4 above, we have $\mathrm{Q_r}(R) = \mathcal{C}(R) = \operatorname{Fract} R$, the classical field of fractions of R, and similarly for S. Moreover, $R\mathcal{C}_\varphi = R_P$ is the localization of R at the prime $P = \operatorname{Ker}\varphi$ and the map $R\mathcal{C}_\varphi \to S\mathcal{C}(S)$ is the usual map $R_P \to \operatorname{Fract} S$.

1.6. Extended centroids and primitive ideals. By Schur's Lemma, the endomorphism ring $\operatorname{End}_R V$ of any simple R-module V_R is a division ring. The following lemma is well-known in the special case of noetherian (or Goldie) rings (see, e.g., Dixmier [11, 4.1.6]); for general rings, the lemma was apparently first observed by Martindale [21, Theorem 12]. Since the latter result is stated in terms of the so-called complete ring of quotients, we include the proof for the reader's convenience.

Lemma 5. *Let V_R be a simple R-module, and let $P = \operatorname{ann}_R V$ be its annihilator. Then the canonical embedding $\mathcal{Z}(R/P) \hookrightarrow \mathcal{Z}(\operatorname{End}_R V)$ extends to an embedding of fields*

$$\mathcal{C}(R/P) \hookrightarrow \mathcal{Z}(\operatorname{End}_R V) \ .$$

Proof. We may assume that $P = 0$. For a given $q \in \mathcal{C}(R)$, we wish to define an endomorphism $\delta_q \in \mathcal{Z}(\operatorname{End}_R V)$. To this end, note that every $x \in V$ can be written as $x = vd$ for suitable $d \in D_q$, $v \in V$. Define

$$\delta_q(x) = v(dq) \in V \ .$$

To see that this is well-defined, assume that $vd = v'd'$ holds for $v, v' \in V$ and $d, d' \in D_q$. Then $(v(dq) - v'(d'q))\, D_q = (vd - v'd')\,(qD_q) = 0$ and so $v(dq) - v'(d'q) = 0$. It is straightforward to check that $\delta_q \in \operatorname{End}_R V$. Moreover, for any $\delta \in \operatorname{End}_R V$ and $vd \in V$, one computes

$$\delta\delta_q(vd) = \delta(v(dq)) = \delta(v)(dq) = \delta_q(\delta(v)d) = \delta_q\delta(vd) \ .$$

Thus, $\delta_q \in \mathcal{Z}(\operatorname{End}_R V)$. The map $\mathcal{C}(R) \to \mathcal{Z}(\operatorname{End}_R V)$, $q \mapsto \delta_q$, is easily seen to be additive. Furthermore, for $q, q' \in \mathcal{C}(R)$, $d \in D_q$, $d' \in D_{q'}$ and $v \in V$, one has

$$\delta_{qq'}(vd'd) = v(d'dqq') = v(d'q')(dq) = \delta_q(\delta_{q'}(vd')d) = \delta_q(\delta_{q'}(vd'd)) \ .$$

Thus, the map is a ring homomorphism; it is injective because $\mathcal{C}(R)$ is a field. □

1.7. Rational algebras and ideals. An algebra R over some field $\Bbbk$ will be called *rational* (or *$\Bbbk$-rational*) if R is prime and $\mathcal{C}(R) = \Bbbk$. A prime ideal P of R will be called rational if R/P is a rational $\Bbbk$-algebra. In view of §1.2.1, the notion of rationality is left-right symmetric.

We remark that rational $\Bbbk$-algebras are called *closed over* $\Bbbk$ in [13] where such algebras are investigated in a non-associative context. Alternatively, one could define a prime $\Bbbk$-algebra R to be rational if the field extension $\mathcal{C}(R)/\Bbbk$ is algebraic; for noetherian (or Goldie) algebras, this version of rationality is adopted in many places in the literature (e.g., [6]). However, we will work with the above definition throughout.

1.7.1. By §1.3 the central closure $R\mathcal{C}(R)$ of any prime ring R is $\mathcal{C}(R)$-rational.

1.7.2. The Schur division rings $\operatorname{End}_R V$ considered in §1.6 are division algebras over $\Bbbk$, and their centers are extension fields of $\Bbbk$. We will say that the algebra R satisfies the *weak Nullstellensatz* if $\mathcal{Z}(\operatorname{End}_R V)$ is algebraic over $\Bbbk$ for every simple R-module V_R.

Proposition 6. *If R is a $\Bbbk$-algebra satisfying the weak Nullstellensatz and $\Bbbk$ is algebraically closed then all primitive ideals of R are rational.*

Proof. By hypothesis, $\mathcal{Z}(\operatorname{End}_R V) = \Bbbk$ holds for every simple R-module V_R. It follows from Lemma 5 that $P = \operatorname{ann}_R V$ satisfies $\mathcal{C}(R/P) = \Bbbk$. □

For an affine commutative $\Bbbk$-algebra R, the Schur division algebras in question are just the quotients R/P, where P is a maximal ideal of R. The classical weak Nullstellensatz is equivalent to the statement that R/P is always algebraic over $\Bbbk$; see, e.g., Lang [18, Theorem IX.1.4]. Thus affine commutative algebras do satisfy the weak Nullstellensatz.

Many noncommutative algebras satisfying the weak Nullstellensatz are known; see [23, Chapter 9] for an overview. In fact, as long as the cardinality of the base field $\Bbbk$ is larger than $\dim_{\Bbbk} R$, the weak Nullstellensatz is guaranteed to hold; see [23, Corollary 9.1.8] or [6, II.7.16]. This applies, for example, to any countably generated algebra over an uncountable field $\Bbbk$.

1.8. Scalar extensions. We continue to let R denote an algebra over some field $\Bbbk$. For any given $\Bbbk$-algebra A, we have an embedding

$$\mathrm{Q_r}(R) \otimes_{\Bbbk} A \hookrightarrow \mathrm{Q_r}(R \otimes_{\Bbbk} A)$$

which extends the canonical embedding $R \otimes_{\Bbbk} A \hookrightarrow \mathrm{Q_r}(R \otimes_{\Bbbk} A)$. For, let $q \in \mathrm{Q_r}(R)$ be represented by the map $f\colon I_R \to R_R$ with $I \in \mathscr{E}(R)$. Then $I \otimes_{\Bbbk} A \in \mathscr{E}(R \otimes_{\Bbbk} A)$. Sending q to the class of the map $f \otimes \mathrm{Id}_A$ we obtain a ring homomorphism $\mathrm{Q_r}(R) \to \mathrm{Q_r}(R \otimes_{\Bbbk} A)$ extending the canonical embedding $R \hookrightarrow R \otimes_{\Bbbk} A \hookrightarrow \mathrm{Q_r}(R \otimes_{\Bbbk} A)$. By Proposition 2(ii),(iii), the image of $\mathrm{Q_r}(R)$ in $\mathrm{Q_r}(R \otimes_{\Bbbk} A)$ commutes with A and the resulting map $\mathrm{Q_r}(R) \otimes_{\Bbbk} A \to \mathrm{Q_r}(R \otimes_{\Bbbk} A)$ is injective. Moreover, since $f \otimes \mathrm{Id}_A$ is an $(R \otimes_{\Bbbk} A, R \otimes_{\Bbbk} A)$-bimodule map if f is an (R, R)-bimodule map, the embedding of $\mathrm{Q_r}(R)$ into $\mathrm{Q_r}(R \otimes_{\Bbbk} A)$ sends $\mathcal{C}(R)$ to $\mathcal{C}(R \otimes_{\Bbbk} A)$. Thus, if A is commutative, this yields an embedding

$$\mathcal{C}(R) \otimes_{\Bbbk} A \hookrightarrow \mathcal{C}(R \otimes_{\Bbbk} A) \; . \tag{4}$$

The following lemma is the associative case of [13, Theorem 3.5].

