ON THE GLOBAL DIMENSION OF FIXED RINGS

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Abstract. Let $G$ be a finite group acting on a $k$-algebra $R$, and let $S = R^G$ denote the fixed subring of $R$. Our main interest is in the case where $|G|$ is not invertible in $R$. Instead, we assume that $R$ is flat over $S$ and that the trivial $kG$-module $k$ has a periodic projective resolution. (For a field $k$ of characteristic $p$, the latter condition holds precisely if the Sylow $p$-subgroups of $G$ are cyclic or generalized quaternion.) We use a periodicity result for Ext-groups, established here in a more general setting that is independent of group actions, to estimate the global dimension of $S$ in this case.

Introduction

Let $G$ be a finite group acting by automorphisms on a $k$-algebra $R$ ($k$ some commutative ring), and let $S = R^G$ denote the fixed subring of $R$. In this note, we are concerned with bounding the (right) global dimension of $S$ in terms of the global dimension of $R$ and other related data.

In case the global dimension of $R$ is at most 1 and $|G|$ is invertible in $R$, one knows that $r.gldim(S) \leq r.gldim(R)$, by results of Levitzki [L] and Bergman [Be]. The situation becomes worse for larger global dimensions and for $|G|^{-1} \notin R$. For example, if $R = k[x,y]$ is the polynomial ring over a field $k$ of characteristic $\neq 2$, and $G = C_2$ acts on $R$ by $x \mapsto -x$, $y \mapsto -y$, then $R$ has global dimension 2 whereas $S = k[x^2,xy,y^2]$ has infinite global dimension [R]. Also, taking $R$ to be the ring of $2 \times 2$-matrices over a field $k$ with char $k = 2$, and $G = \langle (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \rangle \cong C_2$ acting by conjugation on $R$, one obtains $S \cong k \oplus ke$ with $e^2 = 0$. So $r.gldim(R) = 0$, but $r.gldim(S) = \infty$.

Nevertheless, if $|G|^{-1} \in R$, then one has the following estimate (see 2.2 below):

$$r.gldim(S) \leq r.gldim(R) + pdim(R_S).$$
The aim of this article is to establish a similar estimate without the assumption on $|G|$. Our main result is as follows.

**Theorem.** Put $\overline{S} = S/\text{Tr}(R)$, where $\text{Tr}: R \to S$ is the usual trace map, and

$$\rho = \text{r. gldim}(R) + \text{pdim}(R_S) \quad \text{and} \quad \sigma = \text{r. gldim}(\overline{S}) + \text{pdim}(\overline{S}_S).$$

Assume that

1. $S^r R$ is flat, and
2. the trivial $kG$-module $k$ has a resolution $0 \to k \to X_c \to \cdots \to X_1 \to k \to 0$ with all $X_i$ projective over $kG$.

Then either $\text{r. gldim}(S) \leq \max\{\rho, \sigma\}$ or $\text{r. gldim}(S) = \infty$.

Assumption (2) here is satisfied if $|G|^{-1} \in k$ (see 2.3). More interestingly, if $k$ is a field of characteristic $p$, then (2) holds precisely if the Sylow $p$-subgroups of $G$ are cyclic or generalized quaternion. Thus the important special case of one automorphism of finite order acting on $R$ is covered by (2). For further examples, see 2.4 below. The precise meaning of (1) is less clear, but $R$ is known to be projective over $S$ in some cases that arise naturally (see 2.6).

The above result is a consequence of a more precise periodicity result for Ext-groups that holds in the situation of the above theorem (see Theorem 2.7). In §1, we establish such a periodicity result for Ext-groups in a more general abstract setting, independent of group actions. This result is then applied, in §2, to the case of fixed subrings.

### 1. Periodicity for ext

1.1 **Lemma.** Let $S \to \overline{S}$ be a surjective ring homomorphism with kernel $I$. Let $M_S$ be an $S$-module with $M \cdot I^n = 0$ for some $n \geq 1$. Then $\text{pdim}(M_S) \leq \text{r. gldim}(S) + \text{pdim}(\overline{S}_S)$.

