COMPLETELY PRIME PRIMITIVE IDEALS IN GROUP ALGEBRAS OF FINITELY GENERATED NILPOTENT-BY-FINITE GROUPS

M. Lorenz

Mathematisches Institut
Justus Liebig-Universität
D-6300 Giessen, W. Germany

Introduction.

The complete classification of all irreducible representations of a given finitely generated (f.g.) nilpotent-by-finite group $G$ appears at present inaccessible. But it is at least possible to describe the kernels of all simple left $k[G]$-modules, which are the primitive ideals of the group algebra $k[G]$. In Section 1 we show that the primitive ideals of the group algebra $k[G]$ of a f.g. nilpotent-by-finite group $G$ are maximal (Cor. 1.4). This result was proved previously by A.E. Zalesskij [10] for f.g. nilpotent groups.
Section 2 deals with completely prime primitive ideals of $k[G]$, i.e. primitive ideals $P$ such that the factor $k[G]/P$ has no zero divisors. In Theorem (2.5) it is shown that, in case $k$ is algebraically closed, the centre $Z$ of $G$ controls the set of those completely prime primitive ideals $P$ of $k[G]$ for which the natural map $G \rightarrow U(k[G]/P)$ is injective (they are called faithful): $P = (P \cap k[Z]) k[G]$. This result fails to be true if the condition on $P$ to be completely prime is omitted, even if we replace the centre of $G$ by the FC-centre $\Delta(G)$ (2.3).

Stronger results are obtained for supersoluble groups, i.e. groups having a finite normal series with cyclic factors. In Theorem (3.2) we establish a bijection between the set of faithful completely prime primitive ideals of $k[G]$ and the set $Z_{\text{inj}}^*$ of embeddings of the centre $Z$ of $G$ into the multiplicative group $k^*$ of the algebraic closed field $k$. The main ingredient of its proof is a zero divisor theorem stated as Proposition (3.1): Given a supersoluble group $G$ and a normal subgroup $N$ such that $G/N$ is torsion free. If $I_N$ is a $G$-stable completely prime ideal of $k[N]$, then the extended ideal $I_N k[G]$ of $k[G]$ is completely prime too.

Possibly a similar result can be proved for f.g. nilpotent-by-finite groups, in which case the
results of Section 3 would carry over to f.g. nilpotent-by-finite groups.

Notations. All rings considered contain a unit element. \( U(R) \) denotes the group of invertible elements of the ring \( R \). We write \( Q(R) \) for the classical quotient ring of \( R \) (if it exists). The unspecified word "module" will mean "unitary left module" and "ideal" stands for "two-sided ideal".

The set of prime ideals of \( R \) is denoted by \( \text{Spec}(R) \), the set of primitive ideals by \( \text{Priv}(R) \) and the set of completely prime primitive ideals by \( \text{Priv}_c(R) \).

If \( G \) is a group then \( Z(G) \) will denote its centre and \( \Delta(G) \) its FC-centre (sometimes shortly written \( \Delta \)). Let \( H \leq G \) be a subgroup of \( G \). Then

\[ G_G(H) := \{ g \in G : [H:C_H(g)] < m \} \]

In particular \( G_G(G) = \Delta(G) \).

1. Primitive Ideals.

(1.1) We recall the notion of a crossed product of a group \( G \) over a ring \( R \) (see Bovdi [2]): Given maps \( \alpha : G \to \text{Aut}(R) \), \( \gamma : G \times G \to U(R) \) such that

\[
\gamma(x,y) \gamma(xy,z) = \gamma(x,y)\alpha(x) \gamma(x,yz)\\
\gamma(x,y) \gamma^\alpha(xy) = \gamma^\alpha(y)\alpha(x) \gamma(x,y)
\]
for $r \in R$ and $x,y,z \in G$, the crossed product $R_{\alpha}^Y[G]$ is defined to be the free $R$-module with basis 
\(\{x : x \in G\}\). The multiplication in $R_{\alpha}^Y[G]$ is given by the rule
\[(rx) \cdot (sy) = rs^\alpha(x) \gamma(x,y) xy \quad (r,s \in R; x,y \in G).
\]

$R_{\alpha}^Y[G]$ is an associative ring with unit element $\gamma(1,1)^{-1}$.

(1.2) Lemma. Let $G$ be a finite group and let $R$ be a ring such that $Q(R)$ exists and

\[Q(R) \cong \bigoplus_{i=1}^{n} M_{n_i}(D), \quad D \text{ a skew field.}
\]


**Proof.** First notice that the automorphisms in $\alpha(G) \subset \text{Aut}(R)$ can be extended to $Q = Q(R)$. Thus we have a crossed product $Q_{\alpha}^Y[G]$ and a natural inclusion $R_{\alpha}^Y[G] \subset Q_{\alpha}^Y[G]$. The conditions on $G$ and $Q$ imply that $Q_{\alpha}^Y[G]$ is finite dimensional over $D$. Therefore any $x \in R_{\alpha}^Y[G]$ satisfies a relation

\[d_n x^n + \ldots + d_1 x + d_0 = 0 \quad (d_i \in D).\]

Let $c$ be a common denominator for the elements $d_1 \in D \subset Q : cd_1 = r_i \in R$. Then $r_n x^n + \ldots + r_1 x + r_0 = 0$. 

