N-HOMOGENEOUS SUPERALGEBRAS

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ABSTRACT. We develop the theory of N-homogeneous algebras in a super setting, with particular emphasis on the Koszul property. To any Hecke operator $\mathscr{R}$ on a vector superspace, we associate certain superalgebras $\mathsf{S}_{\mathscr{R},N}$ and $\Lambda_{\mathscr{R},N}$ generalizing the ordinary symmetric and Grassmann algebra, respectively. We prove that these algebras are N-Koszul. For the special case where $\mathscr{R}$ is the ordinary supersymmetry, we derive an N-generalized super-version of MacMahon's classical "master theorem".

CONTENTS

INTRODUCTION

0.1. The theory of N-homogeneous algebras owes its existence primarily to the concerns of noncommutative geometry. In fact, as has been expounded by Manin in his landmark publications [36], [37], quadratic algebras (the case $N = 2$) provide a convenient framework for the investigation of quantum group actions on noncommutative spaces. Moreover, certain Artin-Schelter regular algebras [1], natural noncommutative analogs of ordinary polynomial algebras, can be presented as associative algebras defined by cubic relations ($N = 3$). The latter algebras, as well as many of the quadratic algebras studied by Manin, enjoy the additional "Koszul property" which will be of central importance in the present article; it will be reviewed in detail in 0.6 below.

Motivated by these examples and others, Berger [5] initiated the systematic investigation of N-homogeneous algebras for all $N \geq 2$, introducing in particular a natural extension of the notion of Koszul algebra from the familiar quadratic setting to general N-homogeneous

2000 *Mathematics Subject Classification.* Primary 16S37, 05A19.

PHH is supported by the DFG through a Heisenberg-Fellowship.

ML's research is supported in part by NSA Grants H98230-05-1-0025 and H98230-07-1-0008 and by Leverhulme Research Interchange Grant F/00158/X.

algebras. Article [5] gives examples of N-Koszul algebras for all $N \geq 2$; these are the so-called N-symmetric algebras, the special case $N = 2$ being the ordinary symmetric (polynomial) algebra. Following the general outline of Manin's lecture notes [37] on the case of quadratic algebras, Berger, Dubois-Violette and Wambst developed the categorical aspects of N-homogeneous algebras in [7].

0.2. Current interest in N-homogeneous algebras is fueled in part by the fact that they do occur naturally in mathematical physics and in combinatorics. Indeed, Connes and Dubois-Violette [10], [11] introduced a class of 3-homogeneous algebras, called Yang-Mills algebras, which are in fact 3-Koszul. There are two versions of Yang-Mills algebras: in the language of linear superalgebra, the first kind has even (parity $\bar{0}$) algebra generators while the second kind is generated by odd (parity $\bar{1}$) elements.

Combinatorics enters the picture via MacMahon's celebrated "master theorem" [35], specifically the recent quantum generalization of the master theorem due to Garoufalidis, Lê and Zeilberger [20]. As has been pointed out by two of the present authors in [28], the yoga of (quadratic) Koszul algebras leads to a rather effortless and conceptual proof of the quantum master theorem based on the fact that a certain quadratic algebra, known as quantum affine space, is Koszul. Further quantum generalizations and super versions of the master theorem have been obtained by several authors using a variety of approaches; see Foata and Han [17], [18], [19], Konvalinka and Pak [33], Etingof and Pak [16].

0.3. From an algebraic point of view, MacMahon's master theorem (MT) in its various incarnations finds its most natural explanation by the phenomenon of "Koszul duality". Indeed, all versions of MT can be expressed in the form that, for some algebra $\mathcal{B}$, an equation $\Sigma_1 \cdot \Sigma_2 = 1$ holds for suitable power series $\Sigma_1, \Sigma_2 \in \mathcal{B}[\![t]\!]$. Here is a brief outline how one can arrive at such an equation starting with a given N-Koszul algebra $\mathcal{A}$. Associated with $\mathcal{A}$, there is a graded complex, $\mathsf{K}(\mathcal{A})$, which is exact in positive degrees, and a certain endomorphism bialgebra, $\underline{\mathrm{end}}\,\mathcal{A}$, which coacts on all components of $\mathsf{K}(\mathcal{A})$. These components therefore define elements of the representation ring $R_{\underline{\mathrm{end}}\,\mathcal{A}}$ of $\underline{\mathrm{end}}\,\mathcal{A}$, and exactness of $\mathsf{K}(\mathcal{A})$ in positive degrees yields an equation in the power series ring $R_{\underline{\mathrm{end}}\,\mathcal{A}}[\![t]\!]$. Due to the specific form of $\mathsf{K}(\mathcal{A})$, which is constructed from $\mathcal{A}$ together with its so-called dual algebra $\mathcal{A}^!$, the equation in question does indeed state that $\rho_1 \cdot \rho_2 = 1$ holds for suitable $\rho_1, \rho_2 \in R_{\underline{\mathrm{end}}\,\mathcal{A}}[\![t]\!]$. The last step in deriving a MT for $\mathcal{A}$ consists in using (super-)characters to transport the abstract duality equation $\rho_1 \cdot \rho_2 = 1$ from $R_{\underline{\mathrm{end}}\,\mathcal{A}}[\![t]\!]$ to the power series ring over the algebra $\underline{\mathrm{end}}\,\mathcal{A}$, where it takes a more explicit and useable form. Here then is the flow chart of our approach:

$$\begin{array}{c} N\text{-Koszul algebra} \\ \mathcal{A} \end{array} \longrightarrow \begin{array}{c} \text{exact Koszul complex} \\ \mathsf{K}(\mathcal{A}) \end{array} \longrightarrow \begin{array}{c} \text{duality equation} \\ \text{in } R_{\underline{\mathrm{end}}\,\mathcal{A}}[\![t]\!] \end{array} \longrightarrow \text{MT for } \mathcal{A}$$

The actual labor involved in this process consists in the explicit evaluation of (super-)characters at the last arrow above. This step is often facilitated by specializing the bialgebra $\underline{\mathrm{end}}\,\mathcal{A}$, which is highly noncommutative, to a more familiar algebra $\mathcal{B}$ via a homomorphism $\underline{\mathrm{end}}\,\mathcal{A} \to \mathcal{B}$. For example:

- MacMahon's original MT [35] follows in the manner described above by starting with $\mathcal{A} = \mathcal{O}(\Bbbk^d) = \Bbbk[x_1, \ldots, x_d]$, the ordinary polynomial algebra or "affine space", and restricting the resulting MT over $\underline{\mathrm{end}}\,\mathcal{O}(\Bbbk^d)$ to the coordinate ring of $d \times d$-matrices, $\mathcal{O}(\mathrm{Mat}_d(\Bbbk)) = \Bbbk[x^i_j \mid 1 \le i, j \le d]$.

- As was explained in [28], taking "quantum affine space" $\mathcal{O}_{\mathbf{q}}(\Bbbk^d)$ as the point of departure one arrives at the quantum MT of Garoufalidis, Lê and Zeilberger [20] (and Konvalinka and Pak [33] in the multi-parameter case). The endomorphism bialgebra of $\mathcal{O}_{\mathbf{q}}(\Bbbk^d)$ is exactly the algebra of right-quantum matrices as defined in [20].
- Berger's N-symmetric algebra [5] leads to the N-generalization of the MT proved by Etingof and Pak [16] using the above approach, again after restricting to $\mathcal{O}(\mathrm{Mat}_d(\Bbbk))$.

0.4. The present article aims to set forth an extension of the existing theory of N-homogeneous algebras to the category $\mathsf{Vect}^s_{\Bbbk}$ of vector superspaces over some base field $\Bbbk$. While this does not give rise to principal obstacles given that [37] and [7] are at hand as guiding references, the setting of superalgebra requires careful consideration of the order of terms and the so-called "rule of signs" will be ubiquitous in our formulæ. In view of the potential interdisciplinary interest of this material, we have opted to keep our presentation reasonably self-contained and complete.

Therefore, in Sections 1 and 2, we deploy the requisite background material from superalgebra in some detail before turning to N-homogeneous superalgebras in Section 3. The latter section, while following the general outline of [37] and [7] rather closely, also offers explicit discussions of a number of important examples. We interpolate the pure even and pure odd Yang-Mills algebras defined by Connes and Dubois-Violette [10], [11] by a family of superalgebras $\mathcal{YM}^{p|q}$ and give a unified treatment of these algebras. (It turns out, however, that the mixed algebras $\mathcal{YM}^{p|q}$, with p and q both nonzero, are less well-behaved than the pure cases.) Moreover, we discuss a superized version of the N-symmetric algebras of Berger [5]. Finally, in Example 3.4, we introduce new N-homogeneous superversions of the symmetric algebra and the Grassmann algebra of a vector superspace V; these are associated with any Hecke operator $\mathscr{R}\colon V^{\otimes 2} \to V^{\otimes 2}$ and will be denoted by $\mathsf{S}_{\mathscr{R},N}$ and $\Lambda_{\mathscr{R},N}$, respectively.

Sections 4 and 5 contain our main results: Theorem 4.5 shows that the superalgebras $\mathsf{S}_{\mathscr{R},N}$ and $\Lambda_{\mathscr{R},N}$ are in fact N-Koszul, and Theorem 5.4 is superized version of the aforementioned N-generalized MT of Etingof and Pak [16, Theorem 2]. The special case $N = 2$ of Theorem 5.4 is a superization of the original master theorem of MacMahon [35]. The present article was motivated in part by a comment in Konvalinka and Pak [33, 13.4] asking for a "real" super-analog of the classical MT.

0.5. A considerable amount of research has been done by mathematical physicists on various quantum matrix identities. Some of these investigations have been carried out in a super setting; see, e.g., Gurevich, Pyatov and Saponov [23], [24] and the references therein. However, the techniques employed in these articles appear to be quite different from ours.

After submitting this article, we also learned of recent work of Konvalinka [31], [32] which not only concerns MacMahon's MT but also other matrix identities such as the determinantal identity of Sylvester. These identities are proved in [31], [32] by combinatorial means in various noncommutative settings including the right-quantum matrix algebra $\underline{\mathrm{end}}\,\mathcal{O}_q(\Bbbk^d)$.

0.6. We conclude this Introduction by reviewing the precise definitions of N-homogeneous and N-Koszul algebras. Our basic reference is Berger [5]; see also [2], [7], [21].

Let $\mathcal{A}$ be a connected $\mathbb{Z}_{\geq 0}$-graded algebra over a field $\Bbbk$; so $\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{A}_n$ for $\Bbbk$-subspaces $\mathcal{A}_n$ with $\mathcal{A}_0 = \Bbbk$ and $\mathcal{A}_n\mathcal{A}_m \subseteq \mathcal{A}_{n+m}$. Choose a minimal generating set for the algebra $\mathcal{A}$ consisting of homogeneous elements of positive degree; this amounts to choosing a graded basis for a graded subspace $V \subseteq \mathcal{A}_+ = \bigoplus_{n>0} \mathcal{A}_n$ such that $\mathcal{A}_+ = \mathcal{A}_+^2 \oplus V$. The grading of V imparts a grading to the tensor algebra $\mathsf{T}(V)$ of the space V, and we have a graded presentation

$$\mathsf{T}(V)/I \xrightarrow{\sim} \mathcal{A}$$

for some graded ideal I of $\mathsf{T}(V)$, the ideal of *relations* of $\mathcal{A}$.

Recall that a graded vector space $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is said to live in degrees $\geq n_0$ if $M_n = 0$ for all $n < n_0$. Note that the relation ideal I lives in degrees ≥ 2, because $\mathsf{T}(V)_0 \oplus \mathsf{T}(V)_1 \subseteq \Bbbk \oplus V$ and $\Bbbk \oplus V$ injects into $\mathcal{A}$. Fix an integer $N \geq 2$ and define the jump function

$$\nu_N(i) = \begin{cases} \frac{i}{2}N & \text{if } i \text{ is even} \\ \frac{i-1}{2}N + 1 & \text{if } i \text{ is odd} \end{cases} \tag{0.1}$$

The following proposition is identical with [8, Proposition 2.1] except for the fact that we do not a priori assume $\mathcal{A}$ to be generated in degree 1. A proof is given in the Appendix.

Proposition 0.1. *The ideal I of relations of $\mathcal{A}$ lives in degrees $\geq N$ if and only if $\operatorname{Tor}_i^{\mathcal{A}}(\Bbbk,\Bbbk)$ lives in degrees $\geq \nu_N(i)$ for all $i \geq 0$.*

Following Berger [5], the graded algebra $\mathcal{A}$ is said to be *N-Koszul* if $\operatorname{Tor}_i^{\mathcal{A}}(\Bbbk,\Bbbk)$ is concentrated in degree $\nu_N(i)$ for all $i \geq 0$. This implies that the space of algebra generators V is concentrated in degree $\nu_N(1) = 1$; so the algebra $\mathcal{A}$ is *1-generated*. Moreover, choosing a minimal set of homogeneous ideal generators for the relation ideal I amounts to choosing a graded basis for a graded subspace $R \subseteq I$ such that

$$I = R \oplus (V \otimes I + I \otimes V) \tag{0.2}$$

Then $\operatorname{Tor}_2^{\mathcal{A}}(\Bbbk,\Bbbk) \cong R$ and so R must be concentrated in degree $\nu_N(2) = N$ when $\mathcal{A}$ is N-Koszul. To summarize, all N-Koszul algebras are necessarily 1-generated and they have defining relations in degree N; so there is a graded isomorphism

$$\mathcal{A} \cong \mathsf{T}(V)/(R) \qquad \text{with} \quad R \subseteq V^{\otimes N}$$

Such algebras are called *N-homogeneous*.

We remark that Green et al. [21] have studied N-Koszul algebras in the more general context where the grading $\mathcal{A} = \bigoplus_{n\geq 0} \mathcal{A}_n$ is not necessarily connected ($\mathcal{A}_0 = \Bbbk$). In [21, Theorem 4.1], it is shown that an N-homogeneous algebra $\mathcal{A}$ with $\mathcal{A}_0$ split semisimple over $\Bbbk$ is N-Koszul if and only if the Yoneda Ext-algebra $E(\mathcal{A}) = \bigoplus_{n\geq 0} \operatorname{Ext}^n_{\mathcal{A}}(\mathcal{A}_0, \mathcal{A}_0)$ is generated in degrees ≤ 2.

Any N-homogeneous algebra $\mathcal{A}$ whose generating space V carries a $\mathbb{Z}_2$-grading and whose defining relations R are $\mathbb{Z}_2$-graded is naturally a $\Bbbk$-superalgebra, that is, $\mathcal{A}$ has a $\mathbb{Z}_2$-grading ("parity") besides the basic $\mathbb{Z}_{\geq 0}$-grading ("degree"). As will be reviewed below, this extra structure provides us with additional functions on Grothendieck rings, namely superdimension and supercharacters, which lead to natural formulations of the MT in a superized context. Note, however, that the defining property of N-Koszul algebras makes no reference to the $\mathbb{Z}_2$-grading of $\mathcal{A}$. Thus, an N-homogeneous superalgebra is Koszul precisely if it is Koszul as an ordinary N-homogeneous algebra (forgetting the $\mathbb{Z}_2$-grading).

0.7. Throughout $\Bbbk$ is a commutative field and $\otimes$ stands for $\otimes_{\Bbbk}$. Scalar multiplication in $\Bbbk$-vector spaces will often, but not always, be written on the right while linear maps will act from the left. We tacitly assume throughout that $\operatorname{char}\Bbbk \neq 2$; further restrictions on the characteristic of $\Bbbk$ will be stated when required.

1. REVIEW OF LINEAR SUPERALGEBRA

1.1. Vector superspaces. A vector superspace over $\Bbbk$ is a $\Bbbk$-vector space V equipped with a grading by the group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. Thus, we have a decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with $\Bbbk$-subspaces $V_{\bar{0}}$ and $V_{\bar{1}}$ whose elements are called *even* and *odd*, respectively. In general, the $\mathbb{Z}_2$-degree of a homogeneous element $a \in V$ is also called its *parity*; it will be denoted by $\widehat{a} \in \mathbb{Z}_2$. Vector superspaces over $\Bbbk$ form a category $\mathsf{Vect}^s_{\Bbbk}$ whose morphisms are given by the linear maps preserving the $\mathbb{Z}_2$-grading; such maps are also called *even* linear maps.

The *dimension* of an object V of $\mathsf{Vect}^s_{\Bbbk}$ is the usual $\Bbbk$-linear dimension. We shall use the notation

$$d = \dim_{\Bbbk} V\ ,\ p = \dim_{\Bbbk} V_{\bar{0}} \quad \text{and} \quad q = \dim_{\Bbbk} V_{\bar{1}}$$

So $d = p + q$. The *superdimension* of a vector superspace V with $d < \infty$ is defined by

$$\operatorname{sdim} V = p - q \in \mathbb{Z}$$

When working with a fixed basis $\{x_i\}$ of a given V in $\mathsf{Vect}^s_{\Bbbk}$ we shall assume that each x_i is homogeneous; the parity of x_i will be denoted by $\widehat{i}$. The basis $x_1, x_2 \dots$ is called *standard* if $\widehat{i} = \bar{0}$ $(i \le p)$ and $\widehat{i} = \bar{1}$ $(i > p)$.

1.2. Tensors. The tensor product $U \otimes V$ of vector superspaces U and V in $\mathsf{Vect}^s_{\Bbbk}$ is the usual tensor product over $\Bbbk$ of the underlying vector spaces equipped with the natural $\mathbb{Z}_2$-grading: if a, b are homogeneous elements then the parity of $a \otimes b$ is $\widehat{a} + \widehat{b} \in \mathbb{Z}_2$. Instead of the usual symmetry isomorphism $U \otimes V \xrightarrow{\sim} V \otimes U$ for interchanging terms in a tensor product we shall use the *rule of signs*, that is, the following functorial *supersymmetry* isomorphism in $\mathsf{Vect}^s_{\Bbbk}$:

$$c_{U,V} \colon U \otimes V \xrightarrow{\sim} V \otimes U\ , \quad u \otimes v \mapsto (-1)^{\widehat{u}\widehat{v}} v \otimes u \tag{1.1}$$

for u, v homogeneous. (All formulas stated for homogeneous elements only are to be extended to arbitrary elements by linearity.) The supersymmetry isomorphisms $c_{U,V}$ satisfy $c_{V,U} \circ c_{U,V} = \mathrm{Id}_{U \otimes V}$, and they are compatible with the usual associativity isomorphisms $a_{U,V,W} \colon (U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ in $\mathsf{Vect}^s_{\Bbbk}$, that is, they satisfy the "Hexagon Axiom"; see [29, Def. XIII.1.1]. Therefore, $\mathsf{Vect}^s_{\Bbbk}$ is a symmetric tensor category; the unit object is the field $\Bbbk$, with parity $\bar{0}$. See [29, Chap. XIII] or [12] for background on tensor categories.

1.3. Homomorphisms. The space $\operatorname{Hom}_{\Bbbk}(V, U)$ of all $\Bbbk$-linear maps between vector superspaces V and U is again an object of $\mathsf{Vect}^s_{\Bbbk}$, with grading $\operatorname{Hom}_{\Bbbk}(V, U)_{\bar{0}} = \operatorname{Hom}_{\Bbbk}(V_{\bar{0}}, U_{\bar{0}}) \oplus \operatorname{Hom}_{\Bbbk}(V_{\bar{1}}, U_{\bar{1}})$ and $\operatorname{Hom}_{\Bbbk}(V, U)_{\bar{1}} = \operatorname{Hom}_{\Bbbk}(V_{\bar{0}}, U_{\bar{1}}) \oplus \operatorname{Hom}_{\Bbbk}(V_{\bar{1}}, U_{\bar{0}})$; so

$$\operatorname{Hom}_{\Bbbk}(V, U)_{\bar{0}} = \operatorname{Hom}_{\mathsf{Vect}^s_{\Bbbk}}(V, U)$$

In particular, the linear dual $V^* = \operatorname{Hom}_{\Bbbk}(V, \Bbbk)$ belongs to $\mathsf{Vect}^s_{\Bbbk}$. Given homogeneous bases $\{x_j\}$ of V and $\{y_i\}$ of U we can describe any $f \in \operatorname{Hom}_{\Bbbk}(V, U)$ by its matrix $F = (F^i_j)$:

$$f(x_j) = \sum_i y_i F^i_j \tag{1.2}$$

When f is an even map then $F^i_j = 0$ unless $\widehat{i} + \widehat{j} = \bar{0}$.

For finite-dimensional vector superspaces, we have the following functorial isomorphisms in $\mathsf{Vect}^s_{\Bbbk}$ (see, e.g., [43, I.8]):

$$U \otimes V^* \cong \operatorname{Hom}_{\Bbbk}(V, U) \tag{1.3}$$

via $(u \otimes f)(v) = u\langle f, v\rangle$, and

$$V_1^* \otimes \ldots \otimes V_m^* \cong (V_m \otimes \ldots \otimes V_1)^* \tag{1.4}$$

via $\langle f_1 \otimes \ldots \otimes f_m, v_m \otimes \ldots \otimes v_1\rangle = \prod_i \langle f_i, v_i\rangle$. Here, we use the notation $\langle f, v\rangle = f(v)$ for the evaluation pairing

$$\mathrm{ev}_V = \langle\,.\,,\,.\,\rangle \colon V^* \otimes V \to \Bbbk$$

in $\mathsf{Vect}^s_{\Bbbk}$. Similarly, we have a pairing

$$V \otimes V^* \xrightarrow{c_{V,V^*}} V^* \otimes V \xrightarrow{\mathrm{ev}_V} \Bbbk$$

which yields an isomorphism

$$V \xrightarrow{\sim} V^{**} \tag{1.5}$$

in $\mathsf{Vect}^s_{\Bbbk}$.

The isomorphism (1.3) (which is valid as long as one of U or V is finite-dimensional) has the following explicit description. Fix homogeneous bases $\{x_j\}$ of V and $\{y_i\}$ of U and let $F = (F^i_j)$ be the matrix of a given $f \in \operatorname{Hom}_{\Bbbk}(V, U)$ with respect to these bases, as in (1.2). Let $\{x^j\}$ be the dual basis of V^*, defined by $\langle x^j, x_\ell\rangle = \delta^j_\ell$ (Kronecker delta). Then the image of f in $U \otimes V^*$ is given by $\sum_{i,j} y_i \otimes x^j F^i_j$. Note also that x_i and x^i have the same parity.

Finally, if U, V and W are vector superspaces, with U finite-dimensional, then the isomorphism $\operatorname{Id} \otimes c_{W,U^*} \colon V \otimes W \otimes U^* \xrightarrow{\sim} V \otimes U^* \otimes W$ together with (1.3) yields an isomorphism

$$\operatorname{Hom}_{\Bbbk}(U, V \otimes W) \xrightarrow{\sim} \operatorname{Hom}_{\Bbbk}(U, V) \otimes W \tag{1.6}$$

in $\mathsf{Vect}^s_{\Bbbk}$ which is explicitly given by $(f \otimes w)(u) = (-1)^{\widehat{w}\widehat{u}} f(u) \otimes w$. Similarly, for vector superspaces U, U', V, V' with U, U' finite-dimensional, there is an isomorphism

$$\operatorname{Hom}_{\Bbbk}(U \otimes U', V \otimes V') \xrightarrow{\sim} \operatorname{Hom}_{\Bbbk}(U, V) \otimes \operatorname{Hom}_{\Bbbk}(U', V') \tag{1.7}$$

in $\mathsf{Vect}^s_{\Bbbk}$ given by $(f \otimes g)(u \otimes v) = (-1)^{\widehat{g}\widehat{u}} f(u) \otimes g(v)$.