**Proof.** Arguing by induction on $n$, one reduces to the case $n = 1$, where the assertion is well known (e.g., [Ro, Theorem 9.32]). $\square$

1.2 **Lemma.** Let $S$ be a ring and let

$$T: 0 \to T_{c+1} \to T_c \to \cdots \to T_1 \to T_0 \to 0$$

be a complex of right $S$-modules, with $c \geq 1$. Assume that

$$\delta = \max_{0 \leq n \leq c+1} \{\text{pdim} H_n(T)\} \quad \text{and} \quad \tau = \max_{1 \leq m \leq c} \{\text{pdim}(T_m)\}$$

are both finite. Let $V_S$ be an $S$-module. Then there are homomorphisms

$$f^q: \text{Ext}^q_{S} (T_0, V) \to \text{Ext}^q_{S} (T_{c+1}, V) \quad (q > \tau)$$

such that $f^q$ is epi for $q > \max\{\tau, \delta\}$, and $f^q$ is an isomorphism for $q > \max\{\tau, \delta + 1\}$. 

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Proof. We first introduce a number of (cochain-) complexes. Let \( Q = (Q^i)_{i \geq 0} \) be an injective resolution of \( V_S \). So \( H^i(Q) = 0 \) for \( i > 0 \) and \( H^0(Q) \cong V \). Form the double complex

\[
B = \text{Hom}_S(T, Q) = (B^{p,q})_{p,q \geq 0}, \quad B^{p,q} = \text{Hom}_S(T_p, Q^q)
\]

and its associated total complex

\[
C = \text{Tot}(B) = (C^n)_{n \geq 0}, \quad C^n = \bigoplus_{p+q=n} B^{p,q}.
\]

Note that \( B^{p,q} = 0 \) for \( p > c + 1 \) as \( T_p = 0 \) in this case.

**Step 1.** \( H^n(C) = 0 \) for \( n > p + c + 1 \).

*Proof.* Since \( Q \) is injective, there is a Künneth spectral sequence \( \{E_r\} \) with

\[
E_2^{p,q} = \text{Ext}^p_S(H_q(T), H^j(Q)) \Rightarrow H^{p+q}(C)
\]

(see [Ro, Theorem 11.34] or [G, Theorem 5.4.1]). Thus

\[
E_2^{p,q} = \text{Ext}^p_S(H_q(T), V) = 0
\]

if either \( q > c + 1 \) or \( p > \delta \). Therefore, \( H^{p+q}(C) = 0 \) if \( p + 9 > \delta + c + 1 \).

**Step 2.** The maps \( f^q \).

Consider the first filtration of \( C \), as in [C-E, p. 330] (omitting the subscript \( I \)):

\[
(F^p C)^n = \bigoplus_{r+s=n} B^{r,s}.
\]

The corresponding spectral sequence \( \{E'_r\} \) converges to \( H^*(C) \) and has \( E'_2 \)-term (notation as in [C-E, pp. 330–331])

\[
E'_2 \cong H_1H_{\Pi}(B),
\]

where

\[
H_{\Pi}^{p,q}(B) = H^q(B^{p,*}) = H^q(\text{Hom}_S(T_p, Q)) = \text{Ext}^q_S(T_p, V).
\]

In particular, since \( T_p = 0 \) for \( p > c + 1 \), we have \( H_{\Pi}^{p,*}(B) = 0 \) for \( p > c + 1 \) and so \( (E'_2)^{p,q} = 0 \) for \( p > c + 1 \). Since the differential \( d_r \) of \( E'_r \) has bidegree \( (r,1-r) \), it follows that \( d_{c+2} = 0 \) and so

\[
E'_\infty = E'_c = E_{c+2}.
\]