(1.3) **Proposition.** Let $k$ be a field, $G$ a group and $N$ a normal subgroup of finite index in $G$.

If for every primitive ideal $P$ of $k[N]$ the factor $k[N]/_{P}$ is a simple Noetherian ring then the same is true for every primitive ideal of $k[G]$.

**Proof.** Let $P \in \text{Priv}(k[G])$. Then, by Clifford theory,

$$P_N := P \cap k[N] = \bigcap_{i=1}^{n} x_i^M,$$

where $M$ denotes a primitive and hence maximal ideal of $k[N]$ and $x_i$ are suitable elements of $G$ such that $\{x_i^M : i = 1, 2, \ldots, n\}$ is a complete irredundant set of $G$-conjugates of $M$.

The isomorphism $k[N]/_{P_N} \cong \bigoplus_{i=1}^{n} k[N]/_{x_i^M} \cong n \bigoplus_{i=1}^{n} k[N]/M$ shows that $k[N]/_{P_N}$ is a Noetherian ring. Therefore the finitely generated $k[N]/_{P_N}$-module $k[G]/_{P}$ is Noetherian too.

In order to prove the maximality of $P$ it suffices to consider the image $\overline{P}$ of $P$ in $k[G]/_{P_N}k[G]$.

The ring $k[G]/_{P_N}k[G]$ carries the structure of a crossed product:

$$k[G]/_{P_N}k[G] \cong (k[N]/_{P_N})^{Y}[G/N].$$

Furthermore, as was shown above,

$$k[N]/_{P_N} \cong \bigoplus_{i=1}^{n} R_i$$

is a finite direct sum of pairwise isomorphic simple Noetherian rings.
Hence \( Q(k[N]/P_N) \cong \bigoplus_{i=1}^{n} \Sigma \oplus Q(P_i) \cong \bigoplus_{i=1}^{n} \Sigma \oplus M_i(D) \), D a skew field. Thus Lemma (1.2) can be applied to \( k[G]/P_N k[G] \).

Suppose that \( \overline{Q} \) is a maximal ideal of \( k[G]/P_N k[G] \) such that \( \overline{F} \not\subseteq \overline{Q} \). By (Goldie [5], Theorem 3.9) \( \overline{Q} \) contains an element \( c \) that is regular modulo \( \overline{F} \). Lemma (1.2) yields a linear relation

\[
r_n r_n + \ldots + r_1 c + r_0 \in \overline{F} \quad (r_i \in k[N]/P_N).
\]

Choose \( n \) to be minimal. Then \( r_0 \in \overline{Q} \), but \( r_0 \notin \overline{F} \), for otherwise \( (r_n c^{n-1} + \ldots + r_1) c \in \overline{F} \) and, since \( c \) is regular modulo \( \overline{F} \), \( r_n c^{n-1} + \ldots + r_1 \in \overline{F} \) contradicting the minimality of \( n \). Thus we have seen that

\( \overline{Q} \cap k[N]/P_N \neq 0 \). Furthermore \( \overline{Q} \cap k[N]/P_N \) is obviously stable under conjugation with elements of \( G \).

But this is impossible since, by assumption on \( M \),

the ideal \( P_N = \bigcap_{i=1}^{n} M_i \) is maximal among the \( G \)-stable ideals of \( k[N] \). This contradiction proves the maximality of \( P \).

(1.4) Since group algebras of f.g. nilpotent groups are Noetherian (P. Hall [6], Theorem 1) and their primitive ideals are maximal (Zalesskij [10], Theorem 3) we immediately obtain the following
Corollary. Let G be a f.g. nilpotent-by-finite group and let k be a field. Then every primitive ideal of the group algebra k[G] is maximal.

2. Completely Prime Primitive Ideals.

(2.1) An ideal I of the group algebra k[G] is called annihilator free if for every infinite subgroup X ≤ G and every element a ∈ k[G] the inclusion αω(k[X]) ⊆ I implies that a ∈ I. Here ω(k[X]) denotes the augmentation ideal of k[X]. - The following theorem is due to A.E.Zalesskij. For a proof see [9], 11.4.2.

Theorem (Zalesskij). Let G be a group and let H be a nilpotent normal subgroup of G. Set W := DG(H). If I is an ideal of k[G] such that I ∩ k[W] is annihilator free then I = (I ∩ k[W]) k[G].