1.4. Supertrace. Let V be a finite-dimensional object of $\mathsf{Vect}^s_{\Bbbk}$. The *supertrace* is the map

$$\mathrm{str} \colon \operatorname{End}_{\Bbbk}(V) \underset{(1.3)}{\xrightarrow{\sim}} V \otimes V^* \underset{(1.3)}{\longrightarrow} \Bbbk \tag{1.8}$$

in $\mathsf{Vect}^s_{\Bbbk}$. In order to describe the supertrace in terms of matrices, fix a basis $\{x_i\}$ of V consisting of homogeneous elements and let $F = (F^i_j)$ be the matrix of $f \in \operatorname{End}_{\Bbbk}(V)$ as in (1.2). Then

$$\mathrm{str}(f) = \sum_i (-1)^{\widehat{i}} F^i_i$$

where $\widehat{i}$ is the parity of x_i (and of the dual basis vector $x^i \in V^*$) as in §1.1. Thus,

$$\mathrm{str}(\mathrm{Id}_V) = \operatorname{sdim} V.1_{\Bbbk}$$

1.5. Action of the symmetric group. Given vector superspaces $V_1, \dots, V_n$, we can consider the morphism

$$c_i \colon V_1 \otimes \cdots \otimes V_i \otimes V_{i+1} \otimes \cdots \otimes V_n \longrightarrow V_1 \otimes \cdots \otimes V_{i+1} \otimes V_i \otimes \cdots \otimes V_n$$

in $\mathsf{Vect}^s_{\Bbbk}$ which interchanges the factors V_i and V_{i+1} via $c_{V_i,V_{i+1}}$ and is the identity on all other factors. More generally, for any $\sigma \in \mathfrak{S}_n$, the symmetric group consisting of all permutations of $\{1, 2, \dots, n\}$, one can define a morphism

$$c_\sigma \colon V_1 \otimes \cdots \otimes V_n \longrightarrow V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}$$

in $\mathsf{Vect}^s_{\Bbbk}$ as follows. Recall that $\mathfrak{S}_n$ is generated by the transpositions $\sigma_1, \dots, \sigma_{n-1}$ where σ_i interchanges i and $i+1$ and leaves all other elements of $\{1, 2, \dots, n\}$ fixed. The minimal length of a product in the σ_i's which expresses a given element $\sigma \in \mathfrak{S}_n$ is called the length of σ and denoted $\ell(\sigma)$; it is given by

$$\ell(\sigma) = \#\mathsf{inv}(\sigma) \quad \text{with} \quad \mathsf{inv}(\sigma) = \{(i,j) \mid i < j \text{ but } \sigma(i) > \sigma(j)\}$$

Writing $\sigma \in \mathfrak{S}_n$ as a product of certain σ_i, the analogous product of the maps c_i yields a morphism c_σ as above. This morphism is independent of the way σ is expressed in terms of the transpositions σ_i; see [43, I.4.13] or [29, Theorem XIII.1.3]. If all $v_i \in V_i$ are homogeneous then

$$c_\sigma(v_1 \otimes \cdots \otimes v_n) = (-1)^{\sum_{(i,j)\in\mathsf{inv}(\sigma)} \widehat{v_i}\widehat{v_j}} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} \tag{1.9}$$

For example, if all v_i are even then the $\pm$-sign on the right is $+$, and if all v_i are odd then it is $\operatorname{sgn}(\sigma)$, the signature of σ.

Taking all $V_i = V$ we obtain a representation $c \colon \mathfrak{S}_n \longrightarrow \operatorname{Aut}_{\mathsf{Vect}^s_{\Bbbk}}(V^{\otimes n})$ where $V^{\otimes n} = V \otimes \cdots \otimes V$ (n factors). Letting $\Bbbk[\mathfrak{S}_n]$ denote the group algebra of the symmetric group, this extends uniquely to an algebra map

$$c \colon \Bbbk[\mathfrak{S}_n] \longrightarrow \operatorname{End}_{\mathsf{Vect}^s_{\Bbbk}}(V^{\otimes n}) \tag{1.10}$$

We will write $c_a := c(a)$ for $a \in \Bbbk[\mathfrak{S}_N]$.

For the dual superspace V^*, besides the above representation $c \colon \Bbbk[\mathfrak{S}_n] \longrightarrow \operatorname{End}_{\mathsf{Vect}^s_{\Bbbk}}(V^{*\otimes n})$, we also have the *contragredient representation*

$$c^* \colon \Bbbk[\mathfrak{S}_n] \longrightarrow \operatorname{End}_{\mathsf{Vect}^s_{\Bbbk}}(V^{*\otimes n})$$

for the pairing $\langle\, . \,, . \,\rangle \colon V^{*\otimes n} \otimes V^{\otimes n} \to \Bbbk$ in (1.4). Explicitly,

$$\langle c^*_a(x), y\rangle = \langle x, c_{a^*}(y)\rangle$$

for all $a \in \Bbbk[\mathfrak{S}_n]$, $x \in V^{*\otimes n}$ and $y \in V^{\otimes n}$. Here, $.^* \colon \Bbbk[\mathfrak{S}_n] \to \Bbbk[\mathfrak{S}_n]$ is the involution sending $\sigma \in \mathfrak{S}_n$ to σ^{-1}. These two representations are related by

$$c^*_a = c_{\tau a \tau} \tag{1.11}$$

where $\tau = (1,n)(2,n-1)\cdots \in \mathfrak{S}_n$ is the order reversal involution. One only needs to check (1.11) for the transpositions $a = \sigma_i$, which is straightforward.

1.6. Hecke algebras. We recall some standard facts concerning Hecke algebras; these are suitable deformations of the group algebra $\Bbbk[\mathfrak{S}_n]$ considered above. For details, see [13], [14].

Fix $0 \neq q \in \Bbbk$. The Hecke algebra $\mathscr{H}_{n,q}$ is generated as $\Bbbk$-algebra by elements $T_1, \dots, T_{n-1}$ subject to the relations

$$\begin{aligned} &(T_i + 1)(T_i - q) = 0 \\ &T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\ &T_i T_j = T_j T_i \quad \text{if } |i-j| \geq 2 \end{aligned} \tag{1.12}$$

When $q = 1$, one has an isomorphism $\mathscr{H}_{n,1} \xrightarrow{\sim} \Bbbk[\mathfrak{S}_n]$, $T_i \mapsto \sigma_i$ where σ_i is the transposition $(i, i+1)$ as in §1.5. The algebra $\mathscr{H}_{n,q}$ has a $\Bbbk$-basis $\{T_\sigma \mid \sigma \in \mathfrak{S}_n\}$ so that

(i) $T_{\mathrm{Id}} = 1$ and $T_{\sigma_i} = T_i$;

(ii) $T_\sigma T_{\sigma_i} = \begin{cases} T_{\sigma\sigma_i} & \text{if } \ell(\sigma\sigma_i) = \ell(\sigma) + 1; \\ qT_{\sigma\sigma_i} + (q-1)T_\sigma & \text{otherwise} \end{cases}$

By $\Bbbk$-linear extension of the rule

$$T_\sigma^* := T_{\sigma^{-1}} \qquad (\sigma \in \mathfrak{S}_n)$$

one obtains an involution $.^* \colon \mathscr{H}_{n,q} \to \mathscr{H}_{n,q}$. Moreover, the elements $T_i' := -qT_i^{-1} = q - 1 - T_i$ also satisfy relations (1.12). Therefore,

$$\alpha(T_i) := -qT_i^{-1} \tag{1.13}$$

defines an algebra automorphism $\alpha \colon \mathscr{H}_{n,q} \to \mathscr{H}_{n,q}$ of order 2.

The Hecke algebra $\mathscr{H}_{n,q}$ is always a symmetric algebra, and $\mathscr{H}_{n,q}$ is a split semisimple $\Bbbk$-algebra iff the following condition is satisfied:

$$[n]_q! := \prod_{i=1}^{n} [i]_q \neq 0 \quad \text{where } [i]_q := 1 + q + \dots + q^{i-1} \tag{1.14}$$

More precisely, if (1.14) holds then

$$\mathscr{H}_{n,q} \cong \bigoplus_{\lambda \vdash n} \mathrm{Mat}_{d_\lambda \times d_\lambda}(\Bbbk) \tag{1.15}$$

where λ runs over all partitions of n and d_λ denotes the number of standard λ-tableaux. The only partitions λ with $d_\lambda = 1$ are $\lambda = (n)$ and $\lambda = (1^n)$. The central primitive idempotents of $\mathscr{H}_{n,q}$ for these partitions are given by

$$X_n := \frac{1}{[n]_q!} \sum_{\sigma \in \mathfrak{S}_n} T_\sigma \tag{1.16}$$

and

$$Y_n := \frac{1}{[n]_{q^{-1}}!} \sum_{\sigma \in \mathfrak{S}_n} (-q)^{-\ell(\sigma)} T_\sigma \tag{1.17}$$

These idempotents are usually called the q-*symmetrizer* and the q-*antisymmetrizer*, respectively. One has

$$X_n T_\sigma = T_\sigma X_n = q^{\ell(\sigma)} X_n \quad \text{and} \quad Y_n T_\sigma = T_\sigma Y_n = (-1)^{\ell(\sigma)} Y_n \tag{1.18}$$

for $\sigma \in \mathfrak{S}_n$. Furthermore, $\alpha(X_n) = Y_n$.

For later use, we note the following well-known consequence of (1.18). If M is any $\mathscr{H}_{n,q}$-module, with corresponding representation $\mu\colon \mathscr{H}_{n,q} \to \operatorname{End}_{\Bbbk}(M)$, then

$$\operatorname{Im}(\mu(X_n)) = \bigcap_{i=1}^{n-1} \operatorname{Im}(\mu(T_i) + 1) \tag{1.19}$$

Indeed, (1.18) implies that $X_n = [2]_q^{-1}(T_i + 1)X_n$, which yields the inclusion $\subseteq$. On the other hand, any $m \in \bigcap_{i=1}^{n-1} \operatorname{Im}(\mu(T_i) + 1)$ satisfies $(\mu(T_i) - q)(m) = 0$ for all i, by (1.12). Therefore, $\mu(T_\sigma)(m) = q^{\ell(\sigma)}m$ holds for all $\sigma \in \mathfrak{S}_n$, and hence $\mu(X_n)(m) = \frac{1}{[n]_q!}\sum_{\sigma\in\mathfrak{S}_n} q^{\ell(\sigma)}m = m$. This proves $\supseteq$.

1.7. Hecke operators. Again, let $0 \neq q \in \Bbbk$. A Hecke operator (associated to q) on a vector superspace V is a morphism $\mathscr{R}\colon V^{\otimes 2} \to V^{\otimes 2}$ in $\mathsf{Vect}^s_{\Bbbk}$ satisfying the Hecke equation

$$(\mathscr{R} + 1)(\mathscr{R} - q) = 0$$

and the Yang-Baxter equation

$$\mathscr{R}_1\mathscr{R}_2\mathscr{R}_1 = \mathscr{R}_2\mathscr{R}_1\mathscr{R}_2$$

where $\mathscr{R}_1 := \mathscr{R} \otimes \operatorname{Id}_V \colon V^{\otimes 3} \to V^{\otimes 3}$ and similarly $\mathscr{R}_2 := \operatorname{Id}_V \otimes \mathscr{R}$.

The Hecke equation implies that $\mathscr{R}$ is invertible. Moreover, if $\mathscr{R}$ is a Hecke operator associated to q then so is $-q\mathscr{R}^{-1}$.

Defining $\rho(T_i) := \operatorname{Id}_V^{\otimes i-1} \otimes \mathscr{R} \otimes \operatorname{Id}_V^{\otimes n-i-1}$, one obtains a representation

$$\rho = \rho_{n,\mathscr{R}}\colon \mathscr{H}_{n,q} \longrightarrow \operatorname{End}_{\mathsf{Vect}^s_{\Bbbk}}(V^{\otimes n}) \tag{1.20}$$

The representations $\rho_{n,\mathscr{R}}$ and $\rho_{n,-q\mathscr{R}^{-1}}$ are related by $\rho_{n,-q\mathscr{R}^{-1}} = \rho_{n,\mathscr{R}} \circ \alpha$, where α is the automorphism of $\mathscr{H}_{n,q}$ defined in (1.13).

Example 1.1. The supersymmetry operator $c_{V,V}\colon V^{\otimes 2} \to V^{\otimes 2}$ in (1.1) is a Hecke operator associated to $q = 1$, as is its negative, $-c_{V,V}$. The representation $\rho_{c_{V,V}}$ of $\mathscr{H}_{n,1} = \Bbbk[\mathfrak{S}_n]$ in (1.20) is identical with (1.10).

Example 1.2 (superized Drinfel'd-Jimbo [38], [27])**.** Let $x_1, \dots, x_d$ be a standard basis of V as in §1.1. The super analog $\mathscr{R} = \mathscr{R}^{DJ}$ of the standard Drinfel'd-Jimbo Hecke operator is defined as follows. Writing

$$\mathscr{R}(x_i \otimes x_j) = \sum_{k,l} x_k \otimes x_l \mathscr{R}^{k,l}_{i,j}$$

the matrix components $\mathscr{R}^{k,l}_{i,j} \in \Bbbk$ are given by

$$\mathscr{R}^{k,l}_{i,j} = \frac{q^2 - q^{2\varepsilon_{i,j}}}{1 + q^{2\varepsilon_{i,j}}}\delta^{k,l}_{i,j} + (-1)^{\widehat{i}\widehat{j}}\frac{q^{\varepsilon_{i,j}}(q^2+1)}{1 + q^{2\varepsilon_{i,j}}}\delta^{l,k}_{i,j}$$

Here, $\varepsilon_{i,j} = \operatorname{sgn}(i - j)$. Thus,

$$\begin{aligned} \mathscr{R}^{ii}_{ii} &= q^2 && \text{if } \widehat{i} = \bar{0} \\ \mathscr{R}^{ii}_{ii} &= -1 && \text{if } \widehat{i} = \bar{1} \\ \mathscr{R}^{ij}_{ij} &= q^2 - 1 && \text{if } i < j \\ \mathscr{R}^{ji}_{ij} &= (-1)^{\widehat{i}\widehat{j}} q && \text{if } i \neq j \end{aligned} \tag{1.21}$$

and $\mathscr{R}^{k,l}_{i,j} = 0$ in all other cases. One checks that $\mathscr{R}$ is a Hecke operator that is associated to q^2.

2. THE SUPERCHARACTER

2.1. Superalgebras, supercoalgebras etc. An algebra $\mathcal{A}$ in $\mathsf{Vect}^s_{\Bbbk}$ is called a *superalgebra* over $\Bbbk$; this is just an ordinary $\Bbbk$-algebra such that the unit map $\Bbbk \to \mathcal{A}$ and the multiplication

$$\mu\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

are morphisms in $\mathsf{Vect}^s_{\Bbbk}$. In other words, $\mathcal{A}$ is a $\mathbb{Z}_2$-graded $\Bbbk$-algebra in the usual sense: $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ with $\Bbbk$-subspaces $\mathcal{A}_{\bar{0}}$ and $\mathcal{A}_{\bar{1}}$ such that $\mathcal{A}_{\bar{r}}\mathcal{A}_{\bar{s}} \subseteq \mathcal{A}_{\overline{r+s}}$. Homomorphisms of superalgebras, by definition, are algebra maps in $\mathsf{Vect}^s_{\Bbbk}$, that is, they preserve the $\mathbb{Z}_2$-grading.

If V is a vector superspace in $\mathsf{Vect}^s_{\Bbbk}$ then the tensor algebra $\mathsf{T}(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ is a superalgebra via the $\mathbb{Z}_2$-grading of each $V^{\otimes n}$ as in §1.2. In general, if $\mathcal{A}$ is any superalgebra, then by selecting a $\mathbb{Z}_2$-graded subspace $V \subseteq \mathcal{A}$ which generates the algebra $\mathcal{A}$, we obtain a canonical isomorphism of superalgebras

$$\mathsf{T}(V)/(R) \xrightarrow{\sim} \mathcal{A} \tag{2.1}$$

where (R) is the two-sided ideal of $\mathsf{T}(V)$ that is generated by a $\mathbb{Z}_2$-graded linear subspace $R \subseteq \mathsf{T}(V)$.

Given superalgebras $\mathcal{A}$ and $\mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is the superalgebra with the usual additive structure and grading and with multiplication $\mu_{\mathcal{A}\otimes\mathcal{B}}$ defined by using the supersymmetry map (1.1): $\mu_{\mathcal{A}\otimes\mathcal{B}} = (\mu_{\mathcal{A}} \otimes \mu_{\mathcal{B}}) \circ (\mathrm{Id}_A \otimes c_{\mathcal{B},\mathcal{A}} \otimes \mathrm{Id}_{\mathcal{B}})$ or, explicitly,

$$(a \otimes b)(a' \otimes b') = (-1)^{\widehat{a'}\widehat{b}} aa' \otimes bb'$$

for homogeneous $a' \in \mathcal{A}$ and $b \in \mathcal{B}$. In other words, the canonical images of $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{A} \otimes \mathcal{B}$ *supercommute*, in the sense that the supercommutator

$$[a, b] = ab - (-1)^{\widehat{a}\widehat{b}} ba \tag{2.2}$$

vanishes for any pair of homogeneous elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Supercoalgebras, superbialgebras etc. are defined similarly as suitable objects of $\mathsf{Vect}^s_{\Bbbk}$ such that all structure maps are maps in $\mathsf{Vect}^s_{\Bbbk}$. The compatibility between the comultiplication Δ and the multiplication of a superbialgebra $\mathcal{B}$ amounts to the following rule:

$$\Delta(ab) = \sum_{(a),(b)} (-1)^{\widehat{a}_{(2)}\widehat{b}_{(1)}} a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}$$

for homogeneous elements $a, b \in \mathcal{B}$. Here we use the Sweedler notation $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$ and $a_{(1)}, a_{(2)}$ are chosen homogeneous with $\widehat{a}_{(1)} + \widehat{a}_{(2)} = \widehat{a}$.

Example 2.1 (Symmetric superalgebra [40, 3.2.5])**.** The symmetric superalgebra of a given V in $\mathsf{Vect}^s_{\Bbbk}$ is defined by

$$\mathsf{S}(V) = \mathsf{T}(V)/\left([v, w]_{\otimes} \mid v, w \in V\right)$$

where $[v, w]_{\otimes}$ is the supercommutator (2.2) in $\mathsf{T}(V)$. Ignoring parity, $\mathsf{S}(V)$ is isomorphic to $\mathsf{S}(V_{\bar{0}}) \otimes \mathsf{\Lambda}(V_{\bar{1}})$, where $\mathsf{S}(\,.\,)$ and $\mathsf{\Lambda}(\,.\,)$ denote the ordinary symmetric and exterior (Grassmann) algebras, respectively. The symmetric superalgebra is a Hopf superalgebra: comultiplication $\Delta\colon \mathsf{S}(V) \to \mathsf{S}(V) \otimes \mathsf{S}(V)$ is given by $\Delta(v) = v \otimes 1 + 1 \otimes v$ for $v \in V$ and extension to all of $\mathsf{S}(V)$ by multiplicativity. Similarly, the counit $\varepsilon\colon \mathsf{S}(V) \to \Bbbk$ is given by $\varepsilon(v) = 0$ and the antipode $\mathcal{S}\colon \mathsf{S}(V) \to \mathsf{S}(V)$ by $\mathcal{S}(v) = -v$ for $v \in V$.

2.2. Comodules. We refer to [29, Chap. III] for background on comodules, comodule algebras etc.

Given a superbialgebra $\mathcal{B}$, we let $\mathsf{Comod}^s_{\mathcal{B}}$ denote the category of all right $\mathcal{B}$-comodules and $\mathcal{B}$-comodule maps in $\mathsf{Vect}^s_{\Bbbk}$. Thus, for any object V in $\mathsf{Comod}^s_{\mathcal{B}}$, we have a "coaction" morphism

$$\delta_V\colon V \to V \otimes \mathcal{B}$$

in $\mathsf{Vect}^s_{\Bbbk}$. If $x_1, \ldots, x_d$ is a fixed basis of V consisting of homogeneous elements, with $\widehat{i}$ denoting the parity of x_i as before, then we will write

$$\delta_V(x_j) = \sum_i x_i \otimes b^i_j \quad \text{with} \quad b^i_j \in B_{\widehat{i}+\widehat{j}} \tag{2.3}$$

The tensor product of vector superspaces makes $\mathsf{Comod}^s_{\mathcal{B}}$ into a tensor category: if U and V are in $\mathsf{Comod}^s_{\mathcal{B}}$ then $\mathcal{B}$ coacts on $U \otimes V$ by

$$\delta_{U\otimes V}\colon U \otimes V \xrightarrow{\delta_U\otimes\delta_V} U \otimes \mathcal{B} \otimes V \otimes \mathcal{B} \xrightarrow{c_{B,V}} U \otimes V \otimes \mathcal{B} \otimes \mathcal{B} \xrightarrow{\mathrm{Id}\otimes\mu_{\mathcal{B}}} U \otimes V \otimes \mathcal{B} \tag{2.4}$$

If $\mathcal{B}$ is supercommutative as a superalgebra then the supersymmetry $c_{U,V}$ is a $\mathcal{B}$-comodule morphism, i.e., $\delta_{V\otimes U} \circ c_{U,V} = (c_{U,V} \otimes \mathrm{Id}_{\mathcal{B}}) \circ \delta_{U\otimes V}$. Therefore $\mathsf{Comod}^s_{\mathcal{B}}$ is a *symmetric* tensor category in this case .

2.3. The supercharacter map. Let $\mathcal{B}$ denote a superbialgebra and let V be a finite dimensional object in $\mathsf{Comod}^s_{\mathcal{B}}$. The coaction δ_V is an even map in $\mathrm{Hom}_{\Bbbk}(V, V \otimes \mathcal{B})$. Consider the following morphism in $\mathsf{Vect}^s_{\Bbbk}$:

$$\chi^s\colon \mathrm{End}_{\Bbbk}(V) \xrightarrow{\delta_V\circ(\,.\,)} \mathrm{Hom}_{\Bbbk}(V, V \otimes \mathcal{B}) \xrightarrow[(1.6)]{\sim} \mathrm{End}_{\Bbbk}(V) \otimes \mathcal{B} \xrightarrow{\mathrm{str}\otimes\mathrm{Id}} \Bbbk \otimes \mathcal{B} = \mathcal{B} \tag{2.5}$$

where str is the supertrace as in (1.8). This map will be called the *supercharacter* map of V. Forgetting parity and viewing all elements as even, the supertrace becomes the ordinary trace and the supercharacter becomes the usual character. These will be denoted by tr and χ, respectively.