Moreover, by definition of \( \tau \), \( H_{\Pi}^{p,q}(B) = 0 \) for \( q > \tau \) and \( p \neq 0, c + 1 \). Consequently, for \( q > \tau \), we have

\[
(E'_2)^{p,q} = \begin{cases} 
0 & \text{if } p \neq 0, c + 1 \\
\text{Ext}^q_S(T_0, V) & \text{if } p = 0 \\
\text{Ext}^q_S(T_{c+1}, V) & \text{if } p = c + 1
\end{cases} \quad (q > \tau).
\]
By considering the bidegree of the differential $d_r$ of $E'_r$, one sees that

$$q \geq \tau + c \Rightarrow d_r^{0,q} = 0 \quad \text{for} \quad 2 \leq r \leq c$$

$$\Rightarrow (E'_2)^{0,q} = (E'_3)^{0,q} = \cdots = (E'_{c+1})^{0,q}$$

and

$$q \leq \tau \Rightarrow d_r^{c+1-r,q-(1-r)} = 0 \quad \text{for} \quad 2 \leq r \leq c$$

$$\Rightarrow (E'_2)^{c+1,q} = (E'_3)^{c+1,q} = \cdots = (E'_{c+1})^{c+1,q}.$$ 

Therefore, for $q \geq \tau$, the differential $d_{c+1}$ yields a homomorphism

$$(E'_2)^{0,q+c} \sim (E'_{c+1})^{0,q+c} \xrightarrow{d_{c+1}} (E'_{c+1})^{c+1,q} \sim (E'_2)^{c+1,q}.$$ 

For $q > \tau$, this is the required homomorphism

$$f^q : \text{Ext}^q_{S,c+1}(T_0, V) \to \text{Ext}^q_{S,c+1}(T_{c+1}, V).$$

**Step 3. Surjectivity and injectivity.**

Since $H^n(C) = 0$ for $n > \delta + c + 1$, by Step 1, we know that $(E'_\infty)^{0,q} = 0$ for $p + q > \delta + c + 1$. In particular,

$$0 = (E'_\infty)^{c+1,q} = (E'_{c+1})^{c+1,q} = (E'_{c+1})^{c+1,q}/d_{c+1}(E'_{c+1})^{0,q+c}$$

for $c + 1 + q > \delta + c + 1$, and so $f^q$ is surjective for $q > \delta, q > \tau$. Also,

$$0 = (E'_\infty)^{0,q+c} = (E'_{c+2})^{0,q+c} = \text{Ker}(d_{c+1}|(E'_{c+1})^{0,q+c}).$$

for $q + c > \delta + c + 1$. Thus $f^q$ is injective for $q > \delta + 1, q > \tau$. This proves the lemma. □

1.3 **Proposition.** Let $S \subseteq R$ be an inclusion of rings and let $S \to \overline{S}$ be a surjective ring homomorphism. Assume that

$$\rho = \text{r.gl.dim}(R) + \text{pdim}(R_S) \quad \text{and} \quad \sigma = \text{r.gl.dim}(\overline{S}) + \text{pdim}(\overline{S}_S)$$

are both finite. Suppose that there is a complex of $(S,S)$-bimodules

$$P : 0 \to P_{c+1} \to P_c \to \cdots \to P_1 \to P_0 \to 0$$

with $c \geq 1$, such that

(1) each $s_i P_i$ is flat;

(2) for $1 \leq i \leq c$, $P_i$ is an $(S,S)$-bimodule direct summand of some $(S,R)$-bimodule;

(3) the right $S$-action on $H_n(P)$ factors through $\overline{S}$.

Then, for any two $S$-modules $V_S$ and $W_S$, there exist homomorphisms

$$f^q : \text{Ext}^q_{S,c+1}(W \otimes_S P_0, V) \to \text{Ext}^q_{S,c+1}(W \otimes_S P_{c+1}, V) \quad (q > \rho)$$

such that $f^q$ is epi for $q > \max\{\rho, \sigma\}$ and an isomorphism for $q > \max\{\rho, \sigma + 1\}$.