Lemma 7. *Assume that R is rational. Then, for every field extension $K/\Bbbk$, the K-algebra $R_K = R \otimes_{\Bbbk} K$ is rational.*

Proof. By Lemma 3(b), we know that R_K is prime. Moreover, for any given $q \in \mathcal{C}(R_K)$, we may choose an element $0 \neq x \in D_q \cap R$. Fix a $\Bbbk$-basis $\{k_i\}$ for K. The map

$$q_i\colon I = RxR \xrightarrow{q\cdot} R_K \overset{\text{proj}}{\twoheadrightarrow} R \otimes k_i \xrightarrow{\sim} R$$

is an (R, R)-bimodule map. Hence q_i is multiplication with some $c_i \in \Bbbk$, by hypothesis on R, and all but finitely many c_i are zero. Therefore, the map $I \xrightarrow{q\cdot} R_K$ is multiplication with $k = \sum_i c_i k_i \in K$. Consequently, $q = k \in K$. ☐

2. Group actions

In this section, we assume that a group G acts by automorphisms on the ring R; the action will be written as $G \times R \to R$, $(g, r) \mapsto g.r$.

2.1. Let M be a set with a left G-action $G \times M \to M$, $(g, m) \mapsto g.m$. For any subset X of M,

$$G_X = \operatorname{stab}_G X = \{g \in G \mid g.X = X\}$$

will denote the isotropy group of X. Furthermore, we put

$$(X : G) = \bigcap_{g \in G} g.X \; ;$$

this is the largest G-stable subset of M that is contained in X. We will be primarily concerned with the situation where $M = R$ and X is an ideal of R in which case $(X : G)$ is also an ideal of R.

2.2. G-primes. The ring R is said to be *G-prime* if $R \neq 0$ and the product of any two nonzero G-stable ideals of R is again nonzero. A G-stable ideal I of R is called G-prime if R/I is a G-prime ring for the G-action on R/I coming from the given action of G on R. In the special case where the G-action on R is trivial, G-primes of R are just the prime ideals of R in the usual sense. Recall that the collection of all G-prime ideals of R is denoted by G-$\operatorname{Spec} R$ while $\operatorname{Spec} R$ is the collection of all ordinary primes of R.

Proposition 8. (a) *There is a well-defined map*

$$\operatorname{Spec} R \longrightarrow G\text{-}\operatorname{Spec} R \; , \qquad P \mapsto (P : G) \; .$$

(b) *Assume that, for each $r \in R$, the G-orbit $G.r$ generates a finitely generated ideal of R. Then the map in (a) is surjective. In particular, all G-primes of R are semiprime in this case.*

Proof. It is straightforward to check that $(P : G)$ is G-prime for any prime ideal P of R; so (a) is clear.

For (b), consider a G-prime ideal I of R. We will show that there is an ideal P of R which is maximal subject to the condition $(P : G) = I$; the ideal P is then easily seen to be prime. In order to prove the existence of P, we use Zorn's Lemma. So let $\{I_j\}$ be a chain of ideals of R such that $(I_j : G) = I$ holds for all j. We need to show that the ideal $I_* = \bigcup_j I_j$ satisfies $(I_* : G) = I$. For this, let $r \in (I_* : G)$ be given. Then the ideal $(G.r)$ that is generated by $G.r$ is contained in $(I_* : G)$ and $(G.r)$ is a finitely generated G-stable ideal of R. Therefore, $(G.r) \subseteq (I_j : G)$ for some j and so $r \in I$, as desired. □

For brevity, we will call G-actions satisfying the finiteness hypothesis in (b) above *locally ideal finite*. Clearly, all actions of finite groups as well as all group actions on noetherian rings are locally ideal finite. Another important class of examples are the *locally finite* actions in the usual sense: by definition, these are G-actions on some $\Bbbk$-algebra R such that the G-orbit of each $r \in R$ generates a finite-dimensional $\Bbbk$-subspace of R. This includes the rational actions of algebraic groups to be considered in Section 3. In all these cases, Proposition 8 is a standard result; the argument given above is merely a variant of earlier proofs.

2.3. G-primes and the extended centroid. The G-action on R extends uniquely to an action of G on $\mathrm{Q_r}(R)$: if $q \in \mathrm{Q_r}(R)$ is represented by $f\colon I_R \to R_R$ $(I \in \mathscr{E})$ then $g.q \in \mathrm{Q_r}(R)$ is defined to be the class of the map $g.f\colon g.I \to R$ that is given by $(g.f)(g.x) = g.f(x)$ for $x \in I$. Therefore, G also acts on the extended centroid $\mathcal{C}(R)$ of R. As usual, the ring of G-invariants in $\mathcal{C}(R)$ will denoted by $\mathcal{C}(R)^G$.

Proposition 9. *If R is G-prime then $\mathcal{C}(R)^G$ is a field. Conversely, if R is semiprime and $\mathcal{C}(R)^G$ is a field then R is G-prime.*

Proof. We follow the outline of the proof of [1, Theorem 5].

First assume that R is G-prime and let $0 \neq q \in \mathcal{C}(R)^G$ be given. Then qD_q is a nonzero G-stable ideal of R, and hence $\mathrm{l.ann}_R(qD_q) = 0$ because R is G-prime. So $qD_q \in \mathscr{E}(R)$. Moreover, $\mathrm{ann}_R(q) = \{r \in R \mid rq = 0\} \subseteq \mathrm{l.ann}_R(qD_q)$ and so $\mathrm{ann}_R(q) = 0$. Therefore, the map $D_q \to qD_q$, $r \mapsto qr = rq$, is an (R, R)-bimodule isomorphism which is G-equivariant. The class of the inverse map belongs to $\mathcal{C}(R)^G$ and is the desired inverse for q.

Next, assume that R is semiprime but not G-prime. Then there exists a nonzero G-stable ideal I of R such that $J = \mathrm{l.ann}_R(I) \neq 0$. Since R is semiprime, the sum $I + J$ is direct and $I + J \in \mathscr{E}(R)$. Define maps $f, f'\colon I + J \to R$ by $f(i + j) = i$ and $f'(i + j) = j$. Letting q and q' denote the classes of f and f', respectively, in $\mathrm{Q_r}(R)$ we have $f, f' \in \mathcal{C}(R)^G$ and $ff' = 0$. Therefore, $\mathcal{C}(R)^G$ is not a field. □

The following technical lemma will be crucial. Recall that G_I denotes the isotropy group of I.

Lemma 10. *Let P be a prime ideal of R.*

(a) *For every subgroup of $H \leq G$, the canonical map $R/(P : G) \twoheadrightarrow R/(P : H)$ induces an embedding of fields*

$$\mathcal{C}(R/(P : G))^G \hookrightarrow \mathcal{C}(R/(P : H))^{G_{(P:H)}} .$$

The degree of the field extension is at most $[G : G_{(P:H)}]$.

(b) *If P has a finite G-orbit then we obtain an isomorphism of fields*

$$\mathcal{C}(R/(P : G))^G \xrightarrow{\sim} \mathcal{C}(R/P)^{G_P} .$$

Proof. (a) After factoring out the ideal $(P : G)$ we may assume that $(P : G) = 0$, R is G-prime, and $\mathcal{C}(R)^G$ is a field; see Propositions 8 and 9. Consider the canonical map

$$\varphi\colon R \twoheadrightarrow S := R/(P : H) .$$

Using the notation of Lemma 4, we have $\mathcal{C}(R)^G \subseteq \mathcal{C}_\varphi$. Indeed, for each $q \in \mathcal{C}(R)^G$, the ideal D_q is nonzero and G-stable, and hence $D_q \nsubseteq P$. Therefore, $\varphi(D_q)$ is a nonzero H-stable ideal of the H-prime ring S, and so $\varphi(D_q) \in \mathscr{E}(S)$. The map $\mathcal{C}_\varphi \to \mathcal{C}(S)$ constructed in Lemma 4 yields an embedding embedding $\mathcal{C}(R)^G \hookrightarrow \mathcal{C}(S)$: the image of $q \in \mathcal{C}(R)^G$ is the class of the map $f\colon \varphi(D_q) \to S$ that is defined by $f(\varphi(x)) = \varphi(qx)$ for $x \in D_q$. Since φ is $G_{(P:H)}$-equivariant, one computes, for $x \in D_q$ and $g \in G_{(P:H)}$,

$$\begin{aligned}(g.f)(g.\varphi(x)) &= g.f(\varphi(x)) = g.\varphi(qx) = \varphi(g.(qx))\\ &= \varphi(q(g.x)) = f(\varphi(g.x)) = f(g.\varphi(x)) ;\end{aligned}$$

so $g.f = f$. Therefore the image of $\mathcal{C}(R)^G$ is contained in $\mathcal{C}(S)^{G_{(P:H)}}$.