In particular, we have the element

$$\chi^s_V := \chi^s(\mathrm{Id}_V) \in \mathcal{B}_{\bar{0}}$$

To obtain explicit formulas, fix a basis $x_1, \ldots, x_d$ of V consisting of homogeneous elements and let (F^i_j) and (b^i_j) be the matrices of $f \in \mathrm{End}_{\Bbbk}(V)$ and of δ_V with respect to this basis as in (1.2), (2.3). Then

$$\chi^s(f) = \sum_{i,j} (-1)^{\widehat{i}\widehat{j}} b^i_j F^j_i \tag{2.6}$$

Let $\varepsilon\colon \mathcal{B} \to \Bbbk$ denote the counit of $\mathcal{B}$. Then $x_j = \sum_i x_i \varepsilon(b^i_j)$ holds in (2.3). Hence $\varepsilon(b^i_j) = \delta^i_j.1_{\Bbbk}$ and (2.6) gives

$$\varepsilon(\chi^s(f)) = \mathrm{str}(f) \tag{2.7}$$

When f is even formula (2.6) becomes $\chi^s(f) = \sum_{i,j}(-1)^{\widehat{i}} b^i_j F^j_i$, because $F^j_i = 0$ unless $\widehat{i} + \widehat{j} = \bar{0}$. In particular,

$$\chi^s_V = \sum_i (-1)^{\widehat{i}} b^i_i \tag{2.8}$$

In the following, we let $\mathsf{comod}^s_{\mathcal{B}}$ denote the full subcategory of $\mathsf{Comod}^s_{\mathcal{B}}$ consisting of all objects that are finite-dimensional over $\Bbbk$. The supercharacter has the following properties analogous to standard properties of the ordinary character.

Lemma 2.2. *Let $\mathcal{B}$ denote a superbialgebra and let U, V and W be objects of $\mathsf{comod}^s_{\mathcal{B}}$.*

(a) *If $f\colon V \to U$ and $g\colon U \to V$ are $\mathcal{B}$-comodule maps (not necessarily even) then*

$$\chi^s(f \circ g) = (-1)^{\widehat{f}\widehat{g}} \chi^s(g \circ f)$$

(b) *For $f \in \operatorname{End}_{\Bbbk}(V)$, $g \in \operatorname{End}_{\Bbbk}(U)$ view $f \otimes g \in \operatorname{End}_{\Bbbk}(V \otimes U)$ as in* (1.7). *Then*

$$\chi^s(f \otimes g) = \chi^s(f)\chi^s(g)$$

(c) *Given an exact sequence $0 \to U \xrightarrow{\mu} V \xrightarrow{\mu} W \to 0$ in $\mathsf{comod}^s_{\mathcal{B}}$, let $f \in \operatorname{End}_{\Bbbk}(V)$ be such that $f(\mu(U)) \subseteq \mu(U)$, and let $g \in \operatorname{End}_{\Bbbk}(U)$, $h \in \operatorname{End}_{\Bbbk}(W)$ be the maps induced by f. Then*

$$\chi^s(f) = \chi^s(g) + \chi^s(h)$$

In particular, $\chi^s_V = \chi^s_U + \chi^s_W$. Moreover, if $f \in \operatorname{End}_{\mathsf{comod}^s_{\mathcal{B}}}(V)$ is a projection (i.e., $f^2 = f$) then $\chi^s(f) = \chi^s_{\operatorname{Im} f}$.

Proof. (a) Let T_V denote the map $\operatorname{Hom}_{\Bbbk}(V, V \otimes \mathcal{B}) \longrightarrow \mathcal{B}$ in (2.5); so $\chi^s(f) = T_V(\delta_V \circ f)$. Since f and g are comodule maps, we have $\delta_U \circ f = (f \otimes \mathrm{Id}_{\mathcal{B}}) \circ \delta_V$ and similarly for g. Putting $h = \delta_U \circ f \in \operatorname{Hom}_{\Bbbk}(V, U \otimes \mathcal{B})$ we obtain $\chi^s(f \circ g) = T_U(\delta_U \circ f \circ g) = T_U(h \circ g)$ and $\chi^s(g \circ f) = T_V(\delta_V \circ g \circ f) = T_V((g \otimes \mathrm{Id}_{\mathcal{B}}) \circ h)$. Therefore, we must show that

$$T_U(h \circ g) = (-1)^{\widehat{f}\widehat{g}} T_V((g \otimes \mathrm{Id}_{\mathcal{B}}) \circ h)$$

Using the identification $\operatorname{Hom}_{\Bbbk}(V, U \otimes \mathcal{B}) \cong \operatorname{Hom}_{\Bbbk}(V, U) \otimes \mathcal{B}$ as in (1.6), write $h = \sum_i f_i \otimes b_i$ with $f_i \in \operatorname{Hom}_{\Bbbk}(V, U)$, $b_i \in \mathcal{B}$, and $\widehat{f_i} + \widehat{b_i} = \widehat{h} = \widehat{f}$. Then $h \circ g \in \operatorname{Hom}_{\Bbbk}(U, U \otimes \mathcal{B})$ becomes the element $(\sum_i f_i \otimes b_i) \circ g = \sum_i (-1)^{\widehat{b_i}\widehat{g}} (f_i \circ g) \otimes b_i \in \operatorname{End}_{\Bbbk}(U) \otimes \mathcal{B}$, and $(g \otimes \mathrm{Id}_{\mathcal{B}}) \circ h \operatorname{Hom}_{\Bbbk}(V, V \otimes \mathcal{B})$ becomes $\sum_i (g \circ f_i) \otimes b_i$. The standard identity $\mathrm{str}(f_i \circ g) = (-1)^{\widehat{f_i}\widehat{g}} \mathrm{str}(g \circ f_i)$ (cf., e.g., [40, p. 165 §3(b)]) now yields

$$\begin{aligned}
T_U(h \circ g) &= \sum_i (-1)^{\widehat{b_i}\widehat{g}} \mathrm{str}(f_i \circ g) \otimes b_i \\
&= \sum_i (-1)^{\widehat{b_i}\widehat{g} + \widehat{f_i}\widehat{g}} \mathrm{str}(g \circ f_i) \otimes b_i \\
&= (-1)^{\widehat{f}\widehat{g}} T_V((g \otimes \mathrm{Id}_{\mathcal{B}}) \circ h)
\end{aligned}$$

as desired.

(b) Fix homogeneous $\Bbbk$-bases $\{x_i\}$ and $\{y_\ell\}$ of V and U, respectively, and write $\widehat{x_i} = \widehat{i}$, $\widehat{y_\ell} = \widehat{\ell}$ as usual. Moreover, let (F^i_j) and (G^ℓ_m) be the matrices of f and g for these bases, as in (1.2). Then $\{x_i \otimes y_\ell\}$ is a basis of $V \otimes U$, with $x_i \otimes y_\ell$ having parity $\widehat{i} + \widehat{\ell}$. Moreover,

$$\begin{aligned}(f \otimes g)(x_j \otimes y_m) &= (-1)^{\widehat{g}\widehat{j}} f(x_j) \otimes g(y_m) \\ &= (-1)^{\widehat{g}\widehat{j}} \sum_i x_i F^i_j \otimes \sum_\ell y_\ell G^\ell_m \\ &= \sum_{i,\ell} x_i \otimes y_\ell \Phi^{i,\ell}_{j,m} \qquad \text{with } \Phi^{i,\ell}_{j,m} = (-1)^{(\widehat{\ell}+\widehat{m})\widehat{j}} F^i_j G^\ell_m\end{aligned}$$

because $G^m_\ell = 0$ unless $\widehat{\ell} + \widehat{m} = \widehat{g}$. Similarly, writing $\delta_V(x_j) = \sum_i x_i \otimes b^i_j$ with $b^i_j \in \mathcal{B}_{\widehat{i}+\widehat{j}}$ and $\delta_U(y_m) = \sum_\ell y_\ell \otimes c^\ell_m$ with $c^\ell_m \in \mathcal{B}_{\widehat{\ell}+\widehat{m}}$, one obtains using (2.4)

$$\delta_{V\otimes U}(x_j \otimes y_m) = \sum_{i,\ell} x_i \otimes y_\ell \otimes \Psi^{i,\ell}_{j,m} \qquad \text{with } \Psi^{i,\ell}_{j,m} = (-1)^{(\widehat{i}+\widehat{j})\widehat{\ell}} b^i_j c^\ell_m$$

Therefore, formula (2.6) becomes

$$\begin{aligned}\chi^s(f \otimes g) &= \sum_{i,\ell,j,m} (-1)^{(\widehat{i}+\widehat{\ell})(\widehat{j}+\widehat{m})} \Psi^{i,\ell}_{j,m} \Phi^{j,m}_{i,\ell} \\ &= \sum_{i,\ell,j,m} (-1)^{\widehat{i}\widehat{j}+\widehat{\ell}\widehat{m}} b^i_j F^j_i c^\ell_m G^m_\ell \\ &= \chi^s(f)\chi^s(g)\end{aligned}$$

(c) Choose a basis $\{x_i\}$ of V consisting of homogeneous elements so that $x_i = \mu(y_i)$ for $i \leq \dim U$ and let (F^i_j) be the matrix of f for this basis. Then $F^i_j = 0$ for $i > \dim U$, $j \leq \dim U$. Moreover, the y_i form a basis of U and the $z_i = \pi(x_i)$ form a basis of W, and the matrices of g and h for these bases are $(F^i_j)_{i,j\leq \dim U}$ and $(F^i_j)_{i,j>\dim U}$, respectively. Similarly, if (b^i_j) is the matrix of δ_V with respect to the basis basis $\{x_i\}$ as in (2.3) then $b^i_j = 0$ for $i > \dim U$, $j \leq \dim U$, and the matrices of δ_U and δ_W for the given bases are $(b^i_j)_{i,j\leq \dim U}$ and $(b^i_j)_{i,j>\dim U}$, respectively. Therefore,

$$\begin{aligned}\chi^s(f) &= \sum_{i,j} (-1)^{\widehat{i}\widehat{j}} b^i_j F^j_i \\ &= \sum_{i,j\leq \dim U} (-1)^{\widehat{i}\widehat{j}} b^i_j F^j_i + \sum_{i,j>\dim U} (-1)^{\widehat{i}\widehat{j}} b^i_j F^j_i \\ &= \chi^s(g) + \chi^s(h)\end{aligned}$$

The remaining assertions are clear. □

2.4. The Grothendieck ring. Let $\mathcal{B}$ be a superbialgebra and let

$$R_\mathcal{B} = K_0(\mathsf{comod}^s_\mathcal{B})$$

denote the Grothendieck group of the category $\mathsf{comod}^s_\mathcal{B}$. Thus, for each V in $\mathsf{comod}^s_\mathcal{B}$, there is an element $[V] \in R_\mathcal{B}$ and each short exact sequence $0 \to U \to V \to W \to 0$ in $\mathsf{comod}^s_\mathcal{B}$ gives rise to an equation $[V] = [U] + [W]$ in $R_\mathcal{B}$. The group $R_\mathcal{B}$ is in fact a ring with multiplication given by the tensor product of $\mathcal{B}$-comodules. If $\mathcal{B}$ is supercommutative as a superalgebra then the ring $R_\mathcal{B}$ is commutative; see §2.2.

Both the ordinary dimension and the superdimension are additive on short exact sequences and multiplicative on tensor products. Hence they yield ring homomorphisms

$$\dim, \operatorname{sdim}\colon R_{\mathcal{B}} \to \mathbb{Z}$$

Parts (b) and (c) of Lemma 2.2 and formula (2.7) have the following immediate consequence:

Corollary 2.3. *The map $[V] \mapsto \chi^s_V$ yields a well-defined ring homomorphism $\chi^s\colon R_{\mathcal{B}} \to \mathcal{B}_{\bar{0}}$. Furthermore, the following diagram commutes*

$$\begin{array}{ccc} R_{\mathcal{B}} & \xrightarrow{\chi^s} & \mathcal{B}_{\bar{0}} \\ {\scriptstyle \operatorname{sdim}}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ \mathbb{Z} & \xrightarrow[\text{can.}]{} & \Bbbk \end{array}$$

Forgetting the $\mathbb{Z}_2$-grading, the corollary also gives the more familiar version with χ and $\dim$ in place of χ^s and sdim, respectively.

2.5. General linear supergroup and Berezinian. Let V in $\mathsf{Vect}^s_{\Bbbk}$ be finite-dimensional and fix a standard basis $x_1, \dots, x_d$ with $\widehat{i} = \bar{0}$ $(i \le p)$ and $\widehat{i} = \bar{1}$ $(i > p)$.

2.5.1. For each supercommutative $\Bbbk$-superalgebra $\mathcal{R}$ we denote by $\mathsf{E}(V)(\mathcal{R})$ the set of all $\mathcal{R}$-linear maps $V \otimes \mathcal{R} \to V \otimes \mathcal{R}$ in $\mathsf{Vect}^s_{\Bbbk}$. Using the identification $\operatorname{End}_{\mathcal{R}}(V \otimes \mathcal{R}) \cong \operatorname{Hom}_{\Bbbk}(V, V \otimes \mathcal{R}) \cong \operatorname{End}_{\Bbbk}(V) \otimes \mathcal{R}$ (see (1.6)), we may view $\mathsf{E}(V)(\mathcal{R})$ as the even subspace of $\operatorname{End}_{\Bbbk}(V) \otimes \mathcal{R}$:

$$\mathsf{E}(V)(\mathcal{R}) = (\operatorname{End}_{\Bbbk}(V) \otimes \mathcal{R})_{\bar{0}}$$

This defines a functor $\mathsf{E}(V)$ from the category of supercommutative $\Bbbk$-superalgebras to the category of semigroups.

2.5.2. Tensoring the supertrace $\operatorname{str}\colon \operatorname{End}_{\Bbbk}(V) \to \Bbbk$ of (1.8) with $\operatorname{Id}_{\mathcal{R}}$, we obtain an $\mathcal{R}$-linear supertrace map $\operatorname{str}\colon \operatorname{End}_{\Bbbk}(V) \otimes \mathcal{R} \to \mathcal{R}$ in $\mathsf{Vect}^s_{\Bbbk}$ which restricts to a map $\mathsf{E}(V)(\mathcal{R}) \to \mathcal{R}_{\bar{0}}$. The given standard basis $x_1, \dots, x_d$ of V is an $\mathcal{R}$-basis of $V \otimes \mathcal{R}$. In terms of this basis, an element $\phi \in \mathsf{E}(V)(\mathcal{R})$ is given by

$$\phi(x_j) = \sum_{i=1}^{d} x_i \Phi^i_j \quad \text{with} \quad \Phi^i_j \in \mathcal{R}_{\widehat{i}+\widehat{j}} \tag{2.9}$$

Thus ϕ is described by a *supermatrix* $\Phi = \left(\Phi^i_j\right)$ in *standard form* over $\mathcal{R}$:

$$\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.10}$$

where $A = \left(\Phi^i_j\right)_{i,j \le p}$ and $D = \left(\Phi^i_j\right)_{i,j > p}$ are square matrices with entries in $\mathcal{R}_{\bar{0}}$ while C, D are matrices over $\mathcal{R}_{\bar{1}}$. The supertrace of ϕ is given by

$$\operatorname{str}(\phi) = \sum_i (-1)^{\widehat{i}} \Phi^i_i = \operatorname{tr}(A) - \operatorname{tr}(D) =: \operatorname{str}(\Phi)$$

2.5.3. The functor $\mathsf{E}(V)$ is represented by a supercommutative $\Bbbk$-superbialgebra which coacts on V; this algebra will be denoted by

$$\mathcal{B} = \mathcal{O}(\mathsf{E}(V))$$

Thus, there is a natural isomorphism of $\mathsf{E}(V)$ with the functor $\mathrm{Hom}(\mathcal{B}, ?)$ of parity preserving algebra homorphisms. In particular, the identity map on $\mathcal{B}$ corresponds to an element $\xi \in \mathsf{E}(V)(\mathcal{B})$. Let $X = (x^i_j)_{d\times d}$ be the matrix of ξ, as in (2.9). The elements x^i_j have parity $\widehat{i} + \widehat{j}$ and they form a set of supercommuting algebraically independent generators of $\mathcal{B}$. In fact, $\mathcal{B}$ is isomorphic to the symmetric superalgebra $\mathsf{S}(V^* \otimes V)$, with $x^i_j \mapsto x^i \otimes x_j$, where $\{x^i\} \subseteq V^*$ is the dual basis for the given basis of V.

We can think of X as the *generic supermatrix* with respect to the given basis of V: any supermatrix $\Phi = \left(\Phi^i_j\right)$ as in (2.9) comes from an algebra map $\mathcal{B} \to \mathcal{R}$ via $x^i_j \mapsto \Phi^i_j$. The canonical coaction $\delta\colon V \to V \otimes \mathcal{B}$, the comultiplication Δ and the counit ε of $\mathcal{B}$ are given by

$$\begin{aligned} \delta(x_j) &= \sum_i x_i \otimes x^i_j \\ \Delta(x^i_j) &= \sum_k x^i_k \otimes x^k_j \\ \varepsilon(x^i_j) &= \delta^i_j \end{aligned} \tag{2.11}$$

These formulas can also be written as $\delta(x_1, \dots, x_d) = (x_1, \dots, x_d) \otimes X$, $\Delta(X) = X \otimes X$ and $\varepsilon(X) = 1$.

2.5.4. Similarly, $\mathsf{GL}(V)(\mathcal{R})$ is defined, for any supercommutative $\Bbbk$-superalgebra $\mathcal{R}$, as the set of all *invertible* $\mathcal{R}$-linear endomorphism of $V \otimes \mathcal{R}$ in $\mathsf{Vect}^s_\Bbbk$. The condition for a supermatrix Φ in standard form (as in (2.10)) to be invertible is that A and D are invertible as ordinary matrices over $\mathcal{R}_{\bar{0}}$. In this case, the inverse of Φ is given by

$$\Phi^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

See Berezin [3, Theorem 3.1 and Lemma 3.2]. The element

$$\mathrm{ber}(\Phi) := \det(A)\det(D - CA^{-1}B)^{-1} = \det(D)^{-1}\det(A - BD^{-1}C) \tag{2.12}$$

is called the *superdeterminant* or *Berezinian* of Φ; it is an invertible element of $\mathcal{R}_{\bar{0}}$.

The functor $\mathsf{GL}(V)$ is represented by a supercommutative Hopf superalgebra $\mathcal{O}(\mathsf{GL}(V))$ which is generated over $\mathcal{B} = \mathcal{O}(\mathsf{E}(V))$ by $\det(X_{11})^{-1}$ and $\det(X_{22})^{-1}$, where $X_{11} = \left(x^i_j\right)_{i,j\le p}$ and $X_{22} = \left(x^i_j\right)_{i,j>p}$ are the even blocks of the generic supermatrix X. By [3, Theorem 3.3], the Berezinian $\mathrm{ber}(X)$ is a group-like element in $\mathcal{O}(\mathsf{GL}(V))$.

2.6. Supersymmetric functions and exterior powers. Throughout this section, V will denote a finite-dimensional vector superspace over $\Bbbk$. We assume that the characteristic of $\Bbbk$ is zero.

2.6.1. Let

$$Y_n = \tfrac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma \in \Bbbk[\mathfrak{S}_n]$$

be the *antisymmetrizer* idempotent of the group algebra $\Bbbk[\mathfrak{S}_n]$ and define

$$\Lambda^n V := \operatorname{Im} c_{Y_n} \subseteq V^{\otimes n} \tag{2.13}$$

where $c \colon \Bbbk[\mathfrak{S}_n] \to \operatorname{End}_{\mathsf{Vect}^s_\Bbbk}(V^{\otimes n})$ is as in (1.10). Thus, $\Lambda^n V$ is the space of antisymmetric n-tensors,

$$\Lambda^n V = \{ y \in V^{\otimes n} \mid c_\sigma(y) = \operatorname{sgn}(\sigma) y \text{ for all } \sigma \in \mathfrak{S}_n \}$$

For later use, we describe an explicit basis of $\Lambda^n V$. To this end, fix a standard basis $x_1, \dots, x_d$ of V, with $\widehat{i} = \bar{0}$ for $i \le p$ and $\widehat{i} = \bar{1}$ for $i > p$. Then the products $x_{\mathbf{i}} = x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n}$ for sequences $\mathbf{i} = (i_1, i_2, \dots, i_n) \in \{1, 2, \dots, d\}^n$ form a graded basis of $V^{\otimes n}$ that is permuted up to a $\pm$-sign by the action of $\mathfrak{S}_n$ on $V^{\otimes n}$; see formula (1.9):

$$c_\sigma(x_{\mathbf{i}}) = \operatorname{sgn}_{\mathbf{i}}(\sigma) x_{\sigma(\mathbf{i})} \tag{2.14}$$

with

$$\operatorname{sgn}_{\mathbf{i}}(\sigma) = (-1)^{\sum_{(p,q) \in \mathsf{inv}(\sigma)} \widehat{i_p}\widehat{i_q}} \quad \text{and} \quad \sigma(\mathbf{i}) = (i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2)}, \dots, i_{\sigma^{-1}(n)})$$

Therefore, by elementary properties of monomial group representations, a $\Bbbk$-basis of $\Lambda^n V$ is given by the nonzero elements $c_{Y_n}(x_{\mathbf{i}})$ where $\mathbf{i}$ ranges over a transversal for the $\mathfrak{S}_n$-action on $\{1, 2, \dots, d\}^n$. Such a transversal is provided by the weakly increasing sequences $\mathbf{i} \in \{1, 2, \dots, d\}^n$. Moreover, for a weakly increasing $\mathbf{i}$, it is easily seen from (2.14) that $c_{Y_n}(x_{\mathbf{i}}) = 0$ holds precisely if $i_\ell = i_{\ell+1} \le p$ for some ℓ. Therefore, a basis of $\Lambda^n V$ is given by the elements $c_{Y_n}(x_{\mathbf{i}})$ with $\mathbf{i} = (i_1 < i_2 < \cdots < i_m < i_{m+1} \le \cdots \le i_n) \in \{1, 2, \dots, d\}^n$ and $i_m \le p < i_{m+1}$.

In particular,

$$\dim_\Bbbk \Lambda^n V = \sum_{m+m'=n} \binom{p}{m} \binom{q+m'-1}{m'} \tag{2.15}$$

where $p = \dim_\Bbbk V_{\bar{0}}$ and $q = \dim_\Bbbk V_{\bar{1}}$. Equivalently, the generating power series in $\mathbb{Z}[\![t]\!]$ for the sequence $\dim_\Bbbk \Lambda^n V$ is given by

$$\sum_{n \ge 0} \dim_\Bbbk \Lambda^n V t^n = \frac{(1+t)^p}{(1-t)^q} \tag{2.16}$$

When $q > 0$ then all $\Lambda^n V$ are nonzero. For additional details on exterior powers, see, e.g., [43, Sections I.5 and I.7].

2.6.2. Consider the super bialgebra $\mathcal{B} = \mathcal{O}(\mathsf{E}(V))$ as defined in §2.5.3 and recall that V is in $\mathsf{comod}^s_{\mathcal{B}}$. The representation $c \colon \Bbbk[\mathfrak{S}_n] \to \operatorname{End}_{\mathsf{Vect}^s_\Bbbk}(V^{\otimes n})$ of (1.10) actually has image in $\operatorname{End}_{\mathsf{comod}^s_{\mathcal{B}}}(V^{\otimes n})$, since $\mathcal{B}$ is supercommutative. Therefore, $\Lambda^n V$ also belongs to $\mathsf{comod}^s_{\mathcal{B}}$ and we can define the n^{th} *elementary supersymmetric function* by

$$e_n := \chi^s_{\Lambda^n V} = \chi^s(c_{Y_n}) \in \mathcal{B}_{\bar{0}}$$

Here, the equality $\chi^s_{\Lambda^n V} = \chi^s(c_{Y_n})$ holds by Lemma 2.2(c).