**Proof.** Put $T = W \otimes_S P$, a complex of right $S$-modules. We estimate the numbers $\delta$ and $\tau$ in Lemma 1.2. First, by assumption (2), $T_i$ is an $S$-module
direct summand of some right $R$-module, say $Q_i$ ($i = 1, \ldots, c$). Therefore, by [Ro, Theorem 9.32],
\[ \text{pdim}(T_i)_S \leq \text{pdim}(Q_i)_S \leq \text{pdim}(Q_i)_R + \text{pdim}(R_S) \leq \rho. \]

Hence
\[ \tau \leq \rho. \]

Next, we claim that $H_n(T) \cdot I^{n+1} = 0$ for $0 \leq n \leq c + 1$, where $I$ denotes the kernel of the map $S \to \overline{S}$. Indeed, since $S\mathcal{P}$ is flat, by (1), there is a Künneth spectral sequence $\{E^r\}$ with
\[ E^2_{p,q} = \text{Tor}^S_{p}(W, H_q(\mathcal{P})) \Rightarrow H_{p+q}(T) \]
([Ro, Theorem 11.34] or [M, Chapter XII, Theorem 12.1]). By (3), we have $E^2_{p,q} \cdot I = 0$. The spectral sequence yields a chain
\[ 0 = F^{-1}H_n \subseteq F^0H_n \subseteq \cdots \subseteq F^nH_n = H_n \]
of $S$-submodules of $H_n = H_n(T)$ such that each $F^pH_n/F^{p-1}H_n$ is isomorphic to a subquotient of $E^2_{p,n-p}$. Therefore, $H_n(T) \cdot I^{n+1} = 0$, as we have claimed.

Lemma 1.1 now yields $\delta \leq \sigma$, and so the proposition follows from Lemma 1.2. \(\Box\)

1.4 Remarks. The proof of Proposition 1.3 works without essential changes if (3) is weakened to
\[ (3') \text{ Some power of } I = \text{Ker}(S \to \overline{S}) \text{ annihilates each } H_n(\mathcal{P}). \]
Moreover, arguing somewhat differently (without invoking the spectral sequence) in the last part of the proof, the conclusion of Proposition 1.3 still holds if (1) is replaced by
\[ (1') \text{ Some power of } I \text{ annihilates each } \text{Tor}_i^S(W, P_j), \text{ for all } i > 0. \]
However, these generalizations don’t seem to be useful, at least not for our applications to fixed rings in §2.

1.5 Corollary. In the situation of Proposition 1.3, assume in addition that $S$ is an $(S, S)$-bimodule direct summand of $P_{c+1}$. Then, for any $S$-module $V_S$, either $\text{idim}(V_S) \leq \max\{\rho, \sigma\}$ or $\text{idim}(V_S) = \infty$. Consequently, either $\text{r.gldim}(S) \leq \max\{\rho, \sigma\}$ or $\text{r.gldim}(S) = \infty$.

Proof. Assume that $\text{idim}(V_S) > \max\{\rho, \sigma\}$, so $\text{Ext}^q_S(W, V) \neq 0$ for some $S$-module $W_S$ and some $q > \max\{\rho, \sigma\}$. Proposition 1.3 yields an epimorphism $\text{Ext}^{q+c}_S(W \otimes_S P_0, V) \to \text{Ext}^q_S(W \otimes_S P_{c+1}, V)$ and the latter group is nonzero, since it maps onto $\text{Ext}^q_S(W, V)$, by assumption on $P_{c+1}$. Therefore,
\[ \text{Ext}^{q+c}_S(W \otimes_S P_0, V) \neq 0, \]
and we can continue with $W \otimes_S P_0$ in place of $W$. \(\Box\)
2. Periodic resolutions for groups

2.1 Notations. The following notations will be kept throughout this section. \( R \) will be an algebra over some commutative ring \( k \), and \( G \) will be a finite group acting on \( R \) by \( k \)-algebra automorphisms that will be written \( r \mapsto r^g \) \((r \in R, g \in G)\). We let \( S = R^G \) denote the fixed subring of \( R \). As usual, \( \text{Tr}: R \to S \) is the trace map given by \( \text{Tr}(r) = \sum_{g \in G} r^g \) \((r \in R)\). Since \( \text{Tr} \) is an \((S, S)\)-bimodule map, its image \( \text{Tr}(R) \) is an ideal of \( S \). We put \( S = S/\text{Tr}(R) \). The augmentation ideal of the group algebra \( kG \) will be denoted by \( \text{aug} \), and we put \( \hat{G} = \sum_{g \in G} g \in kG \). Viewing \( R \) as a right \( kG \)-module via the given \( G \)-action, the trace map can be written as \( \text{Tr}(r) = r^{\hat{G}} \) \((r \in R)\).