It remains to show that $[\mathcal{C}(S)^{G_{(P:H)}} : \mathcal{C}(R)^G] \le [G : G_{(P:H)}]$ if the latter number is finite. To this end, put $N = \bigcap_{g\in G} g^{-1}G_{(P:H)}g$; this is a normal subgroup of G which has finite index in G and is contained in $G_{(P:H)}$. Since $(P : H) = (P : G_{(P:H)})$, the foregoing yields embeddings of fields

$$\mathcal{C}(R)^G \hookrightarrow \mathcal{C}(S)^{G_{(P:H)}} = \mathcal{C}(R/(P : G_{(P:H)}))^{G_{(P:H)}} \hookrightarrow \mathcal{C}(R/(P : N))^{N'} ,$$

where $N' := G_{(P:H)} \cap G_{(P:N)}$. The image of $\mathcal{C}(R)^G$ under the composite embedding is contained in $\mathcal{C}(R/(P : N))^{G_{(P:N)}}$ and, by Galois theory,

$$[\mathcal{C}(R/(P : N))^{N'} : \mathcal{C}(R/(P : N))^{G_{(P:N)}}] \le [G_{(P:N)} : N'] \le [G : G_{(P:H)}] .$$

It suffices to show that the image of $\mathcal{C}(R)^G$ is actually equal to $\mathcal{C}(R/(P : N))^{G_{(P:N)}}$. Therefore, replacing H by N, it suffices to show:

$$\text{If } H \trianglelefteq G \text{ and } [G : G_{(P:H)}] < \infty \text{ then } \mathcal{C}(R)^G \text{ maps onto } \mathcal{C}(S)^{G_{(P:H)}}. \tag{5}$$

To this end, we will prove the following

Claim 11. Let $t \in \mathcal{C}(S)^{G_{(P:H)}}$ be given. There exists a G-stable ideal I of R such that $0 \ne \varphi(I) \subseteq D_t$ and such that, for every $x \in I$, there exists an $x' \in R$ satisfying

$$\varphi(g.x') = t\varphi(g.x) \quad \text{for all } g \in G. \tag{6}$$

Note that G-stability of I and the condition $\varphi(I) \subseteq D_t$ ensure that $t\varphi(g.x) \in S$ holds for all $g \in G$, $x \in I$. Moreover, any G-stable ideal I satisfying $0 \ne \varphi(I)$ belongs to $\mathscr{E}(R)$. For, $\operatorname{l.ann}_S \varphi(I) = 0$ since S is H-prime, and hence $\operatorname{l.ann}_R I$ is contained in $(P : G) = 0$. Finally, the element x' is uniquely determined by (6) for any given x. Indeed, if $x'' \in R$ also satisfies (6) then $\varphi(g.x') = \varphi(g.x'')$ holds for all $g \in G$ and so $x' - x'' \in (P : G) = 0$. Therefore, assuming the claim for now, we can define a map

$$f\colon I \to R , \quad x \mapsto x' .$$

It is easy to check that f is G-equivariant. Furthermore, for $r_1, r_2 \in R$,

$$\begin{aligned}\varphi(g.(r_1x'r_2)) &= \varphi(g.r_1)\varphi(g.x')\varphi(g.r_2) = \varphi(g.r_1)t\varphi(g.x)\varphi(g.r_2)\\ &= t\varphi(g.r_1)\varphi(g.x)\varphi(g.r_2) = t\varphi(g.(r_1xr_2)) .\end{aligned}$$

This shows that f is (R, R)-bilinear. Hence, defining q to be the class of f, we obtain the desired element $q \in \mathcal{C}(R)^G$ mapping to our given $t \in \mathcal{C}(S)^{G_{(P:H)}}$, thereby proving (5).

It remains to construct I as in the claim. Put

$$D = \Bigg(\bigcap_{\substack{x,y\in G\\ x^{-1}y\notin G_{(P:H)}}} x.(P:H) + y.(P:H) \Bigg)^{[G:G_{(P:H)}]-1}.$$

Then D is a G-stable ideal of R satisfying $0 \neq \varphi(D)$. For the latter note that the finitely many ideals $x.(P:H) + y.(P:H)$ are H-stable, since H is normal, and none of them is contained in $(P:H)$. By the Chinese remainder theorem [7, 1.3], the image of the map

$$\begin{array}{ccccc} \mu\colon R & \hookrightarrow & \prod_{g\in G/G_{(P:H)}} R/g.(P:H) & \xrightarrow{\sim} & \prod_{g\in G/G_{(P:H)}} S \\ \cup & & \cup & & \cup \\ r & \mapsto & (r + g.(P:H))_{g\in G/G_{(P:H)}} & \mapsto & (\varphi(g^{-1}.r))_{g\in G/G_{(P:H)}} \end{array}$$

contains the ideal $\prod_{g\in G/G_{(P:H)}} \varphi(D)$. Now put $I = (\varphi^{-1}(D_t) : G)D$. This is certainly a G-stable ideal of R satisfying $\varphi(I) \subseteq D_t$. Suppose that $\varphi(I) = 0$. Since $\varphi(D)$ is a nonzero H-stable ideal of the H-prime ring S, we must have

$$(\varphi^{-1}(D_t) : G) = \bigcap_{g\in G/G_{(P:H)}} g.\varphi^{-1}(D_t) \subseteq (P:H)$$

and so $g.\varphi^{-1}(D_t) \subseteq (P:H)$ for some $g \in G$. But then $g.\varphi^{-1}(D_t) \subsetneqq \varphi^{-1}(D_t)$ which is impossible because $\varphi^{-1}(D_t)$ is $G_{(P:H)}$-stable and $G_{(P:H)}$ has finite index in G. Therefore, $\varphi(I) \neq 0$. Finally, if $x \in I$ then $\varphi(g.x) \in D_t\varphi(D)$ for all $g \in G$, and hence $t\varphi(g.x) \in \varphi(D)$. Therefore, $(t\varphi(g^{-1}.x))_{g\in G/G_{(P:H)}} = \mu(x')$ for some $x' \in R$, that is, $\varphi(g^{-1}.x') = (t\varphi(g^{-1}.x))$ holds for all $g \in G/G_{(P:H)}$. Since φ and t are $G_{(P:H)}$-invariant, it follows that $\varphi((gh)^{-1}.x') = (t\varphi((gh)^{-1}.x))$ holds for all $g \in G/G_{(P:H)}$, $h \in G_{(P:H)}$. Therefore, $\varphi(g.x') = t\varphi(g.x)$ for all $g \in G$, as desired.

(b) This is just (5) with $H = 1$. □

2.4. G-rational ideals. Assume now that R is an algebra over some field $\Bbbk$, as in Section 1.7, and that G acts on R by $\Bbbk$-algebra automorphisms. A G-prime ideal I of R will be called *G-rational* if $\mathcal{C}(R/I)^G = \Bbbk$. One can check as in §1.2.1 that the notion of G-rationality is left-right symmetric.

Lemma 10(a) with $H = 1$ immediately implies the following

Proposition 12. *The map* $\operatorname{Spec} R \to G\text{-}\operatorname{Spec} R$, $P \mapsto (P : G)$, *in Proposition 8 restricts to a map* $\operatorname{Rat} R \to G\text{-}\operatorname{Rat} R$.

Unfortunately, the map $\operatorname{Rat} R \to G\text{-}\operatorname{Rat} R$ above need not be surjective, even when the G-action on R is locally ideal finite in the sense of Proposition 8(b):

Example 13. Let $F \supset \Bbbk$ be any non-algebraic field extension satisfying $F^G = \Bbbk$ for some subgroup G of $\operatorname{Gal}(F/\Bbbk)$. For example, F could be chosen to be the rational function field $\Bbbk(t)$ over an infinite field $\Bbbk$ and $G = \Bbbk^*$ acting via $\lambda.f(t) = f(\lambda^{-1}t)$ for $\lambda \in \Bbbk^*$. The G-action on F is clearly locally ideal finite and $\mathrm{Q_r}(F) = \mathcal{C}(F) = F$. Therefore, the zero ideal of F is G-rational, but F has no rational ideals.