Similarly, one defines the n^{th} *super power sum* by

$$p_n := \chi^s(c_{(1,2,\dots,n)}) \in \mathcal{B}_{\bar{0}}$$

where $(1, 2, \ldots, n) \in \mathfrak{S}_n$ the cyclic permutation mapping $1 \mapsto 2 \mapsto 3 \mapsto \ldots \mapsto n \mapsto 1$. In terms of the generic supermatrix X from §2.5.3, one has

$$p_n = \mathrm{str}(X^n)$$

Modulo the space spanned by the Lie commutators $fg - gf$ with $f, g \in \Bbbk[\mathfrak{S}_n]$, the following relation is easily seen to hold in $\Bbbk[\mathfrak{S}_n]$:

$$nY_n \equiv \sum_{i=1}^{n} (-1)^{i-1} (1, 2, \ldots, i) Y_{n-i}$$

(with $Y_0 = 1$). Applying the function $\chi^s \circ c \colon \Bbbk[\mathfrak{S}_n] \to \mathcal{B}_{\bar{0}}$ to this relation and using Lemma 2.2(a),(b), one obtains the *Newton relations*:

$$ne_n = \sum_{i=1}^{n} (-1)^{i-1} p_i e_{n-i}$$

Let t be a formal parameter (of parity $\bar{0}$) and consider the generating functions $P(t) = \sum_{n \geq 1} p_n t^{n-1}$ and $E(t) = \sum_{n \geq 0} e_n t^n$ in $\mathcal{B}_{\bar{0}}[\![t]\!]$. The Newton relations can be written in the form $P(-t) = \frac{d}{dt} \log E(t)$; see, e.g., [34, p. 23]. Combining this with the identity

$$\mathrm{ber}(\exp(tX)) = \exp(\mathrm{str}(tX))$$

due to Berezin ([3, Chapter 3] or [40, p. 167]), one obtains the following expansion for the characteristic function $\mathrm{ber}(1 + tX)$ of generic supermatrix X:

Proposition 2.4. $\mathrm{ber}(1 + tX) = \sum_{n \geq 0} e_n t^n$

This proposition is known; see, e.g., Khudaverdian and Voronov [30, Prop. 1].

3. HOMOGENEOUS SUPERALGEBRAS

3.1. N-homogeneous superalgebras. Let N be an integer with $N \geq 2$. A *homogeneous superalgebra of degree* N or N*-homogeneous superalgebra* is an algebra $\mathcal{A}$ of the form (2.1) with V finite-dimensional and $R \subseteq V^{\otimes N}$:

$$\mathcal{A} = A(V, R) \cong \mathsf{T}(V)/(R)$$

The assumption $R \subseteq V^{\otimes N}$ implies that, besides the usual $\mathbb{Z}_2$-grading ("parity"), $\mathcal{A}$ also has a connected $\mathbb{Z}_+$-grading ("degree"),

$$\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$$

The algebra $\mathcal{A}$ is generated by $\mathcal{A}_1 = V$ and all homogeneous components $\mathcal{A}_n$ are finite-dimensional objects of $\mathsf{Vect}^s_{\Bbbk}$. In fact,

$$\mathcal{A}_n \cong V^{\otimes n}/R_n \quad \text{with} \quad R_n := (R) \cap V^{\otimes n} = \sum_{i+j+N=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \tag{3.1}$$

Note that $R_n = 0$ for $n < N$; so $\mathcal{A}_n \cong V^{\otimes n}$ if $n < N$.

Morphisms of N-homogeneous superalgebras $f \colon \mathcal{A} = A(V, R) \to \mathcal{A}' = A(V', R')$ are morphism of superalgebras which also respect the $\mathbb{Z}_+$-grading. Equivalently, by restricting to degree 1, we have a morphism $f_1 \colon \mathcal{A}_1 = V \to \mathcal{A}'_1 = V'$ in $\mathsf{Vect}^s_{\Bbbk}$ whose N^{th} tensor power satisfies $f_1^{\otimes N}(R) \subseteq R'$. Thus, one has a category $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$ of N-homogeneous $\Bbbk$-superalgebras.

Finally, N-homogeneous superalgebras with $N = 2$ are called *quadratic* superalgebras; for $N = 3$, they are called *cubic*, etc..

3.2. Some examples. In order to explicitly describe a certain N-homogeneous superalgebra $\mathcal{A} = A(V, R)$, we will usually fix a $\mathbb{Z}_2$-graded $\Bbbk$-basis $x_1, \dots, x_d$ of $V = \mathcal{A}_1$ and denote the the parity of x_i by $\widehat{i}$, as in §1.1. The x_i form a set of algebra generators for A. Following Manin [38],[39], the d-tuple $\mathbf{f} = (\widehat{1}, \dots, \widehat{d}) \in \mathbb{Z}_2^d$ is called the *format* of the basis $\{x_i\}$.

Example 3.1 (Quantum superspace [39])**.** For a fixed family $\mathbf{q}$ of scalars $0 \neq q_{ij} \in \Bbbk$ $(1 \leq i < j \leq d)$ and a given format $\mathbf{f} = (\widehat{1}, \dots, \widehat{d}) \in \mathbb{Z}_2^d$ of the basis $x_1, \dots, x_d$, the quadratic superalgebra $\mathcal{A} = \mathsf{S}_{\mathbf{q}}^{\mathbf{f}}$ is defined as the factor of $\mathsf{T}(V)$ modulo the ideal generated by the elements

$$r_i := x_i \otimes x_i \in (V^{\otimes 2})_{\bar{0}} \qquad (\widehat{i} = \bar{1}) \tag{3.2}$$

$$r_{ij} := x_j \otimes x_i - q_{ij}(-1)^{\widehat{i}\widehat{j}} x_i \otimes x_j \in (V^{\otimes 2})_{\widehat{i}+\widehat{j}} \qquad (i < j) \tag{3.3}$$

Thus, the algebra $\mathsf{S}_{\mathbf{q}}^{\mathbf{f}}$ is generated by $x_1, \dots, x_d$ subject to the defining relations

$$x_i x_i = 0 \qquad (\widehat{i} = \bar{1})$$

and

$$x_j x_i = q_{ij}(-1)^{\widehat{i}\widehat{j}} x_i x_j \qquad (i < j).$$

In the special case where all $q_{ij} = 1$, the algebra $\mathsf{S}_{\mathbf{q}}^{\mathbf{f}}$ is the symmetric superalgebra $\mathsf{S}(V)$ of V as in Example 2.1.

The ordered monomials of the form $x_1^{m_1} x_2^{m_2} \dots x_d^{m_d}$, with $\sum_i m_i = n$, $m_i \geq 0$ for all i and $m_i \leq 1$ if $\widehat{i} = \bar{1}$, form a $\Bbbk$-basis of the n^{th} homogeneous component of $\mathsf{S}_{\mathbf{q}}^{\mathbf{f}}$. Therefore,

$$\dim_{\Bbbk}(\mathsf{S}_{\mathbf{q}}^{\mathbf{f}})_n = \sum_{r+s=n} \binom{r+p-1}{p-1} \binom{q}{s} \tag{3.4}$$

where $\dim V_{\bar{0}} = p$ and $\dim V_{\bar{1}} = q$ as usual.Thus, the generating series of the dimensions is

$$\sum_{n \geq 0} \dim_{\Bbbk}(\mathsf{S}_{\mathbf{q}}^{\mathbf{f}})_n t^n = \frac{(1+t)^q}{(1-t)^p}$$

Example 3.2 (Yang-Mills algebras [11],[10])**.** Fix a collection of elements $x_1, \dots, x_d$ $(d \geq 2)$, numbered so as to have parity $\widehat{i} = \bar{0}$ for $i \leq p$ and $\widehat{i} = \bar{1}$ for $i > p$. Let $G = (g_{ij}) \in \mathsf{GL}_d(\Bbbk)$ be an invertible symmetric $d \times d$-matrix satisfying $g_{ij} = 0$ if $\widehat{i} \neq \widehat{j}$ and consider the cubic superalgebra $\mathcal{A}$ that is generated by elements $x_1, \dots, x_d$ subject to the relations

$$\sum_{i,j} g_{ij}[x_i, [x_j, x_k]] = 0 \qquad (k = 1, \dots, d) \tag{3.5}$$

Here $[\,.\,,.\,]$ is the supercommutator (2.2). The algebra $\mathcal{A}$ will be denoted by $\mathcal{YM}^{p|q}$ $(q = d-p)$. In particular, the pure even algebra $\mathcal{YM}^{d|0}$ is the ordinary Yang-Mills algebra introduced in [10] while $\mathcal{YM}^{0|d}$ is the super Yang-Mills algebra as in [11].

As usual, put $V = \sum_i \Bbbk x_i$ and let $[\,.\,,.\,]_{\otimes}$ denote the supercommutator in $\mathsf{T}(V)$. Furthermore, put $r_k = \sum_{i,j} g_{ij}[x_i, [x_j, x_k]_{\otimes}]_{\otimes}$ and $R = \sum_k \Bbbk r_k \subseteq V^{\otimes 3}$; so $\mathcal{YM}^{p|q} = \mathsf{T}(V)/(R)$. Using the symmetry of G, we may replace the r_k by simpler relations as follows. Choose an invertible $d \times d$-matrix $C = (c_{ij})$ with $c_{ij} = 0$ if $\widehat{i} \neq \widehat{j}$ and such that $C^{\text{tr}}GC$ is diagonal, say $\sum_{i,j} c_{ir} g_{ij} c_{js} = g_s \delta_s^r$. Replace the bases $\{x_i\}$ of V and $\{r_k\}$ of R by the new bases $y_i = \sum_j c^{ij} x_j$ and $s_k = \sum_k c^{k\ell} r_\ell$ where $C^{-1} = (c^{ij})$. Note that y_i has parity $\widehat{i}$ and s_k has parity $\widehat{k}$, the parity of r_k. A simple calculation shows that $s_k = \sum_{i \neq k} g_i [y_i, [y_i, y_k]_\otimes]_\otimes$. Thus we obtain the following defining relations for the generators $y_1, \dots y_d$ of $\mathcal{YM}^{p|q}$:

$$\sum_{i \neq k} g_i [y_i, [y_i, y_k]] = 0 \qquad (k = 1, \dots, d) \tag{3.6}$$

The resulting algebras for $d = 2$ are as follows. Putting $x = y_1$ and $y = y_2$ we have two defining relations: $[x, [x, y]] = 0$ and $[y, [y, x]] = 0$. In the pure even case ($\widehat{x} = \widehat{y} = \bar{0}$), the supercommutators are the ordinary Lie commutators. So $\mathcal{YM}^{2|0}$ is the enveloping algebra of the Heisenberg Lie algebra; see [1, (0.4)]. In the pure odd case ($\widehat{x} = \widehat{y} = \bar{1}$), the two relations can be written as $x^2 y = y x^2$ and $y x^2 = x^2 y$. The resulting algebra $\mathcal{YM}^{0|2}$ is a cubic Artin-Schelter algebra of type S_1 [1, (8.6)]. Thus, both unmixed algebras are Artin-Schelter regular of global dimension 3. In the mixed case, however ($\widehat{x} = \bar{0}$, $\widehat{y} = \bar{1}$), the relations say that x commutes with the Lie commutator $[x, y]$ while y anticommutes: $y[x, y] = -[x, y]y$. Thus, $[x, y]$ is a normal element of $\mathcal{YM}^{1|1}$ and $\mathcal{YM}^{1|1}/([x, y])$ is a polynomial algebra in two variables over $\Bbbk$. Moreover, the calculation

$$[x, y]^2 = [x, [x, y]y] = -[x, y[x, y]] = -[x, y]^2$$

shows that $[x, y]^2 = 0$. Thus, the algebra $\mathcal{YM}^{1|1}$ is noetherian with Gelfand-Kirillov dimension 2 and infinite global dimension.

Returning to the case of general $d \geq 2$, we now concentrate on the unmixed algebras introduced by Connes and Dubois-Violette. We will denote these algebras by $\mathcal{YM}^+ = \mathcal{YM}^{d|0}$ and $\mathcal{YM}^- = \mathcal{YM}^{0|d}$. In all formulas below, $+$ applies to $\mathcal{YM}^+$ and $-$ to $\mathcal{YM}^-$. The generators $s_k = \sum_{i \neq k} g_i [y_i, [y_i, y_k]_\otimes]_\otimes$ of the space of relations R can be written as $s_k = \sum_\ell y_\ell \otimes m_{\ell k} = \pm \sum_\ell m_{k\ell} \otimes y_\ell$ with

$$m_{\ell k} = \begin{cases} g_\ell \left(y_\ell \otimes y_k - (1 \pm 1) y_k \otimes y_\ell \right) & \text{for } \ell \neq k \\ \pm \sum_{i \neq k} g_i y_i \otimes y_i & \text{for } \ell = k \end{cases}$$

Thus, putting $Y = (y_1, \dots, y_d)$ and letting M denote the $d \times d$-matrix over $\mathcal{YM}^\pm$ whose (ℓ, k)-entry is the image of $m_{\ell k}$, the defining relations (3.6) can be written as

$$YM = 0 \qquad \text{or} \qquad MY^{\text{tr}} = 0 \tag{3.7}$$

The defining relations (3.6) for $\mathcal{A} = \mathcal{YM}^-$ amount to the even element $\sum_i g_i y_i^2 \in \mathcal{A}_2$ being central in $\mathcal{A}$.

Example 3.3 (N-symmetric superalgebra; cf. [5]). Let $N \geq 2$ be given and let V be a vector superspace V over a field $\Bbbk$ with $\operatorname{char} \Bbbk = 0$ or $\operatorname{char} \Bbbk > N$. Define

$$\mathsf{S}_N(V) = A(V, R) \quad \text{with} \quad R = \Lambda^N V = c_{Y_N}\left(V^{\otimes N}\right) \subseteq V^{\otimes N}$$

where Y_N is the antisymmetrizer idempotent of the group algebra $\Bbbk[\mathfrak{S}_N]$; see (2.13). This defines a functor $\mathsf{S}_N(\,.\,)\colon \mathsf{Vect}_\Bbbk^s \to \mathsf{H}_N\mathsf{Alg}_\Bbbk^s$. Since $2c_{Y_2}$ is the supercommutator in $\mathsf{T}(V)$, the algebra $\mathsf{S}_2(V)$ is just the symmetric superalgebra $\mathsf{S}(V)$ of V; see Example 3.1. The algebra $\mathsf{S}_N(V)$, for a pure even space $V = V_{\bar{0}}$ and general $N \geq 2$, has been introduced in [5].

If $2 \leq M \leq N$ then, viewing $\Bbbk[\mathfrak{S}_M]$ as a subalgebra of $\Bbbk[\mathfrak{S}_N]$ as usual, the antisymmetrizers of $\Bbbk[\mathfrak{S}_N]$ and $\Bbbk[\mathfrak{S}_M]$ satisfy $Y_N = Y_M a$ for some $a \in \Bbbk[\mathfrak{S}_N]$. Therefore,

$$R = c_{Y_N}\left(V^{\otimes N}\right) \subseteq c_{Y_M}\left(V^{\otimes N}\right) = c_{Y_M}\left(V^{\otimes M}\right) \otimes V^{\otimes(N-M)}$$

This shows that the identity map on V extends to an epimorphism of superalgebras $\mathsf{S}_N(V) \twoheadrightarrow \mathsf{S}_M(V)$.

Now assume that $\dim_k V = d$ and fix a standard basis $x_1, \ldots, x_d$ of V, with $\widehat{i} = \bar{0}$ for $i \leq p$ and $\widehat{i} = \bar{1}$ for $i > p$. From the basis for $\Lambda^N V$ exhibited in §2.6.1 we obtain that the algebra $\mathsf{S}_N(V)$ is generated by $x_1, \ldots, x_d$ subject to the relations

$$\sum_{\sigma \in \mathfrak{S}_N} (-1)^{\sum_{(p,q)\in \mathsf{inv}(\sigma)} 1+\widehat{i_p}\widehat{i_q}} x_{i_{\sigma^{-1}(1)}} x_{i_{\sigma^{-1}(2)}} \ldots x_{i_{\sigma^{-1}(N)}} = 0$$

with $1 \leq i_1 < i_2 < \cdots < i_m \leq p = \dim_\Bbbk V_{\bar{0}} < i_{m+1} \leq \cdots \leq i_N \leq d = \dim_\Bbbk V$; see formula (2.14).

Example 3.4. The following construction generalizes Example 3.3. Fix $N \geq 2$ and $0 \neq q \in \Bbbk$ and assume that condition (1.14) is satisfied. Given a Hecke operator $\mathscr{R} \colon V^{\otimes 2} \to V^{\otimes 2}$ on a vector superspace V we define the N-homogeneous superalgebra

$$\Lambda_{\mathscr{R},N} := A(V,R) \quad \text{with} \quad R = \operatorname{Im} \rho_{\mathscr{R}}(X_N) \subseteq V^{\otimes N} \tag{3.8}$$

where $X_N \in \mathscr{H}_{N,q}$ is the q-symmetrizer (1.16) and $\rho_{\mathscr{R}}$ is the representation (1.20) of $\mathscr{H}_{N,q}$. We also put

$$\mathsf{S}_{\mathscr{R},N} := \Lambda_{-q\mathscr{R}^{-1},N} = A(V,R) \quad \text{with} \quad R = \operatorname{Im} \rho_{\mathscr{R}}(Y_N) \subseteq V^{\otimes N} \tag{3.9}$$

where $Y_N \in \mathscr{H}_{N,q}$ is the antisymmetrizer (1.17). The algebra $\mathsf{S}_N(V)$ in Example 3.3 is identical with $\mathsf{S}_{c_{V,V},N}$ ($q = 1$).

3.3. The dual of a homogeneous superalgebra. Let $\mathcal{A} = A(V,R)$ be an N-homogeneous superalgebra. The dual $\mathcal{A}^!$ of $\mathcal{A}$ is defined by

$$\mathcal{A}^! = A(V^*, R^\perp)$$

where, $R^\perp \subseteq V^{*\,\otimes N}$ is the (homogeneous) subspace consisting of all elements that vanish on $R \subseteq V^{\otimes N}$, using (1.4) in order to evaluate elements of $V^{*\,\otimes N}$ on $V^{\otimes N}$. Thus, (3.1) takes the form

$$\mathcal{A}^!_n = V^{*\,\otimes n}/R^\perp_n \quad \text{with} \quad R^\perp_n := \sum_{i+j+N=n} V^{*\,\otimes i} \otimes R^\perp \otimes V^{*\,\otimes j} \tag{3.10}$$

Identifying $V^{*\,\otimes n}$ with the linear dual of $V^{\otimes n}$ via (1.4), we have $V^{*\,\otimes i} \otimes R^\perp \otimes V^{*\,\otimes j} = \left(V^{\otimes j} \otimes R \otimes V^{\otimes i}\right)^\perp$. Hence,

$$R^\perp_n = \left(\bigcap_{i+j+N=n} V^{\otimes j} \otimes R \otimes V^{\otimes i} \right)^\perp \tag{3.11}$$

The canonical isomorphism $V \xrightarrow{\sim} V^{**}$ in (1.5) leads to an isomorphism $V^{\otimes N} \xrightarrow{\sim} V^{**\,\otimes N}$ which maps R onto $R^{\perp\perp}$. Hence,

$$\mathcal{A}^{!!} \cong \mathcal{A} \tag{3.12}$$

Moreover, if $f\colon \mathcal{A} = A(V,R) \to \mathcal{A}' = A(V',R')$ is any morphism in $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$ then the transpose of $f_1\colon V \to V'$ induces a morphism $f^!\colon (\mathcal{A}')^! \to \mathcal{A}^!$ in $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$. Thus, we have a contravariant quasi-involutive *dualization functor* $\mathcal{A} \mapsto \mathcal{A}^!$, $f \mapsto f^!$ on $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$.

Example 3.5. The dual of $A(V,0) = \mathsf{T}(V)$ is $A(V^*, V^{*\otimes N})$; so

$$\mathsf{T}(V)^! = \mathsf{T}(V^*)/\left(V^{*\otimes N}\right)$$

In particular, letting $V = \Bbbk$ be the unit object of $\mathsf{Vect}^s_{\Bbbk}$, we have $A(\Bbbk,0) = \Bbbk[t]$ (polynomial algebra) and $A(\Bbbk,0)^! = \Bbbk[d]/(d^N)$, with t and d both having degree 1 and parity $\bar{0}$.

Example 3.6 (Dual of quantum superspace)**.** We will describe the dual $\mathcal{A}^!$ of quantum superspace $\mathcal{A} = \mathsf{S}^{\mathbf{f}}_{\mathbf{q}}$; see Example 3.1. Fix a homogeneous $\Bbbk$-basis $x_1,\dots,x_d$ with format $\mathbf{f}$ for V, and let $x^1,\dots,x^d$ denote the dual basis of V^*; this basis also has format $\mathbf{f}$. Evaluating an arbitrary element $f = \sum_{\ell,m} f_{\ell m} x^\ell \otimes x^m \in V^{*\otimes 2}$ on one of the generating relations $r_i, r_{ij} \in R$ in (3.2), (3.3) we obtain $\langle f, r_i\rangle = f_{ii}$ and $\langle f, r_{ij}\rangle = f_{ij} - q_{ij}(-1)^{\widehat{i}\widehat{j}} f_{ji}$. Therefore, the space $R^\perp \subseteq V^{*\otimes 2}$ has a basis consisting of the elements $s^\ell := x^\ell \otimes x^\ell$ ($\widehat{\ell} = \bar{0}$) and $s^{\ell,k} := x^\ell \otimes x^k + q_{k\ell}(-1)^{\widehat{k}\widehat{\ell}} x^k \otimes x^\ell$ ($k < \ell$). In summary, $\mathcal{A}^!$ is generated by $x^1,\dots,x^d$ subject to the defining relations

$$x^\ell x^\ell = 0 \qquad (\widehat{\ell} = \bar{0})$$

and

$$x^\ell x^k = -q_{k\ell}(-1)^{\widehat{k}\widehat{\ell}} x^k x^\ell \qquad (k < \ell).$$

Thus, $\mathcal{A}^!$ is isomorphic to quantum superspace $\mathsf{S}^{\mathbf{f}'}_{\mathbf{q}'}$ with $q'_{ij} = (-1)^{\widehat{i}+\widehat{j}} q_{ij}$ and $\mathbf{f}' = \mathbf{f} + (\bar{1},\dots,\bar{1})$ the format obtained from $\mathbf{f}$ by parity reversal in all components.

Example 3.7 (Duals of the Yang-Mills algebras)**.** Continuing with the notation of Example 3.2, we now desribe the algebra $\mathcal{A}^!$ for $\mathcal{A} = \mathcal{YM}^{p|q}$. We assume that $\operatorname{char}\Bbbk = 0$ and work with generators $y_1,\dots,y_d$ of $\mathcal{A}$ satisfying (3.6).