2.2 The case \( |G|^{-1} \in R \). If \( |G|^{-1} \in R \), then the map \( |G|^{-1} \cdot \text{Tr}: R \to S \) is the identity on \( S \). Thus \( S \) is an \((S, S)\)-bimodule direct summand of \( R \), and [K, Theorem 5 on p. 173] implies

\[
\text{r.gldim}(S) \leq \text{r.gldim}(R) + \text{pdim}(R_S).
\]

Thus, in the following, our emphasis will be on the case where \( |G|^{-1} \notin R \), although the case \( |G|^{-1} \in R \) is covered by the following, too.

2.3 Periodic resolutions. Following [A], we say that the trivial \( kG \)-module \( k \) is periodic if there exists an exact sequence of left (say) \( kG \)-modules

\[
0 \to k \xrightarrow{\mu} X_0 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_i} X_i \xrightarrow{\epsilon} k \to 0,
\]

where all \( X_i \) are projective over \( kG \). The smallest such \( c \) is called the period of \( k \). (In [A], these concepts are introduced, in the same way, for arbitrary \( kG \)-modules.) The case where \( k \) is projective over \( kG \) or, equivalently, \( |G|^{-1} \in k \), is included here via the obvious sequence \( 0 \to k \to X_1 = k \oplus k \to k \to 0 \).

2.4 Remarks and Examples. (a) If \( k \) is periodic with period \( c \), then the (Tate) cohomology ring \( \tilde{H}^*(G, k) \) has an invertible element \( u \in \tilde{H}^c(G, k) \). In particular, using the fact that \( \tilde{H}^*(G, k) \) is anticommutative, one sees that, if \( 2 \neq 0 \) in \( \tilde{H}^0(G, k) = k/|G|k \), then \( c \) must be even. But, of course, the case of odd \( c \) also occurs, e.g. \( c = 1 \) when \( |G| = 2 \) and \( 2 = 0 \) in \( k \).

(b) If \( Z \) is periodic for \( ZG \), then any \( k \) is periodic for \( kG \). In this case, all abelian subgroups of \( G \) must be cyclic ([Br, Theorem 9.5 on p. 157]). All finite groups of this type have been classified by Zassenhaus [Z] in the solvable case (see also [W, Theorem 6.1.11]) and by Suzuki [S] in general.

A construction of periodic free \( ZG \)-resolutions \( X \) for \( Z \) starting with a fixed-point-free complex representation of \( G \) (that is, a finite-dimensional \( CG \)-module \( V \) such that \( \{ v \in V : gv = v \} = 0 \) holds for all \( 1 \neq g \in G \)) is described in [Br, p. 154]. The complete classification of all groups admitting a fixed-point-free representation can be found in [W, Theorems 6.1.11 and 6.3.1]. These
groups include in particular the finite subgroups of the multiplicative group of the quaternions $H$:

— the finite cyclic groups,
— the generalized quaternion groups $Q_{4m} = \langle x, y \mid y^2 = x^m, xyx = y \rangle$ ($m \geq 2$),
— the binary tetrahedral group (order 24), the binary octahedral group (order 48), and the binary icosahedral group $SL(2, 5)$ (order 120).

For further examples and more details see [Br, pp. 154–156].

(c) If $k$ is a field of characteristic $p > 0$, then the trivial $kG$-module $k$ is periodic if and only if all elementary abelian $p$-subgroups of $G$ are cyclic (see [A-E]). The latter condition is satisfied if and only if the Sylow $p$-subgroups of $G$ are either cyclic or generalized quaternion (for $p = 2$ only) (see [Br, p. 157]).