2.5. Algebras over a large algebraically closed base field. We continue to assume that R is an algebra over some field $\Bbbk$ and that G acts on R by $\Bbbk$-algebra automorphisms. The following lemma is a version of [25, Lemme 3.3].

Lemma 14. *Let* $I \in \operatorname{Spec} R$ *be given. Put* $C = \mathcal{C}(R/I)$ *and consider the natural map of* C*-algebras*

$$\psi\colon R_C = R \otimes_{\Bbbk} C \twoheadrightarrow (R/I) \otimes_{\Bbbk} C \twoheadrightarrow (R/I)C$$

where $(R/I)C \subseteq \mathrm{Q_r}(R/I)$ *is the central closure of* R/I. *Then:*

(a) $\widetilde{I} = \operatorname{Ker}\psi$ *is a* C*-rational ideal of* R_C.
(b) *If* $I \in G\text{-}\operatorname{Rat} R$ *then, letting* G *act on* R_C *by* C*-linear extension of the action on* R*, we have*

$$(\widetilde{I} : G) = I \otimes_{\Bbbk} C \ .$$

Proof. Part (a) is clear, since $R_C/\widetilde{I} \cong (R/I)C$ is C-rational; see §1.7.1.

For (b), note that the map ψ is G-equivariant for the diagonal G-action on $R_C = R \otimes_{\Bbbk} C$ and the usual G-action on $(R/I)C \subseteq \mathrm{Q_r}(R/I)$. Therefore, $\widetilde{I}$ is stable under all automorphisms $g \otimes g$ with $g \in G$, and hence we have $(g \otimes 1)(\widetilde{I}) = (1 \otimes g^{-1})(\widetilde{I})$. We conclude that

$$(\widetilde{I} : G) = \bigcap_{g \in G} (1 \otimes g)(\widetilde{I}) = I \otimes_{\Bbbk} C \ ,$$

where the last equality uses the fact that $\widetilde{I} \cap R = I$ and our hypothesis $C^G = \Bbbk$; see [5, Cor. to Prop. V.10.6]. □

As an application of the lemma, we offer the following "quick and dirty" existence result for generic rational ideals.

Proposition 15. *Let* R *be a countably generated algebra over an algebraically closed base field* $\Bbbk$ *of infinite transcendence degree over its prime subfield and assume that the group* G *is countably generated. Then every prime ideal* $I \in G\text{-}\operatorname{Rat} R$ *has the form* $I = (P : G)$ *for some* $P \in \operatorname{Rat} R$.

Proof. Let a prime $I \in G\text{-}\operatorname{Rat} R$ be given and let $\Bbbk_0$ denote the prime subfield of $\Bbbk$. By hypothesis on R, we have $\dim_{\Bbbk} R \leq \aleph_0$. Choosing a $\Bbbk$-basis $\mathcal{B}$ of R which contains a $\Bbbk$-basis for I and adjoining the structure constants of R with respect to $\mathcal{B}$ to $\Bbbk_0$, we obtain a countable field K with $\Bbbk_0 \subseteq K \subseteq \Bbbk$. Putting $R_0 = \sum_{b \in \mathcal{B}} Kb$ we obtain a K-subalgebra of R such that $R = R_0 \otimes_K \Bbbk$ and $I = I_0 \otimes_K \Bbbk$, where $I_0 = I \cap R_0$. At the cost of adjoining at most countably many further elements to K, we can also make sure that R_0 is stable under the action of G. Thus, R_0/I_0 is a G-stable K-subalgebra of R/I and $R/I = (R_0/I_0) \otimes_K \Bbbk$. Put $C = \mathcal{C}(R_0/I_0)$ and note that (4) implies that $C^G = K$, because $\mathcal{C}(R/I)^G = \Bbbk$. Thus, $I_0 \in G\text{-}\operatorname{Rat} R_0$ and Lemma 14 yields an ideal $\widetilde{I_0} \in \operatorname{Rat}(R_0 \otimes_K C)$ such that $(\widetilde{I_0} : G) = I_0 \otimes_K C$. Furthermore, since R_0/I_0 is countable, the field C is countable as well; this follows from Proposition 2. By hypothesis on $\Bbbk$, there is a $\Bbbk_0$-embedding of C into $\Bbbk$; see [5, Cor. 1 to Théorème V.14.5]. Finally, Lemma 7 implies that $P = \widetilde{I_0} \otimes_C \Bbbk$ is a rational ideal of $(R_0 \otimes_K C) \otimes_C \Bbbk = R$ satisfying $(P : G) = (I_0 \otimes_K C) \otimes_C \Bbbk = I$, as desired. □

3. Rational actions of algebraic groups

In this section, we work over an algebraically closed base field $\Bbbk$. Throughout, the group G will be an affine algebraic group over $\Bbbk$ and R will be a $\Bbbk$-algebra on which G acts by $\Bbbk$-algebra automorphisms. The Hopf algebra of regular functions on G will be denoted by $\Bbbk[G]$. The notations introduced in Section 2 remain in effect. In addition, $\otimes$ will stand for $\otimes_{\Bbbk}$.

3.1. G-modules. A $\Bbbk$-vector space M is called a *G-module* if there is a linear representation

$$\rho_M : G \longrightarrow \operatorname{GL}(M)$$

satisfying

(a) local finiteness: all G-orbits in M generate finite-dimensional subspaces of M, and
(b) for every finite-dimensional G-stable subspace $V \subseteq M$, the induced group homomorphism $G \to \operatorname{GL}(V)$ is a homomorphism of algebraic groups.

As is well-known, these requirements are equivalent to the existence of a $\Bbbk$-linear map

$$\Delta_M : M \longrightarrow M \otimes \Bbbk[G] \tag{7}$$

which makes M into an $\Bbbk[G]$-comodule; see Jantzen [16, 2.7-2.8] or Waterhouse [41, 3.1-3.2] for details. We will use the Sweedler notation

$$\Delta_M(m) = \sum m_0 \otimes m_1 \qquad (m \in M)$$

as in Montgomery [30]. Writing $\rho_M(g)(m) = g.m$, we have

$$g.m = \sum m_0 m_1(g) \qquad (g \in G, m \in M)\ . \tag{8}$$

Linear representations ρ_M as above are often called *rational*. Tensor products of rational representations of G are again rational, and similarly for sums, subrepresentations and homomorphic images of rational representations.

Example 16. If the group G is finite then G-modules are the same as (left) modules M over the group algebra $\Bbbk G$ and all linear representations of G are rational. Indeed, in this case, $\Bbbk[G]$ is the linear dual of $\Bbbk G$, that is, the $\Bbbk$-vector space of all functions $G \to \Bbbk$ with pointwise addition and multiplication. The map $\Delta_M : M \to M \otimes \Bbbk[G]$ is given by

$$\Delta_M(m) = \sum_{x \in G} x.m \otimes p_x\ ,$$

where $p_x \in \Bbbk[G] = (\Bbbk G)^*$ is defined by $p_x(y) = \delta_{x,y}$ (Kronecker delta) for $x, y \in G$.

3.2. Some properties of G-modules. Let M be a G-module. The coaction Δ_M in (7) is injective. In fact, extending Δ_M to a map

$$\Delta_M : M \otimes \Bbbk[G] \longrightarrow M \otimes \Bbbk[G] \tag{9}$$

by $\Bbbk[G]$-linearity, we obtain an *automorphism* of $M \otimes \Bbbk[G]$: the inverse of Δ_M is the $\Bbbk[G]$-linear extension of the map $(\mathrm{Id}_M \otimes S) \circ \Delta_M : M \longrightarrow M \otimes \Bbbk[G]$, where $S : \Bbbk[G] \to \Bbbk[G]$ is the antipode of $\Bbbk[G]$: $(Sf)(g) = f(g^{-1})$ for $g \in G$.

Furthermore, G-stable cores can be computed with Δ_M as follows.