Let $y^1,\dots,y^d$ denote the basis of V^* given by $\langle y^i, y_j\rangle = \delta^i_j$ and put $\gamma = \frac{1}{d-1}\sum_i g_i^{-1} y^i \otimes y^i \in V^{*\otimes 2}$. Then, for the generators $s_k = \sum_{i\neq k} g_i[y_i,[y_i,y_k]_\otimes]_\otimes$ of R as in Example 3.2, one computes

$$\begin{aligned} \langle y^a \otimes y^b \otimes y^c, s_k\rangle &= g_c\delta^c_b\delta^a_k + (-1)^{\widehat{b}} g_b\delta^b_a\delta^c_k - (-1)^{\widehat{a}\widehat{k}}(1+(-1)^{\widehat{a}})g_a\delta^a_c\delta^b_k \\ \langle y^i \otimes \gamma, s_k\rangle &= \delta^i_k \end{aligned} \tag{3.13}$$

Therefore, the map $\varphi \mapsto \varphi - \sum_k \langle\varphi, s_k\rangle y^k \otimes \gamma$ is an epimorphism $V^{*\otimes 3} \twoheadrightarrow R^\perp \subset V^{*\otimes 3}$. We obtain that the algebra $\mathcal{A}^!$ is generated by $y^1,\dots,y^d$ subject to the relations

$$y^a y^b y^c = (g_c\delta^c_b y^a + (-1)^{\widehat{b}} g_b\delta^b_a y^c - (-1)^{\widehat{a}\widehat{b}}(1+(-1)^{\widehat{a}})g_a\delta^a_c y^b)\mathbf{g} \tag{3.14}$$

where $\mathbf{g} = \frac{1}{d-1}\sum_i g_i^{-1} y^i y^i$ is the image of γ in $\mathcal{A}$.

Since $\mathcal{A}^!$ is 3-homogeneous, we clearly have $\mathcal{A}^!_0 = \Bbbk$, $\mathcal{A}^!_1 = \bigoplus_i \Bbbk y^i = V^*$ and $\mathcal{A}^!_2 = \bigoplus_{i,j} \Bbbk y^i y^j \cong V^{*\otimes 2}$. By (3.13), the elements $y^a\mathbf{g}$ form a $\Bbbk$-basis of $\mathcal{A}^!_3 = V^{*\otimes 3}/R^\perp \cong R^*$. Using the defining relations (3.14) it is not hard to see that $\mathcal{A}^!_4 = \Bbbk\mathbf{g}^2$ and $\mathcal{A}^!_n = 0$ for $n \geq 5$. If $\mathcal{A} = \mathcal{YM}^{p|q}$ is of mixed type (i.e., $p \neq 0$ and $q \neq 0$) then $\mathbf{g}^2 = 0$.

Example 3.8 (Dual of the N-symmetric superalgebra)**.** Recall from Example 3.3 that $\mathsf{S}_N(V) = A(V,R)$ with $R = c_{Y_N}\left(V^{\otimes N}\right)$. Since Y_N is central in $\Bbbk[\mathfrak{S}_N]$ and stable under the inversion involution * of $\Bbbk[\mathfrak{S}_N]$, it follows from (1.11) that

$$\langle x, c_{Y_N}(y)\rangle = \langle c_{Y_N}(x), y\rangle$$

holds for all $x \in V^{*\otimes N}$ and $y \in V^{\otimes N}$. Therefore,

$$R^{\perp} = \operatorname{Ker}_{V^{*\otimes N}}(c_{Y_N}) = (1 - c_{Y_N})\left(V^{*\otimes N}\right)$$

and so

$$\mathsf{S}_N(V)^! = A\left(V^*, (1 - c_{Y_N})(V^{*\otimes N})\right)$$

Note that

$$\bigcap_{i+j+N=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} = c_{Y_n}\left(V^{\otimes n}\right) \tag{3.15}$$

holds for all $n \geq N$. This follows from (1.19). Alternatively, as has been noted in Example 3.3, we have $c_{Y_n}(V^{\otimes n}) \subseteq R \otimes V^{\otimes(n-N)}$. In the same way, one sees that $c_{Y_n}(V^{\otimes n}) \subseteq V^{\otimes i} \otimes R \otimes V^{\otimes j}$ whenever $i + j + N = n$. For the reverse inclusion, note that each $x \in V^{\otimes i} \otimes R \otimes V^{\otimes j}$ satisfies $c_{\sigma_\ell}(x) = -x$ for all transpositions $\sigma_\ell = (\ell, \ell+1) \in \mathfrak{S}_n$ with $i < \ell < i + N$. Hence, the left hand side of (3.15) is contained in the space of antisymmetric n-tensors, $\Lambda^n V = c_{Y_n}(V^{\otimes n})$, thereby proving (3.15). We deduce from (3.10), (3.11) and (2.15) that

$$\dim_{\Bbbk} \mathsf{S}_N(V)^!_n = \begin{cases} d^n & \text{if } n < N \\ \sum_{r+s=n} \binom{p}{r}\binom{q+s-1}{s} & \text{if } n \geq N \end{cases} \tag{3.16}$$

where $d = \dim_{\Bbbk} V$, $p = \dim_{\Bbbk} V_{\bar{0}}$ and $q = \dim_{\Bbbk} V_{\bar{1}}$.

3.4. The operations $\circ$ and $\bullet$ on $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$. Let $\mathcal{A} = A(V,R)$ and $\mathcal{A}' = A(V',R')$ be N-homogeneous superalgebras. Following [37] and [7] we define the white and black products $\mathcal{A} \circ \mathcal{A}'$ and $\mathcal{A} \bullet \mathcal{A}'$ by

$$\mathcal{A} \circ \mathcal{A}' = A\left(V \otimes V', c_{\pi_N}\left(R \otimes V'^{\otimes N} + V^{\otimes N} \otimes R'\right)\right)$$
$$\mathcal{A} \bullet \mathcal{A}' = A\left(V \otimes V', c_{\pi_N}\left(R \otimes R'\right)\right)$$

where $\pi_N \in \mathfrak{S}_{2N}$ is the inverse of the permutation

$$(1, 2, \dots, 2N) \mapsto (1, N+1, 2, N+2, \dots, k, N+k, \dots, N, 2N)$$

Explicitly, $c_{\pi_N} : V^{\otimes N} \otimes V'^{\otimes N} \longrightarrow (V \otimes V')^{\otimes N}$ is the morphism in $\mathsf{Vect}^s_{\Bbbk}$ that is given by

$$c_{\pi_N}\left(v_1 \otimes \dots v_N \otimes v'_1 \otimes \dots v'_N\right) = (-1)^{\sum_i \sum_{j>i} \widehat{v'_i}\widehat{v}_j}(v_1 \otimes v'_1) \otimes \dots (v_N \otimes v'_N) \tag{3.17}$$

Hence, $c_{\pi_N}(R \otimes R')$ and $c_{\pi_N}\left(R \otimes V'^{\otimes N} + V^{\otimes N} \otimes R'\right)$ are homogeneous subspaces of $(V \otimes V')^{\otimes N}$ and so $\mathcal{A} \circ \mathcal{A}'$ and $\mathcal{A} \bullet \mathcal{A}'$ belong to $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$.

Under the isomorphism $(V'^* \otimes V^*)^{\otimes N} \xrightarrow{\sim} (V \otimes V')^{*\otimes N}$ which comes from (1.4), the relations $c_{\pi_N}\left(R'^{\perp} \otimes R^{\perp}\right)$ of $\mathcal{A}'^! \bullet \mathcal{A}^!$ map onto the relations $\left(c_{\pi_N}\left(R \otimes V'^{\otimes N} + V^{\otimes N} \otimes R'\right)\right)^{\perp}$ of $(\mathcal{A} \circ \mathcal{A}')^!$. In fact, by (1.11) we have $c^*_{\pi_N} = c_{\pi_N}$, because $\pi_N \tau = \tau \pi_N$, and so $\langle x, y\rangle = \langle c_{\pi_N}(x), c_{\pi_N}(y)\rangle$ holds for all $x \in V'^{*\otimes N} \otimes V^{*\otimes N}$ and $y \in V^{\otimes N} \otimes V'^{\otimes N}$. Therefore, canonically,

$$(\mathcal{A} \circ \mathcal{A}')^! \cong \mathcal{A}'^! \bullet \mathcal{A}^! \quad \text{and} \quad (\mathcal{A} \bullet \mathcal{A}')^! \cong \mathcal{A}'^! \circ \mathcal{A}^! \tag{3.18}$$

the two identities being equivalent by (3.12).

By definition of $\circ$, the canonical isomorphisms $\Bbbk \otimes V \cong V \cong V \otimes \Bbbk$ in $\mathsf{Vect}^s_{\Bbbk}$ give isomorphisms $A(\Bbbk, 0) \circ \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \circ A(\Bbbk, 0)$ in $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$, and (3.18) yields similar isomorphisms for $\bullet$, with $A(\Bbbk, 0)^! = \Bbbk[d]/(d^N)$ replacing $A(\Bbbk, 0) = \Bbbk[t]$; see Example 3.5.

The supersymmetry isomorphism $c_{V,V'} \colon V \otimes V' \xrightarrow{\sim} V' \otimes V$ in $\mathsf{Vect}^s_{\Bbbk}$ (see (1.1)) yields isomorphisms

$$\mathcal{A} \circ \mathcal{A}' \cong \mathcal{A}' \circ \mathcal{A} \quad \text{and} \quad \mathcal{A} \bullet \mathcal{A}' \cong \mathcal{A}' \bullet \mathcal{A} \tag{3.19}$$

in $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$. To see this, note that the following diagram of isomorphisms in $\mathsf{Vect}^s_{\Bbbk}$ commutes:

$$\begin{array}{ccc}
V^{\otimes N} \otimes V'^{\otimes N} & \xrightarrow{c_{\pi_N}} & (V \otimes V')^{\otimes N} \\
{\scriptstyle c_{V^{\otimes N}, V'^{\otimes N}}}\downarrow & & \downarrow{\scriptstyle c^{\otimes N}_{V,V'}} \\
V'^{\otimes N} \otimes V^{\otimes N} & \xrightarrow[c_{\pi_N}]{} & (V' \otimes V)^{\otimes N}
\end{array}$$

with $v_1 \otimes \dots v_N \otimes v'_1 \otimes \dots v'_N \mapsto (-1)^{\sum_i \sum_{j \geq i} \widehat{v'_i}\widehat{v}_j} (v'_1 \otimes v_1) \otimes \dots (v'_N \otimes v_N)$ in both composites. Therefore, putting $R_{\mathcal{A} \circ \mathcal{A}'} = c_{\pi_N} \left(R \otimes V'^{\otimes N} + V^{\otimes N} \otimes R' \right)$ and similarly for $R_{\mathcal{A}' \circ \mathcal{A}}$ etc., we have

$$\begin{aligned}
c^{\otimes N}_{V,V'} \left(R_{\mathcal{A} \circ \mathcal{A}'} \right) &= \left(c_{\pi_N} \circ c_{V^{\otimes N}, V'^{\otimes N}} \right) \left(R \otimes V'^{\otimes N} + V^{\otimes N} \otimes R' \right) \\
&= c_{\pi_N} \left(R' \otimes V^{\otimes N} + V'^{\otimes N} \otimes R \right) \\
&= R_{\mathcal{A}' \circ \mathcal{A}}
\end{aligned}$$

In the same way, one sees that $c^{\otimes N}_{V,V'} \left(R_{\mathcal{A} \bullet \mathcal{A}'} \right) = R_{\mathcal{A}' \bullet \mathcal{A}}$. This proves (3.19).

Similarly, the associativity isomorphism $a_{V,V',V''} \colon (V \otimes V') \otimes V'' \cong V \otimes (V' \otimes V'')$ in $\mathsf{Vect}^s_{\Bbbk}$ leads to isomorphisms

$$(\mathcal{A} \circ \mathcal{A}') \circ \mathcal{A}'' \cong \mathcal{A} \circ (\mathcal{A}' \circ \mathcal{A}'') \quad \text{and} \quad (\mathcal{A} \bullet \mathcal{A}') \bullet \mathcal{A}'' \cong \mathcal{A} \bullet (\mathcal{A}' \bullet \mathcal{A}'') \tag{3.20}$$

in $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$. This is a consequence of the following commutative diagram of isomorphisms in $\mathsf{Vect}^s_{\Bbbk}$:

$$\begin{array}{ccccc}
\left(V^{\otimes N} \otimes V'^{\otimes N} \right) \otimes V''^{\otimes N} & \xrightarrow[c_{\pi_N} \otimes \mathrm{Id}]{} & (V \otimes V')^{\otimes N} \otimes V''^{\otimes N} & \xrightarrow[c_{\pi_N}]{} & ((V \otimes V') \otimes V'')^{\otimes N} \\
{\scriptstyle a_{V^{\otimes N}, V'^{\otimes N}, V''^{\otimes N}}}\downarrow & & & & \downarrow{\scriptstyle a^{\otimes N}_{V,V',V''}} \\
V^{\otimes N} \otimes \left(V'^{\otimes N} \otimes V''^{\otimes N} \right) & \xrightarrow[\mathrm{Id} \otimes c_{\pi_N}]{} & V^{\otimes N} \otimes (V' \otimes V'')^{\otimes N} & \xrightarrow[c_{\pi_N}]{} & (V \otimes (V' \otimes V''))^{\otimes N}
\end{array}$$

Finally, the compatibility between the isomorphisms $c_{V,V'}$ and $a_{V,V',V''}$ (see §1.2) is inherited by the isomorphisms (3.19) and (3.20) in $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$. To summarize:

Proposition 3.9. *The operations $\circ$ and $\bullet$ both make the category $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$ of N-homogeneous $\Bbbk$-superalgebras into a symmetric tensor category, with unit objects $A(\Bbbk, 0) = \Bbbk[t]$ for $\circ$ and $A(\Bbbk, 0)^! = \Bbbk[d]/(d^N)$ for $\bullet$.*

3.5. The superalgebra map $i\colon \mathcal{A}\circ\mathcal{A}' \to \mathcal{A}\otimes\mathcal{A}'$. Let $\mathcal{A} = A(V,R)$ and $\mathcal{A}' = A(V',R')$ be objects of $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$. The superalgebra $\mathcal{A}\otimes\mathcal{A}'$ is generated by $V\oplus V'$ subject to the relations

$$R + R' \subseteq (V\oplus V')^{\otimes N} \qquad \text{and} \qquad [V,V']_{\otimes} \subseteq (V\oplus V')^{\otimes 2}$$

where $[\,.\,,.\,]_{\otimes}$ is the supercommutator (2.2) in the tensor algebra, as usual. Thus, $\mathcal{A}\otimes\mathcal{A}'$ is not N-homogeneous when $N\geq 3$. Nonetheless, there always is an injective superalgebra homomorphism $i\colon \mathcal{A}\circ\mathcal{A}' \to \mathcal{A}\otimes\mathcal{A}'$ which is defined as follows. The linear embedding $V\otimes V' \hookrightarrow \mathsf{T}(V)\otimes\mathsf{T}(V')$ extends uniquely to a superalgebra map

$$\tilde{\imath}\colon \mathsf{T}(V\otimes V') \to \mathsf{T}(V)\otimes\mathsf{T}(V') \tag{3.21}$$

which doubles degrees: the restriction of $\tilde{\imath}$ to degree n is the embedding

$$\mathsf{T}(V\otimes V')_n = (V\otimes V')^{\otimes n} \xrightarrow{c_{\pi_n}^{-1}} V^{\otimes n}\otimes V'^{\otimes n} \subseteq (\mathsf{T}(V)\otimes\mathsf{T}(V'))_{2n}$$

in $\mathsf{Vect}^s_{\Bbbk}$, where c_{π_n} is as in (3.17). Thus, $\tilde{\imath}$ identifies the superalgebra $\mathsf{T}(V\otimes V')$ with the (super) *Segre product* $\bigoplus_{n\geq 0} V^{\otimes n}\otimes V'^{\otimes n}$ of $\mathsf{T}(V)$ and $\mathsf{T}(V')$.

The map $\tilde{\imath}$ sends $R_{\mathcal{A}\circ\mathcal{A}'} = c_{\pi_N}\left(R\otimes V'^{\otimes N} + V^{\otimes N}\otimes R'\right) \subseteq (V\otimes V')^{\otimes N}$ to $R\otimes V'^{\otimes N} + V^{\otimes N}\otimes R'$, the kernel of the canonical epimorphism $V^{\otimes N}\otimes V'^{\otimes N} \twoheadrightarrow \mathcal{A}_N\otimes\mathcal{A}'_N$. Thus:

Proposition 3.10. *The algebra map $\tilde{\imath}$ in* (3.21) *passes down to yield an injective homomorphism $\Bbbk$-superalgebras $i\colon \mathcal{A}\circ\mathcal{A}' \rightarrowtail \mathcal{A}\otimes\mathcal{A}'$ which doubles degree. The image of i is the super Segre product $\bigoplus_{n\geq 0}\mathcal{A}_n\otimes\mathcal{A}'_n$ of $\mathcal{A}$ and $\mathcal{A}'$.*

3.6. Internal $\underline{\mathrm{Hom}}$. The isomorphisms (1.3) and (1.4) together with associativity lead to a functorial isomorphism

$$\mathrm{Hom}_{\Bbbk}(U\otimes V, W^*) \cong \mathrm{Hom}_{\Bbbk}(U,(V\otimes W)^*)$$

in $\mathsf{Vect}^s_{\Bbbk}$. Explicitly, if $g\in\mathrm{Hom}_{\Bbbk}(U\otimes V,W^*)$ and $g'\in\mathrm{Hom}_{\Bbbk}(U,(V\otimes W)^*)$ correspond to each other under the above isomorphism then

$$\langle g(u\otimes v), w\rangle = \langle g'(u), v\otimes w\rangle \tag{3.22}$$

holds for all $u\in U$, $v\in V$ and $w\in W$.

In particular, by restricting to $\bar{0}$-components, we have a $\Bbbk$-linear isomorphism

$$\mathrm{Hom}_{\mathsf{Vect}^s_{\Bbbk}}(U\otimes V, W^*) \cong \mathrm{Hom}_{\mathsf{Vect}^s_{\Bbbk}}(U,(V\otimes W)^*) \tag{3.23}$$

This isomorphism leads to

Proposition 3.11. *There is a functorial isomorphism*

$$\mathrm{Hom}_{\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}}(\mathcal{A}\bullet\mathcal{B},\mathcal{C}) \cong \mathrm{Hom}_{\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}}(\mathcal{A},\mathcal{C}\circ\mathcal{B}^!)$$

Proof. We follow Manin [37, 4.2]. Let $\mathcal{A} = A(U,R)$, $\mathcal{B} = A(V,S)$ and $\mathcal{C} = A(W,T)$ be N-homogeneous superalgebras. We will prove the proposition in the following equivalent form; see (3.12) and (3.18):

$$\mathrm{Hom}_{\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}}(\mathcal{A}\bullet\mathcal{B},\mathcal{C}^!) \cong \mathrm{Hom}_{\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}}(\mathcal{A},(\mathcal{B}\bullet\mathcal{C})^!)$$

Recall that $\mathcal{C}^! = A(W^*,T^{\perp})$ and $(\mathcal{B}\bullet\mathcal{C})^! = A((V\otimes W)^*,(c_{\pi_N}(S\otimes T))^{\perp})$. Let $g\colon U\otimes V\to W^*$ be a morphism in $\mathsf{Vect}^s_{\Bbbk}$ and let $g'\colon U\to(V\otimes W)^*$ be the morphism in $\mathsf{Vect}^s_{\Bbbk}$ that corresponds to g under (3.23). We must show that, for homogeneous subspaces $R \subseteq U^{\otimes N}$, $S \subseteq V^{\otimes N}$ and $T \subseteq W^{\otimes N}$,

$$g^{\otimes N}\left(c_{\pi_N}(R \otimes S)\right) \subseteq T^{\perp} \quad \Leftrightarrow \quad g'^{\otimes N}\left(R\right)\left(c_{\pi_N}(S \otimes T)\right)^{\perp}$$

Identifying $T^{\perp\perp}$ with T as in §3.3, the first inclusion is equivalent to

$$\langle g^{\otimes N}\left(c_{\pi_N}(R \otimes S)\right), T\rangle = 0 \tag{3.24}$$

while the second inclusion states that

$$\langle g'^{\otimes N}(R), c_{\pi_N}(S \otimes T)\rangle = 0 \tag{3.25}$$

But (3.22) shows that (3.24) and (3.25) are equivalent, which proves the proposition. □

Proposition 3.11 says that the tensor category $(\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}, \bullet)$ has an internal $\underline{\mathrm{Hom}}$ which is given by

$$\underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}) = \mathcal{B} \circ \mathcal{A}^!$$

Explicitly, $\underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B})$ is an object of $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$ which represents the functor $(\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk})^{\mathrm{op}} \to \mathsf{Sets}$, $\mathcal{X} \mapsto \mathrm{Hom}_{\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}}(\mathcal{X} \bullet \mathcal{A}, \mathcal{B})$; so there is an isomorphism of functors

$$\mathrm{Hom}_{\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}}(? \bullet \mathcal{A}, \mathcal{B}) \cong \mathrm{Hom}_{\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}}(?, \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}))$$

By general properties of $\underline{\mathrm{Hom}}$ (see [12, Def. 1.6]), the morphism $\mathrm{Id}_{\underline{\mathrm{Hom}}(\mathcal{A},\mathcal{B})}$ corresponds to a morphism

$$\mu\colon\ \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}) \bullet \mathcal{A} \to \mathcal{B} \tag{3.26}$$

in $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$ satisfying the following universal property: for any morphism $f\colon \mathcal{X} \bullet \mathcal{A} \to \mathcal{B}$ in $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$ there exists a unique morphism $g\colon \mathcal{X} \to \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X} \bullet \mathcal{A} & & \\ {\scriptstyle g \bullet \mathrm{Id}_{\mathcal{A}}}\downarrow & \overset{f}{\searrow} & \\ \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}) \bullet \mathcal{A} & \xrightarrow[\mu]{} & \mathcal{B} \end{array}$$

In degree 1, the map μ is simply $\mathrm{Id}_V \otimes \mathrm{ev}_U\colon V \otimes U^* \otimes U \longrightarrow V \otimes \Bbbk = V$.

From $\underline{\mathrm{Hom}}(\mathcal{B}, \mathcal{C}) \bullet \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}) \bullet \mathcal{A} \overset{\mathrm{Id}\,\bullet\mu}{\longrightarrow} \underline{\mathrm{Hom}}(\mathcal{B}, \mathcal{C}) \bullet \mathcal{B} \overset{\mu}{\longrightarrow} \mathcal{C}$ one obtains in this way a composition morphism

$$m\colon\ \underline{\mathrm{Hom}}(\mathcal{B}, \mathcal{C}) \bullet \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B}) \to \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{C}) \tag{3.27}$$

in $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$. The morphisms μ and m satisfy the obvious associativity properties.