2.5 Lemma. Assume that the trivial $kG$-module $k$ is periodic with period $c$. Then there exists a complex of $(S, S)$-bimodules

$P: 0 \to P_{c+1} \to P_c \to \cdots \to P_1 \to P_0 \to 0$

such that

1. $P_0 \cong S \cong P_{c+1}$;
2. for $1 \leq i \leq c$, $P_i$ is an $(S, S)$-bimodule direct summand of some $R^{(\alpha_i)}$;
3. the left and right $S$-actions on $H_*(P)$ factor through $\overline{S}$.

Proof. Let

$X: 0 \to k \xrightarrow{\mu} X_c \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_2} X_1 \xrightarrow{\phi_1} k \to 0$

be a periodic $kG$-resolution as in 2.3. Expand $X$ into a complete periodic resolution

$\hat{X}: \cdots \to X_{c+1} = X_1 \xrightarrow{\phi_1+\mu\phi} X_c \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_2} X_1 \xrightarrow{\phi_1+\mu\phi} X_0 = X_c \to \cdots$.

Consider the complex $Q = R \otimes_{kG} \hat{X}$, with maps $\psi_i = \text{id}_R \otimes \phi_i$. The $(S, S)$-bimodule structure on $R$ makes $Q$ a complex of $(S, S)$-bimodules. Moreover, since each $X_i$ is a direct summand of some $kG^{(\alpha_i)}$, $Q_i = R \otimes_{kG} X_i$ is an $(S, S)$-bimodule direct summand of $R \otimes_{kG} kG^{(\alpha_i)} = R^{(\alpha_i)}$. The homology of $Q$ is given by

$H_i(Q) = \hat{H}_{i-1}(G, R)$,

the Tate homology of $G$ with coefficients in $R$, up to an index shift. The $(S, S)$-bimodule structure on $H_*(Q) = \hat{H}_{*+1}(G, R)$ can be viewed as the (left and right) cap product action of $S = H^0(G, R)$ on $\hat{H}_*(G, R)$, and this action factors through $\overline{S} = \hat{H}^0(G, R)$.

We now modify $X$ and $Q$ to obtain $P$. Note that $\text{Im}(\mu) \subseteq \text{ann}_{X_c}(\omega G) = \hat{G} \cdot X_c$ and so, identifying $k$ with $k \cdot \hat{G} \subseteq kG$, we have $\mu(\hat{G}) = \hat{G} \cdot \xi$ for some $\xi \in X_c$. Define a $kG$-linear map $\mu': kG \to X_c$ by $\mu'(1) = \xi$, so $\mu'|_{k\hat{G}} = \mu$. Therefore, $\phi_c \circ \mu'|_{k\hat{G}} = 0$, and hence $\text{Im}(\phi_c \circ \mu') \subseteq \text{ann}_{X_c}(\hat{G}) = (\omega G) \circ X_{c-1}$.
Furthermore, define $\varepsilon': X_1 \to kG$ to be the composite $X_1 \xrightarrow{\varepsilon} k \cong k \cdot \hat{G} \hookrightarrow kG$. Note that $\phi'_1: X_1 \to X_0 = X_c$ factors as $f_1 = \mu' \circ \varepsilon'$.

Now define $(S, S)$-bimodule maps $\hat{\mu}: R = R \otimes_{kG} kG \to Q_c = R \otimes_{kG} X_c$ by $\hat{\mu} = \text{id}_R \otimes \mu'$ and $\hat{\varepsilon}: Q_1 = R \otimes_{kG} X_1 \to R = R \otimes_{kG} kG$ by $\hat{\varepsilon} = \text{id}_R \otimes \varepsilon'$. Then the image of $\hat{\varepsilon}$ is equal to the image of $R \otimes_{kG} kG \to R \otimes_{kG} kG = R$, that is $\hat{\varepsilon}(Q_1) = \text{Tr}(R)$.