Lemma 17. *For any $\Bbbk$-subspace V of M, we have $(V : G) = \Delta_M^{-1}(V \otimes \Bbbk[G])$*

Proof. Fix a $\Bbbk$-basis $\{v_i\}$ of V and let $\{w_j\}$ be a $\Bbbk$-basis for a complement of V in M. For $m \in M$, we have $\Delta_M(m) = \sum_i v_i \otimes f_i + \sum_j w_j \otimes h_j$ with uniquely determined $f_i, h_j \in \Bbbk[G]$. Moreover,

$$\Delta_M(m) \in V \otimes \Bbbk[G] \iff \text{all } h_j = 0 \iff \forall g \in G : g.m = \sum_i v_i f_i(g) \in V\ .$$

This proves the lemma. $\square$

3.3. Regular representations and intertwining formulas. The right and left *regular representations* of G are defined by

$$\begin{aligned} \rho_r : G &\longrightarrow \mathrm{GL}(\Bbbk[G])\,, \qquad (\rho_r(x)f)\,(y) = f(yx)\ , \\ \rho_\ell : G &\longrightarrow \mathrm{GL}(\Bbbk[G])\,, \qquad (\rho_\ell(x)f)\,(y) = f(x^{-1}y)\ . \end{aligned} \tag{10}$$

for $x, y \in G$. Both regular representations are rational. The right regular representation comes from the comultiplication $\Delta : \Bbbk[G] \to \Bbbk[G] \otimes \Bbbk[G]$ of the Hopf algebra $\Bbbk[G]$; in the usual Sweedler notation, it is given by $\Delta f = \sum f_1 \otimes f_2$, where $f(xy) = \sum f_1(x) f_2(y)$ for $x, y \in G$. Similarly, the left regular representation comes from $\left(S \otimes \mathrm{Id}_{\Bbbk[G]}\right) \circ \Delta \circ S : \Bbbk[G] \to \Bbbk[G] \otimes \Bbbk[G]$.

Now let M be a G-module. Then the rational representations $1_M \otimes \rho_\ell \colon G \to \mathrm{GL}(M \otimes \Bbbk[G])$ and $\rho_M \otimes \rho_\ell \colon G \to \mathrm{GL}(M \otimes \Bbbk[G])$ are intertwined by the automorphism Δ_M of (9): for all $g \in G$, we have

$$\Delta_M \circ (1_M \otimes \rho_\ell)(g) = (\rho_M \otimes \rho_\ell)(g) \circ \Delta_M \ . \tag{11}$$

Similarly,

$$\Delta_M \circ (\rho_M \otimes \rho_r)(g) = (1_M \otimes \rho_r)(g) \circ \Delta_M \ . \tag{12}$$

To prove (12), for example, one checks that both sides of the equation send $m \otimes f \in M \otimes \Bbbk[G]$ to the function $G \to M$, $x \mapsto xg.mf(xg)$.

3.4. Rational group actions. The action of G on the $\Bbbk$-algebra R is said to be *rational* if it makes R a G-module in the above sense. The map

$$\Delta_R \colon R \to R \otimes \Bbbk[G]$$

is then a map of $\Bbbk$-algebras; equivalently, R is a right $\Bbbk[G]$-comodule algebra. Since rational actions are locally finite, they are certainly locally ideal finite in the sense of Proposition 8(b). Therefore, the G-primes of R are exactly the ideals of R of the form $(P : G)$ for $P \in \operatorname{Spec} R$. In particular, G-prime ideals of R are semiprime; for a more precise statement, see Corollary 21 below. Moreover, the $\Bbbk[G]$-linear extension of Δ_R is an automorphism of $\Bbbk[G]$-algebras

$$\Delta_R \colon R \otimes \Bbbk[G] \xrightarrow{\sim} R \otimes \Bbbk[G] \ . \tag{13}$$

We now consider the extended G-action on the Amitsur-Martindale ring of quotients $\mathrm{Q_r}(R)$; see §2.3. This action is usually not rational, even if G acts rationally on R. Part (b) of the following lemma, for classical quotient rings of semiprime Goldie rings, is due to Mœglin and Rentschler [28, I.22].

Lemma 18. *Assume that G acts rationally on R. Then:*

(a) *The centralizer $C_G(T) = \{g \in G \mid g.q = q \ \forall q \in T\}$ of every subset $T \subseteq \mathrm{Q_r}(R)$ is a closed subgroup of G.*

(b) *Let $V \subseteq \mathrm{Q_r}(R)$ be a G-stable $\Bbbk$-subspace of $\mathrm{Q_r}(R)$. The G-action on V is rational if and only if it is locally finite.*

Proof. (a) In view of Proposition 2(iii), the condition for an element $g \in G$ to belong to $C_G(T)$ can be stated as

$$\forall q \in T, r \in D_q : (q - g.q)g.r = 0 \ ,$$

where D_q is as in (3). Using the notation of (8), we have

$$(q - g.q)g.r = q(g.r) - g.(qr) = \sum qr_0 r_1(g) - \sum (qr)_0 (qr)_1(g) \ .$$

Thus, putting $f_{r,q} = \sum qr_0 \otimes r_1 - \sum (qr)_0 \otimes (qr)_1 \in \mathrm{Q_r}(R) \otimes \Bbbk[G]$, we see that $g \in C_G(T)$ if and only if $f_{r,q}(g) = 0$ holds for all $q \in T$ and all $r \in D_q$. Since each equation $f_{r,q}(g) = 0$ defines a closed subset of G, part (a) follows.

(b) Necessity is clear. So assume that the G-action on V is locally finite. Put $S = R \otimes \Bbbk[G]$ and consider the $\Bbbk[G]$-algebra automorphism $\Delta_R \in \operatorname{Aut}(S)$ as in (13) and its extension $\Delta \in \operatorname{Aut}(\mathrm{Q_r}(S))$. We must show that, under the canonical embedding $\mathrm{Q_r}(R) \hookrightarrow \mathrm{Q_r}(S)$ as in §1.8, we have

$$\Delta(V) \subseteq V \otimes \Bbbk[G] \ . \tag{14}$$

Since the action of G on V is locally finite, we may assume that V is finite-dimensional. Therefore, the ideal $D_V = \bigcap_{q \in V} D_q$ belongs to $\mathscr{E}(R)$ and D_V is G-stable, since V is. Lemma 17 implies that $\Delta(D_V \otimes \Bbbk[G]) = D_V \otimes \Bbbk[G]$, and hence

$$\Delta(V)(D_V \otimes \Bbbk[G]) = \Delta(V(D_V \otimes \Bbbk[G])) \subseteq S \ .$$

This shows that the subspace $\Delta(V) \subseteq \mathrm{Q_r}(S)$ actually is contained in $\mathrm{Q_r}(R) \otimes \Bbbk[G]$, and (14) follows from Lemma 17, since $V = (V : G)$. □

From now on, the G-action on R is understood to be rational.

3.5. Connected groups. The group G is *connected* if and only if the algebra $\Bbbk[G]$ is a domain. In this case,

$$\Bbbk(G) = \operatorname{Fract} \Bbbk[G]$$

will denote the field of rational functions on G. The group G acts on $\Bbbk(G)$ by the natural extensions of the right and left regular actions ρ_r and ρ_ℓ on $\Bbbk[G]$; see §3.3.

Part (a) of the following result is due to Chin [8, Corollary 1.3]; the proof given below has been extracted from [40, 3.6]. The proof of part (c) follows the outline of the arguments in [28, I.29, 2^e étape].

Proposition 19. *Assume that G is connected. Then:*

(a) *$(P : G)$ is prime for every $P \in \operatorname{Spec} R$. Therefore, the G-primes of R are exactly the G-stable primes of R.*
(b) *Assume that R is prime and every nonzero ideal I of R satisfies $(I : G) \neq 0$. Then G acts trivially on $\mathcal{C}(R)$.*
(c) *If R is G-rational then the field extension $\mathcal{C}(R)/\Bbbk$ is finitely generated. In fact, there is a G-equivariant $\Bbbk$-embedding of fields $\mathcal{C}(R) \hookrightarrow \Bbbk(G)$, with G acting on $\Bbbk(G)$ via the right regular representation ρ_r.*

Proof. (a) It suffices to show that $(P : G)$ is prime for each prime P; the last assertion is then a consequence of Proposition 8.