3.7. The superbialgebra $\underline{\mathrm{end}}\,\mathcal{A}$. Following Manin [37, 4.2] we define

$$\underline{\mathrm{hom}}(\mathcal{A}, \mathcal{B}) = \underline{\mathrm{Hom}}(\mathcal{A}^!, \mathcal{B}^!)^! = \mathcal{A}^! \bullet \mathcal{B}$$

for $\mathcal{A}$, $\mathcal{B}$ in $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$. Applying the dualization functor to (3.26), (3.27) and recalling (3.18), we obtain morphisms

$$\delta_\circ\colon \mathcal{A} \to \mathcal{B} \circ \underline{\mathrm{hom}}(\mathcal{B}, \mathcal{A})$$

$$\Delta_\circ\colon\ \underline{\mathrm{hom}}(\mathcal{A}, \mathcal{C}) \to \underline{\mathrm{hom}}(\mathcal{A}, \mathcal{B}) \circ \underline{\mathrm{hom}}(\mathcal{B}, \mathcal{C})$$

in $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$. The associativity properties of μ and m translate into corresponding coassociativity properties for $\delta_\circ$ and $\Delta_\circ$. Following $\delta_\circ$ and $\Delta_\circ$ by the algebra map i of Proposition 3.10, we obtain superalgebra maps

$$\delta\colon \mathcal{A} \to \mathcal{B} \otimes \underline{\mathrm{hom}}(\mathcal{B}, \mathcal{A}) \tag{3.28}$$

$$\Delta\colon\ \underline{\mathrm{hom}}(\mathcal{A}, \mathcal{C}) \to \underline{\mathrm{hom}}(\mathcal{A}, \mathcal{B}) \otimes \underline{\mathrm{hom}}(\mathcal{B}, \mathcal{C}) \tag{3.29}$$

Now take $\mathcal{A} = \mathcal{B} = \mathcal{C} = A(V, R)$ and put $\underline{\mathrm{end}}\,\mathcal{A} = \underline{\mathrm{hom}}(\mathcal{A}, \mathcal{A})$; so

$$\underline{\mathrm{end}}\,\mathcal{A} = \mathcal{A}^! \bullet \mathcal{A} = A(V^* \otimes V, c_{\pi_N}(R^\perp \otimes R)) \tag{3.30}$$

Then (3.29) yields a coassociative superalgebra map

$$\Delta\colon\ \underline{\mathrm{end}}\,\mathcal{A} \to \underline{\mathrm{end}}\,\mathcal{A} \otimes \underline{\mathrm{end}}\,\mathcal{A}$$

Moreover, by Proposition 3.11, the morphism $\mathcal{A}^! \xrightarrow{\mathrm{Id}} \mathcal{A}^! \cong \Bbbk[t] \circ \mathcal{A}^!$ corresponds to a morphism $\underline{\mathrm{end}}\,\mathcal{A} = \mathcal{A}^! \bullet \mathcal{A} \to \Bbbk[t]$ in $\mathsf{H}_N\mathsf{Alg}^s_{\Bbbk}$. Following this morphism by the map $t \mapsto 1$ we obtain a superalgebra map

$$\varepsilon\colon\ \underline{\mathrm{end}}\,\mathcal{A} \to \Bbbk$$

which in degree 1 is the usual evaluation pairing $\mathrm{ev}_V\colon V^* \otimes V \to \Bbbk$ in $\mathsf{Vect}^s_{\Bbbk}$. Finally, (3.28) provides us with a superalgebra map

$$\delta_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A} \otimes \underline{\mathrm{end}}\,\mathcal{A} \tag{3.31}$$

Note that $\delta_{\mathcal{A}}$ maps the degree n-component of $\mathcal{A}$ according to

$$\mathcal{A}_n \xrightarrow{\delta_\circ} (\mathcal{A} \circ \underline{\mathrm{end}}\,\mathcal{A})_n \xrightarrow{i} \mathcal{A}_n \otimes (\underline{\mathrm{end}}\,\mathcal{A})_n \hookrightarrow \mathcal{A}_n \otimes \underline{\mathrm{end}}\,\mathcal{A} \tag{3.32}$$

Fixing a graded $\Bbbk$-basis $x_1, \dots, x_d$ of V and denoting the dual basis of V^* by $x^1, \dots, x^d$ as before, $\underline{\mathrm{end}}\,\mathcal{A}$ has algebra generators

$$z^i_j := x^i \otimes x_j \tag{3.33}$$

of degree-1 and parity $\widehat{i} + \widehat{j}$. In terms of these generators, the maps ε, $\delta_{\mathcal{A}}$ and Δ are given by

$$\begin{aligned} \varepsilon(z^j_i) &= \delta^j_i & &\text{or} & \varepsilon(Z) &= 1 \\ \delta_{\mathcal{A}}(x_j) &= \sum_i x_i \otimes z^i_j & &\text{or} & \delta_{\mathcal{A}}(x_1, \dots, x_d) &= (x_1, \dots, x_d) \otimes Z \\ \Delta(z^i_j) &= \sum_k z^i_k \otimes z^k_j & &\text{or} & \Delta(Z) &= Z \otimes Z \end{aligned} \tag{3.34}$$

where $Z = (z^i_j)_{d \times d}$.

Proposition 3.12. *Let $\mathcal{A} = A(V, R)$ be an N-homogeneous $\Bbbk$-superalgebra.*

(a) *With Δ as comultiplication and ε as counit, the superalgebra $\underline{\mathrm{end}}\,\mathcal{A}$ becomes a superbialgebra. Moreover, $\delta_{\mathcal{A}}$ makes $\mathcal{A}$ into a graded right $\underline{\mathrm{end}}\,\mathcal{A}$-comodule superalgebra.*

(b) *Given any $\Bbbk$-superalgebra $\mathcal{B}$ and a morphism of superalgebras $\delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ satisfying $\delta(V) \subseteq V \otimes \mathcal{B}$, there is a unique morphism of superalgebras $\varphi\colon \underline{\operatorname{end}}\,\mathcal{A} \to \mathcal{B}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\delta} & \mathcal{A} \otimes \mathcal{B} \\ & {\scriptstyle \delta_{\mathcal{A}}} \searrow & \uparrow {\scriptstyle \mathrm{Id}_{\mathcal{A}} \otimes \varphi} \\ & & \mathcal{A} \otimes \underline{\operatorname{end}}\,\mathcal{A} \end{array}$$

The proposition is proved as in [37, §5] or [7, Theorem 3].

Example 3.13. When $\mathcal{A} = A(V, 0) = \mathsf{T}(V)$, we have $\underline{\operatorname{end}}\,\mathcal{A} = A(V^* \otimes V, 0) = \mathsf{T}(V^* \otimes V)$; so

$$\underline{\operatorname{end}}\,\mathsf{T}(V) = \mathsf{T}(V^* \otimes V)$$

the free superalgebra generated by the elements z^i_j in (3.33).

Example 3.14. By Examples 3.3 and 3.8, we have

$$\underline{\operatorname{end}}\,\mathsf{S}_N(V) = A\left(V^* \otimes V, c_{\pi_N}\left((1 - c_{Y_N})(V^{*\otimes N}) \otimes c_{Y_N}(V^{\otimes N})\right)\right)$$

For example, the algebra $\underline{\operatorname{end}}\,\mathsf{S}_2(V)$ is generated by the elements z^i_j with parity $\widehat{i} + \widehat{j}$ subject to the relations

$$[z^{i_1}_{j_1}, z^{i_2}_{j_2}] + (-1)^{\widehat{i_1}\widehat{i_2} + (\widehat{i_1} + \widehat{i_2})\widehat{j_1}}[z^{i_2}_{j_1}, z^{i_1}_{j_2}] = 0$$

where $[\,.\,,\,.\,]$ is the supercommutator (2.2). This algebra is highly noncommutative, even for a pure even space V.

Let $\mathcal{O}(\mathsf{E}(V)) = \mathsf{S}(V^* \otimes V)$ be the supercommutative superbialgebra as in §2.5.3, with generators x^i_j. There is a map of superbialgebras

$$\varphi\colon\ \underline{\operatorname{end}}\,\mathsf{S}_N(V) \to \mathcal{O}(\mathsf{E}(V))\,, \qquad z^i_j \mapsto x^i_j \tag{3.35}$$

Indeed, write $\mathcal{B} = \mathcal{O}(\mathsf{E}(V))$ for brevity and recall the coaction $\delta\colon V \to V \otimes \mathcal{B}$, $x_j \mapsto \sum_i x_i \otimes x^i_j$ from (2.11). Since $c_{Y_N} \in \operatorname{End}_{\mathsf{comod}^{\mathsf{s}}_{\mathcal{B}}}(V^{\otimes N})$ (see §2.6.2), the map δ extends to a map of superalgebras

$$\delta\colon \mathsf{S}_N(V) \to \mathsf{S}_N(V) \otimes \mathcal{B}$$

Therefore, Proposition 3.12(b) yields the desired φ. Note that the coaction of $\underline{\operatorname{end}}\,\mathsf{S}_N(V)$ on V, when restricted along φ, becomes the canonical coaction of $\mathcal{O}(\mathsf{E}(V))$ on V; see (2.11) and (3.34).

4. N-KOSZUL SUPERALGEBRAS

Throughout this section, we fix an N-homogeneous superalgebra $\mathcal{A} = A(V, R)$.

4.1. The graded dual $\mathcal{A}^{!*}$. The graded dual

$$\mathcal{A}^{!*} = \bigoplus_n \mathcal{A}_n^{!\,*}$$

of $\mathcal{A}^!$ has a natural structure of a graded right $\underline{\mathrm{end}}\,\mathcal{A}$-comodule. Indeed, the linear dual $\mathcal{A}_n^{!\,*}$ of the degree n-component of $\mathcal{A}^!$ embeds into $V^{\otimes n}$ as follows. Recall from (3.11) that

$$\mathcal{A}_n^{!\,*} = \begin{cases} V^{\otimes n} & \text{if } n < N \\ \bigcap_{i+j+N=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} & \text{if } n \geq N \end{cases} \tag{4.1}$$

This identification makes the graded dual $\mathcal{A}^{!*}$ into a graded right $\underline{\mathrm{end}}\,\mathcal{A}$-comodule. For, by (3.32) the coaction $\delta_{\mathcal{A}}$ restricts in degree 1 to a map $V \to V \otimes \underline{\mathrm{end}}\,\mathcal{A}$ which makes $\mathsf{T}(V)$ into a graded right $\underline{\mathrm{end}}\,\mathcal{A}$-comodule superalgebra. The structure map $\mathsf{T}(V) \to \mathsf{T}(V) \otimes \underline{\mathrm{end}}\,\mathcal{A}$ sends $R \to R \otimes \underline{\mathrm{end}}\,\mathcal{A}$. Therefore, each $V^{\otimes i} \otimes R \otimes V^{\otimes j}$ is a $\underline{\mathrm{end}}\,\mathcal{A}$-subcomodule of $V^{\otimes(i+j+N)}$, and hence $\mathcal{A}_n^{!\,*}$ is a $\underline{\mathrm{end}}\,\mathcal{A}$-subcomodule of $V^{\otimes n}$. Finally, for all $n \geq 0$,

$$\mathcal{A}_{n+1}^{!\,*} \subseteq V \otimes \mathcal{A}_n^{!\,*} \quad \text{and} \quad \mathcal{A}_{n+N}^{!\,*} \subseteq V^{\otimes N} \otimes \mathcal{A}_n^{!\,*} \cap R \otimes V^{\otimes n} = R \otimes \mathcal{A}_n^{!\,*} \tag{4.2}$$

4.2. The Koszul complex. The map

$$\mathcal{A} \otimes V^{\otimes(i+1)} \to \mathcal{A} \otimes V^{\otimes i}$$
$$a \otimes (v_1 \otimes \cdots \otimes v_{i+1}) \mapsto av_1 \otimes (v_2 \otimes \cdots \otimes v_{i+1})$$

is a morphism in the category $\mathsf{Comod}^s_{\underline{\mathrm{end}}\,\mathcal{A}}$ of right $\underline{\mathrm{end}}\,\mathcal{A}$-comodules, because the $\underline{\mathrm{end}}\,\mathcal{A}$-coaction $\delta_{\mathcal{A}}$ in (3.31) is a superalgebra map. Furthermore, this map is a left $\mathcal{A}$-module map which preserves total degree, and it restricts to a map of $\underline{\mathrm{end}}\,\mathcal{A}$-subcomodules

$$d\colon \mathcal{A} \otimes \mathcal{A}_{i+1}^{!\,*} \to \mathcal{A}V \otimes \mathcal{A}_i^{!\,*} \hookrightarrow \mathcal{A} \otimes \mathcal{A}_i^{!\,*}$$

which is the $\mathcal{A}$-linear extension of the embedding (4.2). The map d^N sends $\mathcal{A}_{i+N}^{!\,*}$ to $\mathcal{A}R \otimes \mathcal{A}_i^{!\,*} = 0$; so $d^N = 0$. In other words, we have an N*-complex*

$$K(\mathcal{A})\colon \ \ldots \xrightarrow{d} \mathcal{A} \otimes \mathcal{A}_{i+1}^{!\,*} \xrightarrow{d} \mathcal{A} \otimes \mathcal{A}_i^{!\,*} \xrightarrow{d} \ldots \xrightarrow{d} \mathcal{A} \longrightarrow 0 \tag{4.3}$$

in $\mathsf{Comod}^s_{\underline{\mathrm{end}}\,\mathcal{A}}$ consisting of graded-free left $\mathcal{A}$-modules and $\mathcal{A}$-module maps which preserve total degree. Therefore, $K(\mathcal{A})$ splits into a direct sum of N-complexes $K(\mathcal{A})^n = \bigoplus_{i+j=n} \mathcal{A}_i \otimes \mathcal{A}_j^{!\,*}$ in $\mathsf{comod}^s_{\underline{\mathrm{end}}\,\mathcal{A}}$.

Following [7], the *Koszul complex* $\mathsf{K}(\mathcal{A})$ defined by Berger in [5] can be described as the following contraction of $K(\mathcal{A})$:

$$\mathsf{K}(\mathcal{A})\colon \ \ldots \xrightarrow{d^{N-1}} \mathcal{A} \otimes \mathcal{A}_{N+1}^{!\,*} \xrightarrow{d} \mathcal{A} \otimes \mathcal{A}_N^{!\,*} \xrightarrow{d^{N-1}} \mathcal{A} \otimes \mathcal{A}_1^{!\,*} \xrightarrow{d} \mathcal{A} \longrightarrow 0 \tag{4.4}$$

This is an ordinary complex in $\mathsf{Comod}^s_{\underline{\mathrm{end}}\,\mathcal{A}}$ which splits into a direct sum of complexes $\mathsf{K}(\mathcal{A})^n$ in $\mathsf{comod}^s_{\underline{\mathrm{end}}\,\mathcal{A}}$. The i^{th} components of $\mathsf{K}(\mathcal{A})$ and of $\mathsf{K}(\mathcal{A})^n$ are given by

$$\mathsf{K}(\mathcal{A})_i = \mathcal{A} \otimes \mathcal{A}_{\nu(i)}^{!\,*} \quad \text{and} \quad \mathsf{K}(\mathcal{A})_i^n = \mathcal{A}_{n-\nu(i)} \otimes \mathcal{A}_{\nu(i)}^{!\,*}$$

with $\nu(i) = \nu_N(i)$ as in (0.1). The differential on $\mathsf{K}(\mathcal{A})$ is

$$\delta_i\colon \mathsf{K}(\mathcal{A})_i \to \mathsf{K}(\mathcal{A})_{i-1} \quad \text{where} \quad \delta_i = \begin{cases} d^{N-1} & \text{for } i \text{ even} \\ d & \text{for } i \text{ odd} \end{cases}$$

Writing $\mathcal{A}_+ = \bigoplus_{n>0} \mathcal{A}_n = \mathcal{A}V$ as usual, we have

$$\operatorname{Ker} \delta_i \subseteq \mathcal{A}_+ \mathsf{K}(\mathcal{A})_i$$

for all i. Indeed, this is clear for odd i, since $\delta_i = d$ is injective on $\mathcal{A}^{!\,*}_{\nu(i)}$. For even i, the restriction of $\delta_i = d^{N-1}$ to $\mathcal{A}^{!\,*}_{\nu(i)}$ is given by $d^{N-1}\colon \mathcal{A}^{!\,*}_{\nu(i)} = \mathcal{A}^{!\,*}_{\nu(i-1)+N-1} \hookrightarrow V^{\otimes(N-1)} \otimes \mathcal{A}^{!\,*}_{\nu(i-1)} \xrightarrow{\sim} \mathcal{A}_{N-1} \otimes \mathcal{A}^{!\,*}_{\nu(i-1)} \hookrightarrow \mathcal{A} \otimes \mathcal{A}^{!\,*}_{\nu(i-1)}$ where the first embedding comes from (4.2).

Since $\mathcal{A}^{!\,*}_{\nu(1)} = \mathcal{A}^{!\,*}_1 = V$ and $\mathcal{A}^{!\,*}_{\nu(2)} = \mathcal{A}^{!\,*}_N = R$ by (4.1), the start of the Koszul complex, augmented by the canonical map

$$\mathcal{A} \twoheadrightarrow \Bbbk = \mathcal{A}/\mathcal{A}_+$$

is as follows:

$$\mathcal{A} \otimes R \xrightarrow{\delta_2} \mathcal{A} \otimes V \xrightarrow{\delta_1 = \text{mult}} \mathcal{A} \longrightarrow \Bbbk \longrightarrow 0 \tag{4.5}$$

This piece is easily seen to be exact: writing $\mathcal{A} = \mathsf{T}(V)/I$ with $I = (R) = I \otimes V + \mathsf{T}(V) \otimes R$ as in (0.2), the map $\mathsf{T}(V)_+ = \mathsf{T}(V) \otimes V \twoheadrightarrow \mathcal{A} \otimes V \overset{\delta_1}{\twoheadrightarrow} \mathcal{A}_+$ has kernel I. Thus, $\operatorname{Ker} \delta_1 = I/I \otimes V = \operatorname{Im} \delta_2$. Hence (4.5) is the start of the minimal graded-free resolution of the left $\mathcal{A}$-module $\Bbbk$.

4.3. N-homogeneous Koszul superalgebras. Recall from the Introduction that an N-homogeneous superalgebra $\mathcal{A}$ is called *N-Koszul* if $\operatorname{Tor}^{\mathcal{A}}_i(\Bbbk, \Bbbk)$ is concentrated in degree $\nu_N(i)$ for all $i \geq 0$. By [5, Proposition 2.12] or [8, Theorem 2.4], this happens exactly if the Koszul complex $\mathsf{K}(\mathcal{A})$ is exact in degrees $i > 0$ and in view of (4.5), this amounts to exactness of $\mathsf{K}(\mathcal{A})$ in degrees $i \geq 2$. In this case,

$$\mathsf{K}(\mathcal{A}) \longrightarrow \Bbbk \longrightarrow 0$$

is the minimal graded-free resolution of the trivial left $\mathcal{A}$-module $\Bbbk$.

The Yoneda Ext-algebra $E(\mathcal{A}) = \bigoplus_{i \geq 0} \operatorname{Ext}^i_{\mathcal{A}}(\Bbbk, \Bbbk)$ of an N-Koszul superalgebra $\mathcal{A}$ has the following description in terms of the dual algebra $A^!$:

$$\operatorname{Ext}^i_{\mathcal{A}}(\Bbbk, \Bbbk) \cong \mathcal{A}^!_{\nu(i)} \qquad (i \geq 0)$$

Moreover, identifying $\operatorname{Ext}^i_{\mathcal{A}}(\Bbbk, \Bbbk)$ and $\mathcal{A}^!_{\nu(i)}$, the Yoneda product $f \cdot g$ and the $\mathcal{A}^!$-product fg for $f \in \operatorname{Ext}^i_{\mathcal{A}}(\Bbbk, \Bbbk) = \mathcal{A}^!_{\nu(i)}$ and $g \in \operatorname{Ext}^j_{\mathcal{A}}(\Bbbk, \Bbbk) = \mathcal{A}^!_{\nu(j)}$ are related by $f \cdot g = (-1)^{ij} fg$ when $N = 2$, and

$$f \cdot g = \begin{cases} fg & \text{if } i \text{ or } j \text{ is even} \\ 0 & \text{if } i \text{ and } j \text{ are both odd} \end{cases}$$

for $N > 2$; see [21, Theorem 9.1], [8, Proposition 3.1].

Example 4.1. Quadratic algebras having a PBW-basis are 2-Koszul; see, e.g., [41, Chap. 4, Theorem 3.1]. This applies in particular to quantum superspace $\mathcal{A} = A^{\mathbf{f}}_{\mathbf{q}}$; see Example 3.1. A PBW-basis in this case is given by the collection of ordered monomials $x_1^{m_1} x_2^{m_2} \dots x_d^{m_d}$ with $m_i \geq 0$ for all i and $m_i \leq 1$ if $\widehat{i} = \bar{1}$, as in Example 3.1. For a more general result, see [41, Chap. 4, Theorem 8.1].

Example 4.2. The unmixed Yang-Mills algebras $\mathcal{A} = \mathcal{YM}^{\pm}$ (see Example 3.2) were shown to be 3-Koszul in [10], [11]. Indeed, letting $\mathcal{A}[\ell]$ denotes the shift of $\mathcal{A}$ that is defined by $\mathcal{A}[\ell]_n = \mathcal{A}_{\ell+n}$, the defining relations for $\mathcal{A}$ in the form (3.7) imply that the following complex of graded-free left $\mathcal{A}$-modules is exact:

$$0 \longrightarrow \mathcal{A}[-4] \xrightarrow{\cdot Y} \mathcal{A}[-3]^d \xrightarrow{\cdot M} \mathcal{A}[-1]^d \xrightarrow{\cdot Y^{\mathrm{tr}}} \mathcal{A} \longrightarrow \Bbbk \longrightarrow 0 \tag{4.6}$$

The piece $\mathcal{A}[-3]^d \xrightarrow{\cdot M} \mathcal{A}[-1]^d \xrightarrow{\cdot Y^{\mathrm{tr}}} \mathcal{A} \longrightarrow \Bbbk \longrightarrow 0$ is identical with (4.5). Therefore, (4.6) is the minimal graded-free resolution of $\Bbbk$. The resolution shows that each $\mathrm{Tor}_i^{\mathcal{A}}(\Bbbk, \Bbbk)$ is concentrated in degree $\nu_3(i)$, and hence $\mathcal{A}$ is 3-Koszul. It also follows that (4.6) is isomorphic to $\mathsf{K}(\mathcal{A}) \to \Bbbk \to 0$. In particular, (4.6) confirms the dimensions of the corresponding components $\mathcal{A}_n^!$ in Example 3.7. As has been pointed out in [10], [11], it follows from (4.6) that the Hilbert series $H_{\mathcal{A}}(t) = \sum_{n \geq 0} \dim_{\Bbbk} \mathcal{A}_n \, t^n$ of $\mathcal{A} = \mathcal{YM}^{\pm}$ has the form

$$\begin{aligned} H_{\mathcal{A}}(t) &= \frac{1}{1 - dt + dt^3 - t^4} \\ &= \frac{1}{(1-t^2)(1-dt+t^2)} \end{aligned}$$

If $d > 2$ then the series has a pole in the interval $(0, 1)$, and hence $\dim_{\Bbbk} \mathcal{A}_n$ grows exponentially with n. Therefore, $\mathcal{A}$ is not noetherian in this case; see Stephenson and Zhang [42].

The mixed Yang-Mills algebras $\mathcal{A} = \mathcal{YM}^{p|q}$ with $p \neq 0$ and $q \neq 0$, on the other hand, are never 3-Koszul. For $\mathcal{YM}^{1|1}$ this follows from the description given in Example 3.2: this algebra has infinite global dimension. In general, one can check that the so-called extra condition (see (4.10) below) fails for $\mathcal{A}$, and so $\mathcal{A}$ cannot be Koszul, by [5, Prop. 2.7].

Example 4.3. It has been shown in [5, Theorem 3.13] that the N-symmetric algebra $\mathsf{S}_N(V)$ of a pure even space V over a field of characteristic 0 is N-Koszul. An extension of this result will be offered in Theorem 4.5 below.

4.4. Confluence and Koszulity. For the convenience of the reader, we recall the notions of reduction operators and confluence and their relation to the Koszul property. Complete details can be found in Berger [4], [5].