Next, we show that $\hat{\mu}$ is mono on $\text{Tr}(R) \subseteq R$. To see this, embed $X_c$ as a direct summand in a free module $F = kG^{(I)}$, with corresponding map $\lambda: X_c \to F$. Then, putting $\phi = \lambda \circ \mu': kG \to F$ and $\tilde{\phi} = \text{id}_R \otimes \phi: R \to R \otimes_{kG} F \cong R^{(I)}$, we have $\text{Ker}(\hat{\mu}) = \text{Ker}(\tilde{\phi})$. The map $\phi |_{kG}: kG \xrightarrow{\mu} \hat{G} \cdot X_c \xrightarrow{\lambda} \hat{G} \cdot F = (\hat{G}k)^{(I)}$

is $k$-split, since $X$ splits over $k$, and hence $\mu$ does. Therefore, $\phi(\hat{G}) = (\hat{G}_i \xi_i)_{i \in I}$ for certain $\xi_i \in k$ (almost all 0) with $\sum_i \xi_i \tau_i = 1$ for suitable $\tau_i \in k$. Now, for any $r^\hat{G} \in \text{Tr}(R)$, we have in $R \otimes_{kG} F = R^{(I)}$, $\tilde{\phi}(r^\hat{G}) = r^\hat{G} \otimes \phi(1) = r \otimes \phi(\hat{G})$

$= r \otimes (\hat{G}_i \xi_i)_{i \in I} = (r^\hat{G}_i \xi_i)_{i \in I}$,

and so, if $\tilde{\phi}(r^\hat{G}) = 0$, then $r^\hat{G} = \sum_i r^\hat{G}_i \tau_i = 0$, as we have claimed. It follows that $\text{Tr}(R) \cdot \text{Ker}(\hat{\mu}|_S) = 0 = \text{Ker}(\hat{\mu}|_S) \cdot \text{Tr}(R)$.

Finally, we claim that $\hat{\mu}(S) \subseteq \text{Ker}(\psi_c)$. Indeed, since $\text{Im}(\psi_c \circ \mu') \subseteq (\omega G) \cdot X_{c-1}$, it follows that $\psi_c \circ \hat{\mu}(S)$ is contained in the canonical image of $S \otimes_{kG} (\omega G) X_{c-1}$ in $Q_{c-1} = R \otimes_{kG} X_{c-1}$, and this image equals 0, since $S \omega G = 0$. Thus we have the following complex of $(S, S)$-bimodules

$\mathbf{P}: 0 \to P_{c+1} = S \xrightarrow{\mu|_S} P_c = Q_c \xrightarrow{\psi_1} \cdots \xrightarrow{\psi_2} P_1 = Q_1 \xrightarrow{\varepsilon} P_0 = S \to 0$.

Its homology is: $H_0(\mathbf{P}) = \overline{S}$, $H_1(\mathbf{P}) = H_1(\mathbf{Q})$ (because $\psi_1 = \tilde{\mu} \circ \varepsilon$ and $\tilde{\mu}$ is injective on $\text{Im}(\varepsilon) = \text{Tr}(R)$), and $H_i(\mathbf{P}) = H_i(\mathbf{Q})$ ($i = 1, \ldots, c - 1$). Moreover, $H_{c+1}(\mathbf{P}) = \text{Ker}(\hat{\mu}|_S)$ is annihilated, on both sides, by $\text{Tr}(R)$ and $H_c(\mathbf{P}) = \text{Ker}(\psi_c)/\hat{\mu}(S)$ is an image of $H_c(\mathbf{Q}) = \text{Ker}(\psi_c)/\hat{\mu}(Q_c) = \text{Ker}(\psi_c)/\hat{\mu}(\varepsilon(Q_1)) = \text{Ker}(\psi_c)/\hat{\mu}(\text{Tr}(R))$. Thus (3) is satisfied, and so $\mathbf{P}$ has all the required properties.