By §3.4, we know that the homomorphism $\Delta_R \colon R \to R \otimes \Bbbk[G]$ is centralizing. Therefore, there is a map $\operatorname{Spec}(R \otimes \Bbbk[G]) \to \operatorname{Spec} R$, $Q \mapsto \Delta_R^{-1}(Q)$. In view of Lemma 17, it therefore suffices to show that $P \otimes \Bbbk[G]$ is prime whenever P is. But the algebra $\Bbbk[G]$ is contained in some finitely generated purely transcendental field extension F of $\Bbbk$; see Borel [3, 18.2]. Thus, we have a centralizing extension of algebras

$$(R/P) \otimes \Bbbk[G] \subseteq (R/P) \otimes F\ .$$

Since $(R/P) \otimes F$ is clearly prime, $(R/P) \otimes \Bbbk[G]$ is prime as well as desired.

(b) We first prove the following special case of (b) which is well-known; see [38, Prop. A.1].

Claim 20. If R is a field then G acts trivially on R.

Since G is the union of its Borel subgroups ([3, 11.10]), we may assume that G is solvable. Arguing by induction on a composition series of G ([3, 15.1]), we may further assume that G is the additive group $\mathbb{G}_{\mathrm{a}}$ or the multiplicative group $\mathbb{G}_{\mathrm{m}}$. Therefore, $R \otimes \Bbbk[G]$ is a polynomial algebra or a Laurent polynomial algebra over R. In either case, R is the unique largest subfield of $R \otimes \Bbbk[G]$, because $R \otimes \Bbbk[G]$ has only "trivial" units: the nonzero elements of R if $R \otimes \Bbbk[G] = R[t]$, and the elements of the form rt^m with $0 \neq r \in R$ and $m \in \mathbb{Z}$ if $R \otimes \Bbbk[G] = R[t^{\pm 1}]$. Consequently, the map $\Delta_R \colon R \to R \otimes \Bbbk[G]$ has image in $R \otimes 1$ which in turn says that G acts trivially on R. This proves the Claim.

Now let R be a prime $\Bbbk$-algebra such that $(I : G)$ is nonzero for every nonzero ideal I of R. By the Claim, it suffices to show that the G-action on $\mathcal{C}(R)$ is rational, and by Lemma 18 this amounts to showing that G-action on $\mathcal{C}(R)$ is locally finite. So let $q \in \mathcal{C}(R)$ be given and consider the ideal D_q of R as in (3). By hypothesis, we may pick a nonzero element $d \in (D_q : G)$. The G-orbit $G.d$ generates a finite-dimensional $\Bbbk$-subspace $V \subseteq D_q$. Moreover, qV is contained in a finite-dimensional G-stable subspace $W \subseteq R$. Therefore, for all $g, h \in G$, we have $(g.q)(h.d) = g.(q(g^{-1}h.d)) \in W$, and hence $QV \subseteq W$, where $Q \subseteq \mathcal{C}(R)$ denotes the $\Bbbk$-subspace that is generated by the orbit $G.q$. Thus, multiplication gives a linear map $Q \to \mathrm{Hom}_{\Bbbk}(V, W)$ which is injective, because $V \neq 0$ and nonzero elements of $\mathcal{C}(R)$ have zero annihilator in R. This shows that Q is finite-dimensional as desired.

(c) Put $C = \mathcal{C}(R)$ and $K = \Bbbk(G)$, the field of rational functions on G, that is, the field of fractions of the algebra $\Bbbk[G]$. The algebra $R_K = R \otimes K$ is prime by (a) and its proof, and by (4) there is a tower of fields

$$C \hookrightarrow \mathrm{Fract}(C \otimes K) \hookrightarrow \mathcal{C}(R_K) .$$

We will first show that C is a finitely generated field extension of $\Bbbk$. Since $K/\Bbbk$ is finitely generated, the field $\mathrm{Fract}(C \otimes K)$ is certainly finitely generated over C. Thus, it will suffice to construct a C-algebra embedding $C \otimes C \hookrightarrow \mathrm{Fract}(C \otimes K)$.

To construct such an embedding, consider the natural epimorphism of $\mathcal{C}(R_K)$-algebras $RC \otimes_C \mathcal{C}(R_K) \twoheadrightarrow R_K\mathcal{C}(R_K)$. By Lemma 3(b), this map is injective, because it is clearly injective on RC. Thus,

$$RC \otimes_C \mathcal{C}(R_K) \xrightarrow{\sim} R_K\mathcal{C}(R_K) . \tag{15}$$

Let δ be the K-algebra automorphism of R_K that is defined by K-linear extension of the G-coaction $\Delta_R \colon R \otimes \Bbbk[G] \xrightarrow{\sim} R \otimes \Bbbk[G]$ in (13):

$$\delta = \Delta_R \otimes_{\Bbbk[G]} \mathrm{Id}_K \colon R_K \xrightarrow{\sim} R_K . \tag{16}$$

Let $\widetilde{\delta}$ be the unique extension of δ to an automorphism of the central closure $R_K\mathcal{C}(R_K)$ of R_K. Clearly, $\widetilde{\delta}$ sends the $\mathcal{C}(R_K) = \mathcal{Z}(R_K\mathcal{C}(R_K))$ to itself. We claim that

$$\widetilde{\delta}(C) \subseteq \mathrm{Fract}(C \otimes K) ; \tag{17}$$

so $\widetilde{\delta}$ also sends the $\mathrm{Fract}(C \otimes K)$ to itself. In order to see this, pick $q \in C$ and $d \in D_q$. Then

$$\widetilde{\delta}(q)\Delta_R(d) = \widetilde{\delta}(q)\widetilde{\delta}(d) = \widetilde{\delta}(qd) = \Delta_R(qd)$$

holds in $R_K\mathcal{C}(R_K)$. Here, both $\Delta_R(qd)$ and $\Delta_R(d)$ belong to $R_K \subseteq RC \otimes_C (C \otimes K)$. Fixing a C-basis B for RC and writing $\Delta_R(qd) = \sum_{b \in B} bx_b$ and $\Delta_R(d) = \sum_{b \in B} by_b$ with $x_b, y_b \in C \otimes K$, the above equation becomes

$$\sum_{b \in B} b\widetilde{\delta}(q)y_b = \sum_{b \in B} bx_b .$$

Now (15) yields $\widetilde{\delta}(q)y_b = x_b$ for all b, which proves (17). Now, for the desired embedding, consider the C-algebra map

$$\mu \colon C \otimes C \longrightarrow \mathrm{Fract}(C \otimes K) , \quad c \otimes c' \mapsto c\widetilde{\delta}(c') . \tag{18}$$

We wish to show that μ is injective. To this end, note that the G-action ρ_R on R extends uniquely to an action ρ_{RC} on the central closure RC, and the G-action $1_R \otimes \rho_r$ on R_K extends uniquely to the central closure $R_K\mathcal{C}(R_K)$. Denoting this latter action by $\widetilde{\rho_r}$, the intertwining formula (12) implies that $\widetilde{\delta} \circ \rho_{RC}(g) = \widetilde{\rho_r}(g) \circ \widetilde{\delta} \colon RC \to R_K\mathcal{C}(R_K)$ for all $g \in G$. This yields

$$\mu \circ (\mathrm{Id}_C \otimes \rho_C(g)) = \widetilde{\rho_r}(g) \circ \mu \tag{19}$$

for all $g \in G$. Thus, the ideal $\mathrm{Ker}\,\mu$ of $C \otimes C$ is stable under $(1_C \otimes \rho_C)(G)$. Finally, since $C^G = \Bbbk$, we may invoke [5, Cor. to Prop. V.10.6] to conclude that $\mathrm{Ker}\,\mu$ is generated by its intersection with $C \otimes 1$, which is zero. This shows that μ is injective, and hence the field extension $C/\Bbbk$ is finitely generated.