Let V in $\mathsf{Vect}_{\Bbbk}^s$ be given along with a graded basis $X = \{x_1, \dots x_d\}$ that is ordered by $x_1 > x_2 > \cdots > x_d$. The tensors ("monomials") $x_{\mathbf{i}} = x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_N}$ for $\mathbf{i} = (i_1, i_2, \dots, i_N) \in \{1, 2, \dots, d\}^N$ form a basis of $V^{\otimes N}$ which will be given the lexicographical ordering. An *X-reduction operator* on $V^{\otimes N}$ is a projection $S \in \mathrm{End}_{\mathsf{Vect}_{\Bbbk}^s}(V^{\otimes N})$ such that either $S(x_{\mathbf{i}}) = x_{\mathbf{i}}$ or $S(x_{\mathbf{i}}) < x_{\mathbf{i}}$ holds for each $\mathbf{i}$, where the latter inequality means that $S(x_{\mathbf{i}})$ is a linear combination (possibly 0) of monomials $< x_{\mathbf{i}}$. The monomials $x_{\mathbf{i}}$ satisfying $S(x_{\mathbf{i}}) = x_{\mathbf{i}}$ are called *S-reduced*, all other monomials are *S-nonreduced*. We denote by $\mathrm{Red}(S)$ and $\mathrm{NRed}(S)$) the (super) subspaces of $V^{\otimes N}$ that are generated by the S-reduced monomials and the S-nonreduced monomials, respectively; so $V^{\otimes N} = \mathrm{Red}(S) \oplus \mathrm{NRed}(S)$ and $\mathrm{Im}(S) = \mathrm{Red}(S)$.

Let $\mathcal{L}_X(V^{\otimes N})$ denote the collection of all X-reduction operators on $V^{\otimes N}$. The proof of [4, Theorem 2.3] shows that the application $S \mapsto \mathrm{Ker}(S)$ is a bijection between $\mathcal{L}_X(V^{\otimes N})$ and the set of all super subspaces of $V^{\otimes N}$. Hence $\mathcal{L}_X(V^{\otimes N})$ inherits a lattice structure: for $S, S' \in \mathcal{L}_X(V^{\otimes N})$ one has X-reduction operators $S \wedge S'$ and $S \vee S'$ on $V^{\otimes N}$ which are defined by

$$\begin{aligned}\operatorname{Ker}(S \wedge S') &= \operatorname{Ker}(S) + \operatorname{Ker}(S')\\ \operatorname{Ker}(S \vee S') &= \operatorname{Ker}(S) \cap \operatorname{Ker}(S')\end{aligned}$$

A pair (S, S') of X-reduction operators is said to be *confluent* if

$$\operatorname{Red}(S \vee S') = \operatorname{Red}(S) + \operatorname{Red}(S')$$

Since the inclusion $\supseteq$ is always true, confluence of (S, S') is equivalent to the inequality

$$\dim_{\Bbbk} \operatorname{Im}(S \vee S') \leq \dim_{\Bbbk}(\operatorname{Im}(S) + \operatorname{Im}(S')) \tag{4.7}$$

Let $n \geq N$. Any X-reduction operator S on $V^{\otimes N}$ gives rise to X-reduction operators $S_{n,i}$ on $V^{\otimes n}$ which are defined by

$$S_{n,i} := \operatorname{Id}_{V^{\otimes i}} \otimes S \otimes \operatorname{Id}_{V^{\otimes j}} \qquad (i + j + N = n)$$

A monomial $x_{\mathbf{i}} = x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n}$ of length $n \geq N$ is said to be S-reduced if $x_{\mathbf{i}}$ is $S_{n,i}$-reduced for all i, that is, if every connected submonomial of $x_{\mathbf{i}}$ of length N is S-reduced.

Now let $\mathcal{A} = A(V, R)$ be an N-homogeneous superalgebra, and let S be the X-reduction operator on $V^{\otimes N}$ such that $\operatorname{Ker}(S) = R$. The algebra $\mathcal{A}$ is said to be X-*confluent* if the pairs $(S_{N+i,i}, S_{N+i,0})$ of X-reduction operators on $V^{\otimes N+i}$ are confluent for $i = 1, \ldots, N-1$. By (4.7) this amounts to the inequalities

$$\dim_{\Bbbk} \operatorname{Im}(S_{N+i,i} \vee S_{N+i,0}) \leq \dim_{\Bbbk} \left(\operatorname{Im}(S_{N+i,i}) + \operatorname{Im}(S_{N+i,0})\right) \tag{4.8}$$

being satisfied for $i = 1, \ldots, N-1$.

Following Berger [5], we denote by $\mathscr{T}_n$ the lattice of super subspaces of $V^{\otimes n}$ that is generated by the subspaces

$$R_{n,i} := V^{\otimes i} \otimes R \otimes V^{\otimes j} = \operatorname{Ker}(S_{n,i}) \qquad (i + j + N = n) \tag{4.9}$$

The superalgebra $\mathcal{A}$ is said to be *distributive* if the lattices $\mathscr{T}_n$ are distributive for all n, that is, $C \cap (D + E) = (C \cap D) + (C \cap E)$ holds for all $C, D, E \in \mathscr{T}_n$.

The following proposition states the operative facts concerning Koszulity for our purposes. Part (a) is identical with [5, Thm. 3.11] while (b) is [5, Prop. 3.4].

Proposition 4.4. *Let $\mathcal{A} = A(V, R)$ be an N-homogeneous superalgebra.*

(a) *If $\mathcal{A}$ is X-confluent for some totally ordered graded basis X of V then $\mathcal{A}$ is distributive. Moreover, letting S denote the X-reduction operator on $V^{\otimes N}$ such that* $\operatorname{Ker}(S) = R$, *the classes in $\mathcal{A}$ of the S-reduced monomials $x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n}$ with $x_{i_j} \in X$ form a $\Bbbk$-basis of $\mathcal{A}_n$ for all $n \geq N$.*

(b) *Assume that $\mathcal{A}$ is distributive and the following "extra condition" is satisfied*

$$R_{n+N,0} \cap R_{n+N,n} \subseteq R_{n+N,n-1} \qquad (2 \leq n \leq N-1) \tag{4.10}$$

Then $\mathcal{A}$ is N-Koszul.

After these preparations, we are now ready to prove the following result. The quadratic case $N = 2$ is due to Gurevich [22]; see also Wambst [44].

Theorem 4.5. *Let $N \geq 2$ and $0 \neq q \in \Bbbk$ and assume that $[n]_q \neq 0$ for all $n \geq 1$. Then, for every Hecke operator $\mathscr{R}$ associated with q, the N-homogeneous superalgebra $\Lambda_{\mathscr{R},N}$ defined in* (3.8) *is N-Koszul.*

Proof. Put $\mathcal{A} = \Lambda_{\mathscr{R},N}$ and recall that $\mathcal{A} = A(V, R)$ with

$$R = \operatorname{Im} \rho_{N,\mathscr{R}}(X_N) \subseteq V^{\otimes N}$$

The extra condition (4.10) is a consequence of equation (1.19). Indeed, (1.19) implies that the spaces $R_{n,i}$ in (4.9) have the form

$$R_{n,i} = \bigcap_{s=i+1}^{i+N-1} \operatorname{Im}(\rho_{n,\mathscr{R}}(T_s) + 1) \subseteq V^{\otimes n} \tag{4.11}$$

Applying (4.11) with $\rho = \rho_{n+N,\mathscr{R}}$ we see that the left hand side of (4.10) is identical to

$$\bigcap_{i=1}^{N-1} \operatorname{Im}(\rho(T_i) + 1) \cap \bigcap_{i=n+1}^{n+N-1} \operatorname{Im}(\rho(T_i) + 1) = \bigcap_{i=1}^{n+N-1} \operatorname{Im}(\rho(T_i) + 1)$$

where the equality holds because $n + 1 \leq N$. The last expression is clearly contained in $\bigcap_{i=n}^{n+N-2} \operatorname{Im}(\rho(T_i) + 1)$, which is identical to the right hand side of (4.10). This establishes the extra condition (4.10).

In order to prove the distributivity of $\mathcal{A}$, we follow the approach taken in [25]. We first prove the claim for the standard solution $\mathscr{R}^{DJ}$, i.e., the operator given in Example 1.2 with $d = p$ and $q = 0$. As above, fix a basis $X = \{x_1, \dots, x_d\}$ of V, ordered by $x_1 > x_2 > \cdots > x_d$, and consider the basis of $V^{\otimes n}$ consisting of the monomials $x_{\mathbf{i}} = x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n}$ for $\mathbf{i} = (i_1, i_2, \dots, i_n) \in \{1, 2, \dots, d\}^n$ with the lexicographical ordering. By equation (1.21), the action of the generators T_j of the Hecke algebra $\mathscr{H} = \mathscr{H}_{n,q^2}$ on this basis is given by

$$T_j(x_{\mathbf{i}}) = \begin{cases} q^2 x_{\mathbf{i}} & \text{if } i_j = i_{j+1} \\ (q^2 - 1)x_{\mathbf{i}} + q x_{\sigma_j(\mathbf{i})} & \text{if } i_j < i_{j+1} \\ q x_{\sigma_j(\mathbf{i})} & \text{if } i_j > i_{j+1} \end{cases} \tag{4.12}$$

Here, $\sigma_j = (j, j + 1) \in \mathfrak{S}_n$ and $\sigma(\mathbf{i}) = (i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2)}, \dots, i_{\sigma^{-1}(n)})$ for $\sigma \in \mathfrak{S}_n$, as in Example 3.3.

We claim that the $\mathscr{H}$-submodule of $V^{\otimes n}$ that is generated by $x_{\mathbf{i}}$ is given by

$$\mathscr{H}(x_{\mathbf{i}}) = \bigoplus_{\mathbf{i}' \in \mathfrak{S}_n(\mathbf{i})} \Bbbk x_{\mathbf{i}'} \tag{4.13}$$

where $\mathfrak{S}_n(\mathbf{i})$ is the $\mathfrak{S}_n$-orbit of $\mathbf{i}$. Indeed, (4.12) implies that each $T_\sigma(x_{\mathbf{i}})$ with $\sigma \in \mathfrak{S}_n$ is a linear combination of basis vectors $x_{\mathbf{i}'}$ with $\mathbf{i}' \in \mathfrak{S}_n(\mathbf{i})$. Hence, $\subseteq$ certainly holds in (4.13). For the reverse inclusion, let $\mathbf{i}^*$ denote the unique non-decreasing sequence in $\mathfrak{S}_n(\mathbf{i})$; so $x_{\mathbf{i}^*} = \max\{x_{\mathbf{i}'} \mid \mathbf{i}' \in \mathfrak{S}_n(\mathbf{i})\}$. The last formula in (4.12) implies that

$$T(x_{\mathbf{i}}) = q^{r(\mathbf{i})} x_{\mathbf{i}^*} \tag{4.14}$$

where T is a suitable finite product of length $r(\mathbf{i}) \geq 0$ in the generators T_j. Since T is a unit in $\mathscr{H}$, the inclusion $\supseteq$ holds in (4.13), thereby proving the asserted equality.

Furthermore, (4.14) and (1.18) (with q replaced by q^2) give

$$q^{r(\mathbf{i})} X_n(x_{\mathbf{i}}) = X_n(x_{\mathbf{i}^*}). \tag{4.15}$$

These elements are nonzero. For, (4.15) implies that the elements $X_n(x_{\mathbf{i}^*})$ span the image of X_n on $V^{\otimes n}$, and their number is $\binom{d+n-1}{n}$ which is equal to the rank of X_n (cf. [25, Eq. (5)]). It follows that $X_n(V^{\otimes n}) = \operatorname{Im}\rho_{n,\mathscr{R}^{DJ}}(X_n)$ has a $\Bbbk$-basis consisting of the elements

$$\left\{X_n(x_{\mathbf{i}^*}) \,\middle|\, \mathbf{i}^* = (i_1 \le i_2 \le \cdots \le i_n) \in \{1, 2, \dots, d\}^n\right\}$$

Next, writing

$$X_n(x_{\mathbf{i}}) = \sum_{\mathbf{i}' \in \mathfrak{S}_n(\mathbf{i})} \lambda_{\mathbf{i}'} x_{\mathbf{i}'} \tag{4.16}$$

with $\lambda_{\mathbf{i}'} \in \Bbbk$, we claim that

$$\lambda_{\sigma_j(\mathbf{i}')} = \begin{cases} \lambda_{\mathbf{i}'} & \text{if } \mathbf{i}' = \sigma_j(\mathbf{i}') \\ q^{\pm 1}\lambda_{\mathbf{i}'} & \text{otherwise} \end{cases}$$

To prove this, we may assume that $\mathbf{i}' \neq \sigma_j(\mathbf{i}')$. We compute the coefficient of $x_{\sigma_j(\mathbf{i}')}$ in $T_j X_n(x_{\mathbf{i}})$ in two ways: by (1.18) this coefficient is equal to $q^2\lambda_{\sigma_j(\mathbf{i}')}$ while (4.12) yields the expression $q\lambda_{\mathbf{i}'} + (q^2 - q^{1\pm 1})\lambda_{\sigma_j(\mathbf{i}')}$. The claim follows from this. Writing an arbitrary $\sigma \in \mathfrak{S}_n$ as a product of the inversions σ_j, we see that the coefficients $\lambda_{\mathbf{i}'}$ in (4.16) only differ by a nonzero scalar, and hence they are all nonzero since $X_n(x_{\mathbf{i}}) \neq 0$.

By Proposition 4.4, it suffices to check the X-confluence conditions (4.8) $i = 1, \dots, N-1$. So let S be the X-reduction operator on $V^{\otimes N}$ with $\operatorname{Ker}(S) = R$. It is easy to see from the discussion above (with $n = N$) that S is given by $S(x_{\mathbf{i}^*}) = (1 - X_N/\lambda_{\mathbf{i}^*})(x_{\mathbf{i}^*})$ and $S(x_{\mathbf{i}}) = x_{\mathbf{i}}$ for $\mathbf{i} \neq \mathbf{i}^*$. According to (4.11) and the discussion above, the dimension of $(R \otimes V^{\otimes i}) \cap (V^{\otimes i} \otimes R)$ is $\binom{d+N+i-1}{N+i}$. Thus, the dimension of the left hand side of (4.8) is $d^{N+i} - \binom{d+N+i-1}{N+i}$. On the other hand the monomials in $V^{\otimes N+i}$ that belong to $\operatorname{NRed}(S_{N+i,i}) \cap \operatorname{NRed}(S_{N+i,0})$ are exactly those of the form $x_{\mathbf{i}^*}$ with $\mathbf{i}^* \in \{1, \dots, d\}^{N+i}$ non-decreasing. Their number is precisely $\binom{d+N+i-1}{N+i}$. Therefore, the dimension of $\operatorname{Im}(S_{N+i,i}) + \operatorname{Im}(S_{N+i,0}) = \operatorname{Red}(S_{N+i,i}) + \operatorname{Red}(S_{N+i,0})$ is at least $d^{N+i} - \binom{d+N+i-1}{N+i}$. This proves the inequality in (4.8), thereby finishing the proof of the theorem for the case $\mathscr{R} = \mathscr{R}^{DJ}$.

In order to deal with an arbitrary Hecke operator $\mathscr{R}$, recall that $\mathscr{H}_{n,q}$ is split semisimple, having a representative set of simple modules M_λ indexed by the partitions $\lambda \vdash n$; see (1.15). We denote the representation of $\mathscr{H}_{n,q}$ on M_λ by ρ_λ; it does not depend on the operator $\mathscr{R}$ but only on the partition λ.

Let us fix a decomposition

$$V^{\otimes n} = \bigoplus_{t \in T} M_t$$

into simple $\mathscr{H}_{n,q}$-submodules M_t. Since all M_t are invariant under the operators $\rho_{n,\mathscr{R}}(T_j)$, formula (4.11) yields the decomposition

$$R_{n,i} = \bigoplus_{t \in T} \bigcap_{s=i+1}^{i+N-1} (\rho_{n,\mathscr{R}}(T_s) + 1)(M_t) = \bigoplus_{t \in T} R_{n,i} \cap M_t$$

for all i. Therefore, by [25, Lemma 1.2], distributivity of the lattice $\mathscr{T}_n$ that is generated by the subspaces $R_{n,i}$ of $V^{\otimes n}$ is equivalent to distributivity of the lattices $\mathscr{T}_n \cap M_t$ $(t \in T)$ that are generated by the subspaces

$$R_{n,i} \cap M_t = \bigcap_{s=i+1}^{i+N-1} (\rho_{n,\mathscr{R}}(T_s) + 1)(M_t)$$

of M_t. Now, each M_t is isomorphic to M_λ for some $\lambda \vdash n$. Therefore, the lattice $\mathscr{T}_n \cap M_t$ is isomorphic to the lattice of subspaces of M_λ that is generated by the subspaces

$$\bigcap_{s=i+1}^{i+N-1} (\rho_\lambda(T_s) + 1)(M_\lambda)$$

with $i + N \leq n$. Finally, when $d = \dim V > n$, then all simple $\mathscr{H}_{n,q}$-modules M_λ appear in $V^{\otimes n}$; see [15, Proposition 5.1]. Thus, the distributivity of the lattice associated to $\mathscr{R}^{DJ}$, which we have already verified, implies the distributivity of the corresponding lattice for any Hecke operator $\mathscr{R}$. This completes the proof. □

5. Koszul duality and master theorem

In this section, $\mathcal{A} = A(V, R)$ denotes an N-homogeneous superalgebra that is assumed to be N-Koszul ($N \geq 2$).

5.1. By Koszulity, the complexes

$$\mathsf{K}(\mathcal{A})^n\colon \ \ldots \to \mathcal{A}_{n-\nu_N(i)} \otimes \mathcal{A}^{!\,*}_{\nu_N(i)} \to \mathcal{A}_{n-\nu_N(i-1)} \otimes \mathcal{A}^{!\,*}_{\nu_N(i-1)} \to \ldots \to \mathcal{A}_n \to 0$$

are exact for $n > 0$. This yields equations in the Grothendieck ring $R_{\underline{\text{end}}\,\mathcal{A}}$ of the category $\mathsf{comod}^s_{\underline{\text{end}}\,\mathcal{A}}$:

$$\sum_{i \geq 0} (-1)^i [\mathcal{A}_{n-\nu_N(i)}][\mathcal{A}^{!\,*}_{\nu_N(i)}] = 0 \qquad (n > 0) \tag{5.1}$$

In the power series ring $R_{\underline{\text{end}}\,\mathcal{A}}[\![t]\!]$ over the Grothendieck ring $\mathrm{R}_{\underline{\text{end}}\,\mathcal{A}}$, define the Poincaré series

$$P_{\mathcal{A}}(t) = \sum_{n \geq 0} [\mathcal{A}_n] t^n \qquad \text{and} \qquad P_{\mathcal{A}^{!*}}(t) = \sum_{n \geq 0} [A^{!\,*}_n] t^n$$

For any power series $P(t) = \sum_n a_n t^n$, we use the notation

$$P_N(t) := \sum_{n \equiv 0,1 \bmod N} (-1)^{\alpha_N(n)} a_n t^n$$

where $\alpha_N(n) = n - (n \bmod N)$ denotes the largest multiple of N less than or equal to n. Thus, $P_2(t) = P(t)$ and in general

$$\begin{aligned} P_N(-t) &= \sum_{n \equiv 0,1 \bmod N} (-1)^{n \bmod N} a_n t^n \\ &= \sum_{i \geq 0} (-1)^i a_{\nu_N(i)} t^{\nu_N(i)} \end{aligned} \tag{5.2}$$

In particular,

$$P_{\mathcal{A}^{!*},N}(-t) = \sum_{i \geq 0} (-1)^i [\mathcal{A}^{!\,*}_{\nu_N(i)}] t^{\nu_N(i)}$$

Equations (5.1) are equivalent to the following Koszul duality formula:

Proposition 5.1. *For any N-homogeneous Koszul superalgebra $\mathcal{A}$, the identity*

$$P_{\mathcal{A}}(t)P_{\mathcal{A}^{!*},N}(-t) = 1$$

holds in $R_{\underline{\mathrm{end}}\,\mathcal{A}}[\![t]\!]$.

Applying the ring homomorphism $\chi^s[\![t]\!]\colon R_{\underline{\mathrm{end}}\,\mathcal{A}}[\![t]\!] \to (\underline{\mathrm{end}}\,\mathcal{A})_{\bar{0}}[\![t]\!]$, where χ^s is the supercharacter map as in Corollary 2.3, the formula in Proposition 5.1 takes the following form in $(\underline{\mathrm{end}}\,\mathcal{A})_{\bar{0}}[\![t]\!]$:

Corollary 5.2. $$\left(\sum_{\ell} \chi^s_{\mathcal{A}_\ell} t^\ell\right) \cdot \left(\sum_{m \equiv 0,1 \bmod N} (-1)^{m \bmod N} \chi^s_{\mathcal{A}^{!*}_m} t^m\right) = 1$$

Analogous formulas hold with the supercharacter χ^s replaced by the ordinary character χ or by one of the dimensions $\dim$ and sdim.

By (3.32) the coaction of $\underline{\mathrm{end}}\,\mathcal{A}$ on $\mathcal{A}$ sends $\mathcal{A}_n$ to $\mathcal{A}_n \otimes (\underline{\mathrm{end}}\,\mathcal{A})_n$. A similar remark holds for the $\underline{\mathrm{end}}\,\mathcal{A}$-coaction on $\mathcal{A}^{!*}$; see §4.1. Therefore, both factors in Corollary 5.2 actually belong to the Rees subring $\prod_{n \geq 0} B_n t^n$ of $B[\![t]\!]$, where we have put $B = (\underline{\mathrm{end}}\,\mathcal{A})_{\bar{0}}$.

Example 5.3. As an application of the Hilbert series version of Corollary 5.2, we see that the duals $\mathcal{A}^!$ of the Yang-Mills algebras $\mathcal{A} = \mathcal{YM}^{p|q}$ are never 3-Koszul. In fact, by Example 3.7, we have $H_{\mathcal{A}^!}(t) = 1 + dt + d^2t^2 + dt^3 + t^4$ if $p = 0$ or $q = 0$ and $H_{\mathcal{A}^!}(t) = 1 + dt + d^2t^2 + dt^3$ otherwise. In either case, $H_{\mathcal{A}^!}(t)^{-1}$ has a nonzero coefficient at t^5, which rules out Koszulity.

5.2. A master theorem modeled on the N-symmetric superalgebra $\mathsf{S}_N(V)$. We put $\mathcal{A} = \mathsf{S}_N(V)$ and use the notation of Examples 3.3 and 3.8. In particular, we assume that $\mathrm{char}\,\Bbbk = 0$ and work with a fixed basis $x_1, \ldots, x_d$ of $V = \mathcal{A}_1$ so that $\widehat{i} = \bar{0}$ for $i \leq p$ and $\widehat{i} = \bar{1}$ for $i > p$.

From Example 3.3 (see also Proposition 4.4(a)), we know that a basis of $\mathcal{A}_\ell$ is given by the monomials $x_{\mathbf{i}} = x_{i_1}x_{i_2}\ldots x_{i_\ell}$ for sequences $\mathbf{i} = (i_1, \ldots, i_\ell) \in \{1, \ldots, d\}^\ell$ such that $\mathbf{i}$ has no connected subsequence $\mathbf{j} = (j_1, \ldots, j_N)$ of length N satisfying

$$1 \leq j_1 < \ldots < j_m \leq p < j_{m+1} \leq \ldots \leq j_N \leq d = p + q$$

for some m. Adapting notation of Etingof and Pak [16] to our setting, we denote this set of sequences $\mathbf{i}$ by

$$\Lambda(p|q, N)_\ell \tag{5.3}$$

For example, $\Lambda(p|q, 2)_\ell$ consists of all weakly decreasing sequences $\mathbf{i} = (i_1, \ldots, i_\ell)$ with entries from $\{1, \ldots, d\}$ and such that no repetition occurs in the range $\{p + 1, \ldots, d\}$.