2.6 Flatness and projectivity. If the trivial $kG$-module $k$ is periodic, then Lemma 2.5 guarantees that the hypotheses of Proposition 1.3 are satisfied, except possibly for the flatness of all $S P_i$. For this, it suffices to assume that $S R$ is flat. I am not aware of any results specifically in this direction, but there
do exist quite a few results ensuring projectivity of $\mathcal{S}R$. For example, $\mathcal{S}R$ is known to be projective in each of the following cases:

(a) If the skew group ring $T = R \ast G$ is a simple ring (it actually suffices to assume that $T = \hat{T}G$, where $\hat{G} = \sum_{g \in G} g \in T$), then $R$ is finitely generated projective over $\mathcal{S}$, on both sides (e.g., [Mo, proof of Theorem 2.4]). This happens, for example, if $R$ has no nontrivial $G$-invariant ideals and $G$ consists of outer automorphisms of $R$ (use [Mo, Lemma 3.16]).

(b) If $R$ is a finite direct product of simple rings and $|G|^{-1} \in R$, then $R$ is finitely generated projective over $\mathcal{S}$ [H-R].

(c) Assume that $R$ is hereditary and $|G|^{-1} \in R$. Then each of the following implies that $R$ is projective over $\mathcal{S}$, although not necessarily finitely generated in all cases: $R$ is semiprime Noetherian, $R$ is commutative von Neumann regular, $R$ is reduced von Neumann regular and $G$ is solvable [J].

2.7 Theorem. Let $R$ be an algebra over a commutative ring $k$ and let $G$ be a finite group of $k$-algebra automorphisms of $R$. Put $S = R^G$, $\mathcal{S} = S/\text{Tr}(R)$ and assume that $\rho = r.\text{gldim}(R) + \text{pdim}(R_S)$ and $\sigma = r.\text{gldim}(\mathcal{S}) + \text{pdim}(\mathcal{S}_S)$.

are both finite. Assume further that

1. $S$ is flat, and
2. the trivial $kG$-module $k$ is periodic of period $c$.

Then, for any $S$-modules $V_S$ and $W_S$, there exist homomorphisms

\[ f^q: \text{Ext}^q_{S}(W, V) \longrightarrow \text{Ext}^{q+c}_{S}(W, V) \quad (q > \rho) \]

such that $f^q$ is epi for $q > \max\{\rho, \sigma\}$ and is an isomorphism for $q > \max\{\rho, \sigma + 1\}$. Consequently, either $r.\text{gldim}(S) \leq \max\{\rho, \sigma\}$ or $r.\text{gldim}(S) = \infty$.

Proof. In view of the remarks in 2.6, this follows from Lemma 2.5 and Proposition 1.3. $\square$

2.8 The case $|G| = 2$. We end this article by illustrating Theorem 2.7 in the special case where $|G| = 2$, $2k = 0$, and the skew group ring $T = T \ast G$ is simple, or at least $T = \hat{T}G$. In this case, we have $\rho = r.\text{gldim}(R)$ (see 2.6). Moreover, the exact sequence of $S$-modules

\[ 0 \longrightarrow S \xrightarrow{\text{incl.}} R \xrightarrow{\text{Tr}} \mathcal{S} \longrightarrow \mathcal{S} \longrightarrow 0 \]

shows that $\text{pdim}(\mathcal{S}_S) \leq 2$. Therefore, $\sigma \leq r.\text{gldim}(\mathcal{S}) + 2$. Theorem 2.7 now implies that

—either $r.\text{gldim}(S) = \infty$

—or $r.\text{gldim}(S) \leq \max\{r.\text{gldim}(R), r.\text{gldim}(\mathcal{S}) + 2\}$.

For a lower bound, we note that if $\text{Tr}(R) \neq S$, then $\text{pdim}(\mathcal{S}) \geq 2$, because otherwise $\text{Tr}(R)$ would be projective over $S$, and hence $S$ would be an $S$-module direct summand of $R$. But the equality $T = \hat{T}G$ implies that $R = R \cdot \text{Tr}(R)$, which leads to a contradiction, since $\text{Tr}(R) \neq S$.  

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