It remains to construct a G-equivariant embedding $C \hookrightarrow K$, with G acting on $\Bbbk(G)$ via the right regular representation ρ_r as above. For this, we specialize (18) as follows. Write $C = \mathrm{Fract}\,A$ for some affine $\Bbbk$-subalgebra $A \subseteq C$. Then $\mathrm{Fract}(C \otimes K) = \mathrm{Fract}(A \otimes \Bbbk[G])$, and hence

$$\mu(A \otimes A) \subseteq (A \otimes \Bbbk[G])[s^{-1}]$$

for some $0 \neq s \in A \otimes \Bbbk[G]$. By generic flatness (e.g., Dixmier [11, 2.6.3]), there further exists $0 \neq f \in A \otimes A$ so that $(A \otimes \Bbbk[G])[\mu(f)^{-1}s^{-1}]$ is free over $(A \otimes A)[f^{-1}]$ via μ. Now choose some maximal ideal $\mathfrak{m}$ of A with $f \notin \mathfrak{m} \otimes A$. Let $\overline{f}$ denote the image of f in $(A \otimes A)/(\mathfrak{m} \otimes A) \cong A$, and let $\overline{s}$ denote the image of s in $(A \otimes \Bbbk[G])/(\mathfrak{m} \otimes \Bbbk[G]) \cong \Bbbk[G]$. Since $\mu(\mathfrak{m} \otimes A) = \mathfrak{m}\mu(A \otimes A)$, the map $\mu|_{A\otimes A}$ passes down to a map

$$\overline{\mu}\colon A[\overline{f}^{-1}] \longrightarrow B := \Bbbk[G][\overline{\mu}(\overline{f})^{-1}\overline{s}^{-1}]$$

making B a free $A[\overline{f}^{-1}]$-module. Consequently, $\overline{\mu}$ extends uniquely to an embedding of the fields of fractions, $\operatorname{Fract} A[\overline{f}^{-1}] = C \hookrightarrow \operatorname{Fract} B = K$. Finally, (19) implies that this embedding is G-equivariant, which completes the proof of (c). □

Returning to the case of a general affine algebraic group G, we have the following

Corollary 21. *Every $I \in G\text{-}\operatorname{Spec} R$ has the form $I = (Q : G)$ for some $Q \in \operatorname{Spec} R$ with $[G : G_Q] < \infty$. Moreover, $\mathcal{C}(I)^G \cong \mathcal{C}(Q)^{G_Q}$.*

Proof. We know that $I = (P : G)$ for some $P \in \operatorname{Spec} R$; see §3.4. Let G^0 denote the connected component of the identity in G; this is a connected normal subgroup of finite index in G (e.g., Borel [3, 1.2]). Put $Q = (P : G^0)$. Then Proposition 19(a) tells us that Q is prime. Furthermore, $I = (Q : G)$ and $G^0 \subseteq G_Q$; so $[G : G_Q] < \infty$. The isomorphism $\mathcal{C}(I)^G \cong \mathcal{C}(Q)^{G_Q}$ follows from Lemma 10(b). □

3.6. The fibres of the map (2)**.** Assume that G is connected. Our next goal is to give a description of the fibres of the map $\operatorname{Rat} R \to G\text{-}\operatorname{Rat} R$, $P \mapsto (P : G)$ in Proposition 12. Following [6] we denote the fibre over a given $I \in G\text{-}\operatorname{Rat} R$ by $\operatorname{Rat}_I R$:

$$\operatorname{Rat}_I R = \{P \in \operatorname{Rat} R \mid (P : G) = I\}\ .$$

The group G acts on $\operatorname{Rat}_I R$ via the given action ρ_R on R.

Recall that the group G acts on the rational function field $\Bbbk(G)$ by the natural extensions of the regular representations ρ_r and ρ_ℓ. We let

$$\operatorname{Hom}_G(\mathcal{C}(R/I), \Bbbk(G))$$

denote the collection of all G-equivariant $\Bbbk$-algebra homomorphisms $\mathcal{C}(R/I) \to \Bbbk(G)$ with G acting on $\Bbbk(G)$ via the right regular action ρ_r. The left regular action ρ_ℓ of G on $\Bbbk(G)$ yields a G-action on the set $\operatorname{Hom}_G(\mathcal{C}(R/I), \Bbbk(G))$.

Theorem 22. *Let $I \in G\text{-}\operatorname{Rat} R$ be given. There is a G-equivariant bijection*

$$\operatorname{Rat}_I R \longrightarrow \operatorname{Hom}_G(\mathcal{C}(R/I), \Bbbk(G))\ .$$

Proof. Replacing R by R/I, we may assume that $I = 0$. In particular, R is prime by Proposition 19. We will also put $C = \mathcal{C}(R)$ and $K = \Bbbk(G)$ for brevity. For every $P \in \operatorname{Rat} R$ with $(P : G) = 0$, we will construct an embedding of fields

$$\psi_P\colon C \hookrightarrow K$$

such that the following hold:

(a) $\psi_P(g.c) = \rho_r(g)(\psi_P(c))$ and $\psi_{g.P} = \rho_\ell(g) \circ \psi_P$ holds for all $g \in G$, $c \in C$;
(b) if $P, Q \in \operatorname{Rat} R$ are such that $(Q : G) = (P : G) = 0$ but $Q \neq P$ then $\psi_Q \neq \psi_P$;
(c) given a G-equivariant embedding $\psi\colon C \hookrightarrow K$, with G acting on K via ρ_r, we have $\psi = \psi_P$ for some $P \in \operatorname{Rat} R$ with $(P : G) = 0$.

This will prove the theorem.

In order to construct ψ_P, consider the K-algebra $(R/P)_K = (R/P) \otimes K$. This algebra is rational by Lemma 7. We have a centralizing $\Bbbk$-algebra homomorphism

$$\varphi_P\colon R \xrightarrow{\Delta_R} R \otimes \Bbbk[G] \xrightarrow{\text{can.}} (R/P)_K\ , \tag{20}$$

where the canonical map $R \otimes \Bbbk[G] \to (R/P)_K$ comes from the embedding $\Bbbk[G] \hookrightarrow K$ and the epimorphism $R \twoheadrightarrow R/P$. Since $(P : G) = 0$, Lemma 17 implies that φ_P is injective. Since $(R/P)_K$ is prime, it follows that $\mathcal{C}_{\varphi_P} = C$ holds in Lemma 4. Hence φ_P extends uniquely to a centralizing $\Bbbk$-algebra monomorphism

$$\widetilde{\varphi}_P \colon RC \hookrightarrow (R/P)_K\mathcal{C}((R/P)_K) = (R/P)_K \tag{21}$$

sending C to $\mathcal{C}((R/P)_K) = K$. Thus we may define $\psi_P := \widetilde{\varphi}_P|_C \colon C \hookrightarrow K$. It remains to verify properties (a) - (c).

Part (a) is a consequence of the intertwining formulas (11) and (12). Indeed, (12) implies that $\varphi_P(g.r) = \rho_r(g)(\varphi_P(r))$ holds for all $g \in G$ and $r \in R$. In view of Proposition 2(ii), this identity is in fact valid for $\widetilde{\varphi}_P$ and all $r \in RC$, which proves the first of the asserted formulas for ψ_P in (a). For the second formula, consider the map $(\varphi_P)_K$ that is defined by K-linear extension of (20) to $R_K = R \otimes K$; this is the composite

$$(\varphi_P)_K \colon R_K \xrightarrow{\delta} R_K \overset{\text{can.}}{\twoheadrightarrow} (R/P)_K \ , \tag{22}$$

where δ is as in (16). The map $(\rho_R \otimes \rho_\ell)(g)$ gives ring isomorphisms $R_K \xrightarrow{\sim} R_K$ and $(R/P)_K \xrightarrow{\sim} (R/g.P)_K$ such that the following diagram commutes:

$$\begin{tikzcd}
R_K \arrow[r, "\sim"] \arrow[d, two heads, "\text{can.}"'] & R_K \arrow[d, two heads, "\text{can.}"] \\
(R/P)_K \arrow[r, "\sim"] & (R/g.P)_K
\end{tikzcd}$$

The intertwining formula (11) implies that, for all $g \in G$,

$$(\varphi_{g.P})_K \circ (1_R \otimes \rho_\ell)(g) = (\rho_R \otimes \rho_\ell)(g) \circ (\varphi_P)_K \ .$$

Restricting to R we obtain $\varphi_{g.P} = (\rho_R \otimes \rho_\ell)(g) \circ \varphi_P$, and this becomes $\psi_{g.P} = \rho_\ell(g) \circ \psi_P$ on C. This finishes the proof of (a).