In order to evaluate the character $\chi^s_{\mathcal{A}_\ell}$ in Corollary 5.2, recall from (3.34) that the coaction $\delta_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A} \otimes \underline{\mathrm{end}}\,\mathcal{A}$ is given on the generators x_i of $\mathcal{A}$ by

$$\delta_{\mathcal{A}}(x_i) = \sum_j x_j \otimes z^j_i \in \mathcal{A} \otimes \underline{\mathrm{end}}\,\mathcal{A}$$

where $z^j_i = x^j \otimes x_i$ are the canonical generators of the algebra $\underline{\mathrm{end}}\,\mathcal{A}$. For $\mathbf{i} = (i_1, \ldots, i_\ell) \in \Lambda(p|q, N)_\ell$, we have

$$\delta_{\mathcal{A}}(x_{\mathbf{i}}) = \delta_{\mathcal{A}}(x_{i_1})\delta_{\mathcal{A}}(x_{i_2})\ldots\delta_{\mathcal{A}}(x_{i_\ell}) \in \mathcal{A}_\ell \otimes \underline{\mathrm{end}}\,\mathcal{A}$$

Since $\mathcal{A}_\ell \otimes \underline{\mathrm{end}}\,\mathcal{A} = \bigoplus_{\mathbf{i} \in \Lambda(p|q,N)_\ell} x_{\mathbf{i}} \otimes \underline{\mathrm{end}}\,\mathcal{A}$, we can define $Z(\mathbf{i}) \in (\underline{\mathrm{end}}\,\mathcal{A})_{\bar{0}}$ by

$$\delta_{\mathcal{A}}(x_{\mathbf{i}}) = x_{\mathbf{i}} \otimes Z(\mathbf{i}) + (\text{terms supported on } \Lambda(p|q, N)_\ell \setminus \{\mathbf{i}\})$$

Then (2.8) becomes

$$\chi^s_{\mathcal{A}_\ell} = \sum_{\mathbf{i}\in\Lambda(p|q,N)_\ell} (-1)^{\widehat{\mathbf{i}}} Z(\mathbf{i}) \tag{5.4}$$

with $\widehat{\mathbf{i}} = \widehat{i_1} + \cdots + \widehat{i_\ell}$.

Now consider the super bialgebra $\mathcal{B} = \mathcal{O}(\mathsf{E}(V)) = \Bbbk[x^i_j \mid 1 \le i,j \le d]$ defined in §2.5.3 and recall that the x^i_j are supercommuting variables of parity $\widehat{i} + \widehat{j}$ over $\Bbbk$. Restricting the comodule $\mathcal{A}_\ell$ to $\mathcal{B}$ along the map $\varphi\colon \underline{\mathrm{end}}\,\mathsf{S}_N(V) \to \mathcal{B}$, $z^i_j \mapsto x^i_j$ in (3.35) we must replace $Z(\mathbf{i})$ in (5.4) by $X(\mathbf{i}) := \varphi(Z(\mathbf{i})) \in \mathcal{B}_{\bar 0}$. Thus, writing

$$y_i = \sum_j x_j \otimes x^j_i \in \mathcal{A} \otimes \mathcal{B}$$

and $y_{\mathbf{i}} = y_{i_1} \dots y_{i_\ell} \in \mathcal{A}_\ell \otimes \mathcal{B} = \bigoplus_{\mathbf{j}\in\Lambda(p|q,N)_\ell} x_{\mathbf{j}} \otimes \mathcal{B}$ for $\mathbf{i} = (i_1, \dots, i_\ell)$, we have

$$y_{\mathbf{i}} = x_{\mathbf{i}} \otimes X(\mathbf{i}) + (\text{terms supported on } \Lambda(p|q,N)_\ell \setminus \{\mathbf{i}\}) \tag{5.5}$$

As for the supercharacter of $\mathcal{A}^{!*}_m$, recall from (4.1) and (3.15) that, for all $n \ge N$,

$$\mathcal{A}^{!*}_n = \bigcap_{i+j+N=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} = \mathsf{\Lambda}^n V$$

Viewing $\mathcal{A}^{!*}_n = \mathsf{\Lambda}^n V$ as a comodule over $\mathcal{B} = \mathcal{O}(\mathsf{E}(V))$, the supercharacter of $\mathcal{A}^{!*}_n$ is the n^{th} elementary supersymmetric function e_n which we know, by Proposition 2.4, to be identical to the coefficient at t^n of the characteristic function $\mathrm{ber}(1 + tX)$ of the generic supermatrix $X = \left(x^i_j\right)_{1\le i,j\le d}$ of type $p|q$; so the diagonal blocks $X_{11} = \left(x^i_j\right)_{1\le i,j\le p}$ and $X_{22} = \left(x^i_j\right)_{p+1\le i,j\le p+q}$ consist of even entries while all other entries are odd.

To summarize, we obtain the following super-version of [16, Theorem 2].

Theorem 5.4. *Let $X = \left(x^i_j\right)_{d\times d}$ be the generic supermatrix of type $p|q$. Then*

$$\left(\sum_\ell \sum_{\mathbf{i}\in\Lambda(p|q,N)_\ell} (-1)^{\widehat{\mathbf{i}}} X(\mathbf{i})\, t^\ell\right) \cdot \left(\sum_{m\equiv 0,1 \bmod N} (-1)^{m \bmod N} e_m t^m\right) = 1$$

holds in the power series ring $\Bbbk[x^i_j \mid \text{all } i,j\,]_{\bar 0}[\![t]\!]$. Here $\Lambda(p|q,N)_\ell$ and $X(\mathbf{i})$ are defined by (5.3) *and* (5.5), *respectively, and the e_m are the coefficients of the characteristic function $\mathrm{ber}(1 + tX) = \sum_{n\ge 0} e_n t^n$ of X.*

5.3. As an application of Theorem 5.4 , we determine the superdimension Hilbert series

$$H^s_{\mathcal{A}}(t) = \sum_{\ell\ge 0} \mathrm{sdim}_{\Bbbk}\, \mathcal{A}_\ell\, t^\ell$$

for the N-symmetric superalgebra $\mathcal{A} = \mathsf{S}_N(V)$. For the pure even case, this was already done by Etingof and Pak [16] . The notations of §5.2 remain in effect.

In view of Corollary 2.3, the superdimension Poincaré series follows by applying the counit $\varepsilon\colon \mathcal{B} \to \Bbbk$ to the equation in Theorem 5.4. Indeed, by (2.11), the counit ε sends $X \mapsto 1_{d\times d}$, and hence the elements $X(\mathbf{i})$ in (5.5) all map to 1. Therefore, the first factor in Theorem 5.4 becomes

$$H^s_{\mathcal{A}}(t) = \sum_{\ell\geq 0}\left(\sum_{\mathbf{i}\in\Lambda(p|q,N)_\ell}(-1)^{\widehat{\mathbf{i}}}\right)\,t^\ell$$

For the second factor, note that

$$\operatorname{ber}(1 + t\,1_{d\times d}) = (1+t)^{p-q}$$

by (2.12). Thus,

$$\begin{aligned} H^s_{\mathcal{A}}(t) &= \sum_{\ell\geq 0}\left(\sum_{\mathbf{i}\in\Lambda(p|q,N)_\ell}(-1)^{\widehat{\mathbf{i}}}\right)\,t^\ell \\ &= \begin{cases} \left(\displaystyle\sum_{m\equiv 0,1 \bmod N}(-1)^{m \bmod N}\binom{p-q}{m}t^m\right)^{-1} & \text{if } p\geq q \\ \left(\displaystyle\sum_{m\equiv 0,1 \bmod N}(-1)^{\alpha_N(m)}\binom{m+q-p-1}{q-p-1}t^m\right)^{-1} & \text{if } p<q \end{cases} \end{aligned} \tag{5.6}$$

where $\alpha_N(m) = m - (m \bmod N)$ denotes the largest multiple of N less than or equal to m as in §5.1.

5.4. The ordinary Hilbert series $H_{\mathcal{A}}(t) = \sum_{\ell\geq 0}\dim_{\Bbbk}\mathcal{A}_\ell\,t^\ell$ of the N-symmetric superalgebra $\mathcal{A} = \mathsf{S}_N(V)$ is as follows. Recall from §5.2 that

$$\dim_{\Bbbk}\mathcal{A}_\ell = |\Lambda(p|q,N)_\ell|$$

and from (3.16) that

$$\dim_{\Bbbk}\mathcal{A}^!_n = \begin{cases} d^n & \text{if } n<N \\ \sum_{r+s=n}\binom{p}{r}\binom{q+s-1}{s} & \text{if } n\geq N \end{cases}$$

Therefore, the Hilbert series is

$$\begin{aligned} H_{\mathcal{A}}(t) &= \sum_{\ell\geq 0}|\Lambda(p|q,N)_\ell|\,t^\ell \\ &= \left(\sum_{m\equiv 0,1 \bmod N}(-1)^{m \bmod N}\left(\sum_{r+s=m}\binom{p}{r}\binom{q+s-1}{s}\right)\,t^m\right)^{-1} \end{aligned} \tag{5.7}$$

5.5. Less is known about the Hilbert series of the N-homogeneous superalgebras $\mathcal{A} = \Lambda_{\mathscr{R},N}$ associated to an arbitrary Hecke operator $\mathscr{R}\colon V^{\otimes 2}\to V^{\otimes 2}$ on a vector superspace V ; see Example 3.4. Recall that $\mathcal{A} = A(V,R)$ with $R = \operatorname{Im}\rho_{\mathscr{R}}(X_N) \subseteq V^{\otimes N}$. For any N-homogeneous algebra $\mathcal{A} = A(V,R)$, we have

$$\dim_{\Bbbk}\mathcal{A}^!_n = \dim_{\Bbbk}\bigcap_{i+j+N=n}V^{\otimes j}\otimes R\otimes V^{\otimes i}$$

by (3.10) and (3.11). For $R = \operatorname{Im} \rho_{\mathscr{R}}(X_N)$ in particular, (1.19) further implies that

$$\bigcap_{i+j+N=n} V^{\otimes j} \otimes R \otimes V^{\otimes i} = \rho_{\mathscr{R}}(X_n)\left(V^{\otimes n}\right)$$

holds for $n \geq N$. Now [26, Theorem 3.5] implies that

$$H_{\Lambda_{\mathscr{R},2}!}(t) = \frac{\prod_{\ell=1}^{r}(1+a_\ell t)}{\prod_{m=1}^{s}(1-b_m t)}$$

where (r,s) is the birank of $\mathscr{R}$ and a_ℓ and b_m are positive real numbers. For example, in the situation of 5.4, $(r,s) = (p,q)$ and $a_\ell = b_m = 1$.

For any complex power series $P(t)$, the power series $P_N(-t)$ in (5.2) can be written as

$$P_N(-t) = \frac{1}{N}\sum_{i=1}^{N-1}(1-\zeta_N^{-i})P(\zeta_N^i t)$$

where $\zeta_N = e^{2\pi i/N}$. In particular,

$$\begin{aligned} H_{\mathcal{A}^{!*},N}(-t) &= \frac{1}{N}\sum_{i=1}^{N-1}(1-\zeta_N^{-i})\frac{\prod_{\ell=1}^{r}(1+a_\ell\zeta_N^i t)}{\prod_{m=1}^{s}(1-b_m\zeta_N^i t)} \\ &= \frac{Q_{N,\mathbf{a},\mathbf{b}}(t)}{\prod_{m=1}^{s}(1+b_m t+\ldots+b_m^{N-1}t^{N-1})} \end{aligned}$$

for some real polynomial $Q_{N,\mathbf{a},\mathbf{b}}(t)$ with coefficients being polynomial in $\mathbf{a} = (a_\ell)$ and $\mathbf{b} = (b_m)$. Therefore, the Hilbert series of $\mathcal{A}$ has the form

$$H_{\mathcal{A}}(t) = \frac{\prod_{m=1}^{s}(1+b_m t+\ldots+b_m^{N-1}t^{N-1})}{Q_{N,\mathbf{a},\mathbf{b}}(t)} \tag{5.8}$$

Notice that the fraction on the right-hand side is reduced.

In particular, (5.7) has the form

$$H_{\mathcal{A}}(t) = \frac{(1-t^N)^s}{(1-t)^s Q_{N,\mathbf{1},\mathbf{1}}(t)} \tag{5.9}$$

APPENDIX

For lack of a suitable reference, we include here a proof of Proposition 0.1 that was stated in the Introduction. Our proof is based on the proof of [8, Proposition 2.1] and on additional details that were communicated to us by Roland Berger. For the basics concerning graded algebras, we refer the reader to [9, Chap. II §11] or [6].

As in the Introduction, $\mathcal{A} = \bigoplus_{n\geq 0}\mathcal{A}_n$ denotes an arbitrary connected $\mathbb{Z}_{\geq 0}$-graded $\Bbbk$-algebra and V is a graded subspace of $\mathcal{A}_+ = \bigoplus_{n>0}\mathcal{A}_n$ satisfying $\mathcal{A}_+ = V \oplus \mathcal{A}_+^2$. Thus, $\mathsf{T}(V)/I \xrightarrow{\sim} \mathcal{A}$ for some graded ideal I of $\mathsf{T}(V)$. For convenience, we state Proposition 0.1 again:

Proposition. *The relation ideal I of $\mathcal{A}$ lives in degrees $\geq N$ if and only if $\operatorname{Tor}_i^{\mathcal{A}}(\Bbbk,\Bbbk)$ lives in degrees* $\geq \nu_N(i) = \begin{cases} \frac{i}{2}N & \text{if } i \text{ is even} \\ \frac{i-1}{2}N+1 & \text{if } i \text{ is odd} \end{cases}$

Proof. Let

$$P\colon \qquad \cdots \to P_i \xrightarrow{d_i} P_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} \Bbbk \to 0$$

be a minimal graded-free resolution of the trivial left $\mathcal{A}$-module $\Bbbk$. Thus, all P_i have the form $P_i = \mathcal{A} \otimes E_i$ for some graded subspace $E_i \subseteq \operatorname{Ker} d_{i-1}$ which is chosen so that

$$\operatorname{Ker} d_{i-1} = E_i \oplus \mathcal{A}_+ \operatorname{Ker} d_{i-1} \tag{A.10}$$

In particular, we may take $E_0 = \Bbbk$ and $E_1 = V$. The differential $d_i\colon P_i \to P_{i-1}$ is the graded $\mathcal{A}$-module map that is defined by the inclusion $E_i \hookrightarrow P_{i-1}$. By the graded Nakayama Lemma (e.g., [9, p. AII.171, Prop. 6]), our choice of E_i implies that

$$\operatorname{Im} d_i = \mathcal{A}E_i = \operatorname{Ker} d_{i-1} \quad \text{and} \quad \operatorname{Ker} d_i \subseteq \mathcal{A}_+ \otimes E_i = \mathcal{A}_+ P_i \tag{A.11}$$

for all i. Consequently, the complex $\Bbbk \otimes_{\mathcal{A}} P$ has zero differential, and hence

$$\operatorname{Tor}_i^{\mathcal{A}}(\Bbbk, \Bbbk) \cong \Bbbk \otimes_{\mathcal{A}} P_i \cong E_i$$

In particular,

$$\operatorname{Tor}_0^{\mathcal{A}}(\Bbbk, \Bbbk) \cong \Bbbk \quad \text{and} \quad \operatorname{Tor}_1^{\mathcal{A}}(\Bbbk, \Bbbk) \cong V = \mathcal{A}_+/\mathcal{A}_+^2$$

live in degrees $0 = \nu_N(0)$ and $\geq 1 = \nu_N(1)$, respectively. Moreover, the kernel of $d_1\colon P_1 = (\mathsf{T}(V)/I) \otimes V \to P_0 = \mathcal{A}$ is exactly $I/I \otimes V$, and so

$$\operatorname{Tor}_2^{\mathcal{A}}(\Bbbk, \Bbbk) \cong \operatorname{Ker} d_1/\mathcal{A}_+ \operatorname{Ker} d_1 \cong I/\left(V \otimes I + I \otimes V\right)$$

Therefore, I lives in degrees $\geq N$ if and only if $\operatorname{Tor}_2^{\mathcal{A}}(\Bbbk, \Bbbk)$ lives in degrees $\geq N = \nu_N(2)$.

For the remainder of the proof, assume that I lives in degrees $\geq N$. We will show by induction on i that $\operatorname{Tor}_i^{\mathcal{A}}(\Bbbk, \Bbbk) = E_i$ lives in degrees $\geq \nu_N(i)$ for all i. The cases $i \leq 2$ have been checked above. Assume that E_i lives in degrees $\geq \nu_N(i)$ and similarly for E_{i-1}. By (A.11), we know that $E_{i+1} \subseteq \operatorname{Ker} d_i \subseteq \mathcal{A}_+ \otimes E_i$ and so E_{i+1} certainly lives in degrees $\geq \nu_N(i) + 1$. Since $\nu_N(i) + 1 = \nu_N(i+1)$ when i is even (or when i is arbitrary and $N = 2$), we are done in these cases. From now on, we assume that i is odd. We must show that E_{i+1} lives in degrees $\geq \nu_N(i+1) = \frac{i+1}{2}N$. Since $E_{i+1} \subseteq \operatorname{Ker} d_i$, it suffices to show that d_i is injective in degrees $< \frac{i+1}{2}N$, and since E_i lives in degrees $\geq \nu_N(i) = \frac{i-1}{2}N + 1$, our goal is to show that d_i is injective on all homogeneous components $P_{i,n}$ of P_i in degrees $n = \frac{i-1}{2}N + j$ with $j = 1, \ldots, N-1$. Put $m = \frac{i-1}{2}N$ for simplicity and note that

$$P_{i,m+j} = \bigoplus_{\ell=1}^{j} \mathcal{A}_{j-\ell} \otimes E_{i,m+\ell} \tag{A.12}$$

and

$$P_{i-1,m+j} = \bigoplus_{k=0}^{j} \mathcal{A}_{j-k} \otimes E_{i-1,m+k} \tag{A.13}$$

since E_{i-1} lives in degrees $\geq \nu_N(i-1) = m$. The proposition will be a consequence of the following claims:

(a) d_i is injective on all summands $\mathcal{A}_{j-\ell} \otimes E_{i,m+\ell}$ in (A.12), and
(b) the subspaces $d_i\left(\mathcal{A}_{j-\ell} \otimes E_{i,m+\ell}\right) = \mathcal{A}_{j-\ell}E_{i,m+\ell}$ for $\ell = 1, \ldots, j$ form a direct sum inside $P_{i-1,m+j}$.

In order to prove (a), recall that the restriction of d_i to $E_{i,m+\ell}$ is the inclusion

$$E_{i,m+\ell} \hookrightarrow P_{i-1,m+\ell} = \bigoplus_{k=0}^{\ell} \mathcal{A}_{\ell-k} \otimes E_{i-1,m+k}$$

Hence, the effect of d_i on the ℓ^{th} summand in (A.12) is the embedding

$$\mathcal{A}_{j-\ell} \otimes E_{i,m+\ell} \hookrightarrow \bigoplus_{k=0}^{\ell} \mathcal{A}_{j-\ell} \otimes \mathcal{A}_{\ell-k} \otimes E_{i-1,m+k}$$

followed by the map

$$\bigoplus_{k=0}^{\ell} \mathcal{A}_{j-\ell} \otimes \mathcal{A}_{\ell-k} \otimes E_{i-1,m+k} \longrightarrow \bigoplus_{k=0}^{\ell} \mathcal{A}_{j-k} \otimes E_{i-1,m+k} \subseteq P_{i-1,m+j}$$

which is given by the multiplication map $\mathcal{A}_{j-\ell} \otimes \mathcal{A}_{\ell-k} \to \mathcal{A}_{j-k}$. Since $j-k<N$, our hypothesis on I implies that $\mathcal{A}_{j-k} \cong \mathsf{T}(V)_{j-k}$, and similarly $\mathcal{A}_{j-\ell} \cong \mathsf{T}(V)_{j-\ell}$ and $\mathcal{A}_{\ell-k} \cong \mathsf{T}(V)_{\ell-k}$. Therefore, the above multiplication map is identical with the injection $\mathsf{T}(V)_{j-\ell} \otimes \mathsf{T}(V)_{\ell-k} \hookrightarrow \mathsf{T}(V)_{j-k}$ in $\mathsf{T}(V)$. This proves (a).

For (b), we proceed by induction on j. The case $j=1$ being obvious, let $1 \leq j \leq N-2$ and assume that (ii) holds for $1,\ldots,j$. We wish to show that the subspaces $\mathcal{A}_{j+1-\ell}E_{i,m+\ell}$ $(\ell=1,\ldots,j+1)$ of $P_{i-1,m+j+1}$ form a direct sum. First, by (A.10) we have $E_{i,m+j+1} \cap \mathcal{A}_+ \operatorname{Ker} d_{i-1} = 0$ while $\sum_{\ell=1}^{j} \mathcal{A}_{j+1-\ell}E_{i,m+\ell} \subseteq \mathcal{A}_+ \operatorname{Ker} d_{i-1}$. Therefore, it suffices to show that the sum $\sum_{\ell=1}^{j} \mathcal{A}_{j+1-\ell}E_{i,m+\ell}$ is direct. To this end, note that $\mathcal{A}_{j+1-\ell} = \sum_{d\geq 1} V_d \mathcal{A}_{j+1-d-\ell}$ holds for all $\ell \leq j$. Hence,

$$\sum_{\ell=1}^{j} \mathcal{A}_{j+1-\ell}E_{i,m+\ell} = \sum_{d\geq 1} V_d \sum_{\ell=1}^{j} \mathcal{A}_{j+1-d-\ell}E_{i,m+\ell}$$

By induction, $\sum_{\ell=1}^{j} \mathcal{A}_{j+1-d-\ell}E_{i,m+\ell}$ is a direct sum inside $P_{i-1,m+j+1-d}$. Thus, it suffices to show that the sum $\sum_{d\geq 1} V_d P_{i-1,m+j+1-d} \subseteq P_{i-1,m+j+1}$ is direct. But (A.13) gives

$$P_{i-1,m+j+1} = \bigoplus_{k=0}^{j+1} \mathcal{A}_{j+1-k} \otimes E_{i-1,m+k} = \bigoplus_{k=0}^{j+1} \mathsf{T}(V)_{j+1-k} \otimes E_{i-1,m+k}$$

where the last equality holds since all $j+1-k<N$. Therefore,

$$\sum_{d\geq 1} V_d P_{i-1,m+j+1-d} = \bigoplus_{d\geq 1} V_d \otimes \bigoplus_{k=0}^{j+1-d} \mathsf{T}(V)_{j+1-d-k} \otimes E_{i-1,m+k}$$

as desired. This proves (b), thereby completing the proof of the proposition. □

Acknowledgement. The authors wish to thank Roland Berger for his helpful comments throughout the completion of this paper. Work on this article was initiated during a visit of ML to the Université Montpellier 2 in June 2006. ML would like to thank Claude Cibils for arranging this visit and for his warm hospitality in Montpellier, and James Zhang for help with some examples in this article.

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