For (b), let

$$(\widetilde{\varphi}_P)_K \colon (RC)_K = RC \otimes K \twoheadrightarrow (R/P)_K \tag{23}$$

be defined by K-linear extension of (21) and put $\widetilde{P} = \operatorname{Ker}(\widetilde{\varphi}_P)_K$. Let $Q \in \operatorname{Rat} R$ be given such that $(Q : G) = 0$ and let $\widetilde{Q} = \operatorname{Ker}(\widetilde{\varphi}_Q)_K$ be defined analogously. If $Q \neq P$ then $\widetilde{Q}$ and $\widetilde{P}$ and are distinct primes of $(RC)_K$; in fact, $\widetilde{Q} \cap R_K \neq \widetilde{P} \cap R_K$, because the restriction of $(\widetilde{\varphi}_P)_K$ to R_K is given by (22). Since both $\widetilde{Q}$ and $\widetilde{P}$ are disjoint from RC, Lemma 3(c) gives $\widetilde{P} \cap C_K \neq \widetilde{Q} \cap C_K$. This shows that $(\psi_P)_K$ and $(\psi_Q)_K$ have distinct kernels, and so $\psi_P \neq \psi_Q$ proving (b).

Finally, for (c), let $\psi \colon C \hookrightarrow K$ be some G-equivariant embedding. Define a K-algebra map

$$\Psi \colon R_K \longrightarrow S = RC \otimes_C K$$

by K-linear extension of the canonical embedding $R \hookrightarrow RC$. Note that, for $c \in C$,

$$c \otimes 1 = 1 \otimes \psi(c) \tag{24}$$

holds in S. Put

$$P = \delta(\operatorname{Ker} \Psi) \cap R \ ,$$

with δ as in (16). We will show that P is the desired rational ideal.

The algebra S is K-rational, by Lemma 7, and G acts on S via $\rho_{RC} \otimes_C \rho_r$, where ρ_{RC} is the unique extension of the G-action ρ_R from R to the central closure RC. The map Ψ is G-equivariant for this action and the diagonal G-action $\rho_R \otimes \rho_r$ on R_K. Furthermore, by (12), the automorphism $\delta^{-1} \colon R_K \xrightarrow{\sim} R_K$ is equivariant with respect to the G-actions $1_R \otimes \rho_r$ on the first copy of R_K and

$\rho_R \otimes \rho_r$ on the second R_K. Therefore, the composite $\Psi \circ \delta^{-1}\colon R_K \to S$ is equivariant for the G-actions $1_R \otimes \rho_r$ on R_K and $\rho_{RC} \otimes_C \rho_r$ on S. Now consider the centralizing monomorphism of $\Bbbk$-algebras

$$\mu\colon R/P \hookrightarrow R_K/\delta(\operatorname{Ker}\Psi) \xrightarrow[\delta^{-1}]{\sim} R_K/\operatorname{Ker}\Psi \underset{\Psi}{\hookrightarrow} S\ .$$

By the foregoing, we have $\mu(R/P) \subseteq S^G$, the $\Bbbk$-subalgebra of G-invariants in S. Since S is prime, we have $\mathcal{C}_\mu = \mathcal{C}(R/P)$ in Lemma 4. Hence, μ extends uniquely to a monomorphism $\widetilde{\mu}\colon R/P\mathcal{C}(R/P) \hookrightarrow S\mathcal{C}(S) = S$ sending $\mathcal{C}(R/P)$ to $\mathcal{C}(S) = K$. Therefore, $\widetilde{\mu}(\mathcal{C}(R/P)) \subseteq K^G = \Bbbk$, which proves that P is rational. Furthermore, by Lemma 17, we have $(P : G) = \Delta_R^{-1}(P \otimes \Bbbk[G]) \subseteq \delta^{-1}(\delta(\operatorname{Ker}\Psi)) = \operatorname{Ker}\Psi$. Since Ψ is mono on R, we conclude that $(P : G) = 0$. It remains to show that $\psi = \psi_P$. For this, consider the map $\widetilde{\varphi}_P$ of (21); so $\psi_P = \widetilde{\varphi}_P|_C$. For $q \in C$, $d \in D_q$ we have

$$\delta(qd) \bmod P \otimes K = \widetilde{\varphi}_P(qd) = \widetilde{\varphi}_P(q)\widetilde{\varphi}_P(d) = \delta(\psi_P(q)d) \bmod P \otimes K$$

because $\psi_P(q) \in K$ and δ is K-linear. It follows that $\psi_P(q)d - qd \in \operatorname{Ker}\Psi$; so

$$0 = \psi_P(q)\Psi(d) - \Psi(qd) = \psi_P(q)\Psi(d) - qd \otimes_C 1 = \psi_P(q)\Psi(d) - \psi(q)\Psi(d)$$

where the last equality holds by (24). This shows that $\psi_P(q) = \psi(q)$, thereby completing the proof of the theorem. □

3.7. Proof of Theorem 1. We have to prove:

(1) given $I \in G\text{-}\operatorname{Rat} R$, there is a $P \in \operatorname{Rat} R$ such that $I = (P : G)$;
(2) if $P, P' \in \operatorname{Rat} R$ satisfy $(P : G) = (P' : G)$ then $P' = g.P$ for some $g \in G$.

3.7.1. We first show that it suffices to deal with the case of connected groups. Let G^0 denote the connected component of the identity in G, as before, and assume that both (1) and (2) hold for G^0.

In order to prove (1) for G, let $I \in G\text{-}\operatorname{Rat} R$ be given. By Corollary 21, there exists $Q \in \operatorname{Spec} R$ with $I = (Q : G)$, $G^0 \subseteq G_Q$ and $\mathcal{C}(R/Q)^{G_Q} = \Bbbk$. Since G_Q/G^0 is finite, it follows that Q is in fact G^0-rational. Inasmuch as (1) holds for G^0, there exists $P \in \operatorname{Rat} R$ with $Q = (P : G^0)$. It follows that $(P : G) = (Q : G) = I$, proving (1).

Now suppose that $(P : G) = (P' : G)$ for $P, P' \in \operatorname{Rat} R$. Putting $P^0 = (P : G^0)$ we have $(P : G) = \bigcap_{x \in G/G^0} x.P^0 = \bigcap_{x \in G/G^0} (x.P : G^0)$, a finite intersection of G^0-prime ideals of R. Similarly for $P'^0 = (P' : G^0)$. The equality $(P : G) = (P' : G)$ implies that $(P' : G^0) = (x.P : G^0)$ for some $x \in G$. (Note that if $V \subseteq g.V$ holds for some $\Bbbk$-subspace $V \subseteq R$ and some $g \in G$ then we must have $V = g.V$, because the G-action on R is locally finite.) Invoking (2) for G^0, we see that $P' = yx.P$ for some $y \in G^0$, which proves (2) for G.

3.7.2. Now assume that G is connected. In view of Theorem 22, proving (1) amounts to showing that there is a G-equivariant $\Bbbk$-algebra homomorphism $\mathcal{C}(R/I) \to \Bbbk(G)$ with G acting on $\Bbbk(G)$ via the right regular action ρ_r. But this has been done in Proposition 19(c). For part (2), it suffices to invoke Theorem 22 in conjunction with the following result which is the special case of Vonessen [40, Theorem 4.7] for connected G.

Proposition 23. *Let G act on $\Bbbk(G)$ via ρ_r and let F be a G-stable subfield of $\Bbbk(G)$ containing $\Bbbk$. Let $\operatorname{Hom}_G(F, \Bbbk(G))$ denote the collection of all G-equivariant $\Bbbk$-algebra homomorphisms $\varphi\colon F \to \Bbbk(G)$. Then the G-action on $\operatorname{Hom}_G(F, \Bbbk(G))$ that is given by $g.\varphi = \rho_\ell(g) \circ \varphi$ is transitive.*

This completes the proof of Theorem 1. □

3.7.3. It is tempting to try and prove (1) above in the following more direct fashion. Assume that R is G-prime and choose an ideal P of R that is maximal subject to the condition $(P : G) = 0$. This is possible by the proof of Proposition 8(b) and we have also seen that P is prime. I don't know if the ideal P is actually rational. This would follow if the field extension $\mathcal{C}(R)^G \hookrightarrow \mathcal{C}(R/P)^{G_P}$ in Lemma 10 were algebraic in the present situation. Indeed, every ideal I of R with $I \supsetneqq P$ satisfies $(I : G) \neq 0$, and hence $(I : H) \supsetneqq P$. Therefore, Proposition 19(b) tells us that the connected component of the identity of G_P acts trivially on $\mathcal{C}(R/P)$ and so $\mathcal{C}(R/P)$ is finite over $\mathcal{C}(R/P)^{G_P}$.

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