(2.2) An ideal of the group algebra k[G] is called faithful if the natural map G → U(k[G]/I) is injective, i.e. if G(I) := G ∩ (1+I) = <1>. It is completely prime if the factor k[G]/I has no zero divisors. - From (2.1) we derive the following
**Corollary.** Let $G$ be a nilpotent-by-finite group and let $k$ be a field. If $P$ is a faithful ideal of $k[G]$ such that $P \cap k[\Delta]$ is completely prime in $k[\Delta]$ then $P = (P \cap k[\Delta]) k[G]$.

**Proof.** Let $H$ be a nilpotent normal subgroup of finite index in $G$. Then $\mathbb{D}_G(H) = \Delta(G)$. Furthermore the conditions on $P$ obviously imply that $P \cap k[\Delta]$ is annihilator free. Now Theorem (2.1) yields the result.

(2.3) **Remark.** It follows from (2.2) that $\Delta(G)$ controls the set of faithful completely prime ideals of $k[G]$. It is a further consequence of Zaleskij's Theorem (2.1) that, in case $G$ is a f.g. nilpotent group, the FC-centre $\Delta(G)$ even controls the set of all faithful prime ideals of $k[G]$. The following example shows that this result does not extend to group algebras of f.g. nilpotent-by-finite groups:

Let $A = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle$ be a free abelian group of rank 4 and let $t \in \text{Aut}(A)$ be given by $a^t = a$, $b^t = b^{-1}a$, $c^t = cb$, $d^t = d^{-1}c$. Then the semidirect product $G = A \rtimes A^t$ is nilpotent-by-finite, in fact supersoluble, and $\Delta = \Delta(G) = \langle a, b \rangle$. Let $k$ be a field containing an element $\xi \in k^\ast$ of infinite order and define $f \in \text{Hom}(A, k^\ast)$ by $f(a) = f(b) = \xi$, $f(c) = f(d) = 1$. 


The field $k$ is a $k[A]$-module via $f$ and the induced $k[G]$-module $V := k[G] \otimes k_f$ is easily seen to be irreducible. Furthermore $G$ acts faithfully on $V$. Therefore the annihilator $P := \text{Ann}_{k[G]} V$ is a faithful primitive ideal of $k[G]$.

Direct calculation shows that $a := d^{-1}c(b-1) - dc^{-1}(b-1)$ is an element of $P$ that is not contained in $(P \cap k[\Delta]) k[G]$.

(2.4) Lemma. Let $G$ be a polycyclic-by-finite group, $k$ a field and $P$ a primitive ideal of $k[G]$. Then:

a) The centre $Z(k[G]/P)$ is an algebraic extension field of $k$.

b) $k[\Delta]/P \cap k[\Delta]$ is finite dimensional over $k$.

Proof. Part a) is a consequence of P. Hall's work in [7] (cf. [8], Theorem 1.2). We proceed to prove b). Let $A := Z(\Delta)$. Then $[\Delta : A] < \infty$ and $A < G$.

The orbits of the action of $G$ (by conjugation) on $R := k[A]/P \cap k[A]$ are finite. Therefore, if $R^G$ denotes the set of $G$-invariant elements of $R$, $R^G$ is a finitely generated $R^G$-module and $R^G$ is a finitely generated $k$-algebra (see [1], S.33, Theorem 2). Since $R^G = Z(k[G]/P) \cap R$, it follows from part a) that $R^G$ is finite dimensional over $k$. Therefore $\dim_k R < \infty$ and finally $\dim_k k[\Delta]/P \cap k[\Delta] \leq \dim_k k[\Delta]/(P \cap k[\Delta])k[\Delta] = \dim_k R \cdot [\Delta : A] < \infty$. 

(2.5) Theorem. Let \( G \) be a f.g. nilpotent-by-finite group and let \( k \) be an algebraically closed field. If there exists a faithful completely prime primitive ideal \( P \) of \( k[G] \) then:

a) \( \Delta(G) = Z(G) \), \( G/Z(G) \) is torsion free and \( Z(G) \) is embedded into the multiplicative group \( k' \) of \( k \).

b) \( P = (P \cap k[Z(G)]) k[G] \).

Proof. In view of (2.2) it suffices to prove assertion a). Let \( P \) be a faithful completely prime primitive ideal of \( k[G] \). Then \( P_\Delta = P \cap k[\Delta] \) is completely prime in \( k[\Delta] \). Furthermore, by Lemma (2.4), \( k[\Delta]/P_\Delta \) is finite dimensional over \( k \). Since \( k \) is algebraically closed, it follows that \( k[\Delta]/P_\Delta \cong k \).

Thus the elements of \( \Delta \) are central modulo \( P \). Therefore \( \Delta(G) = Z(G) \), since \( P \) was assumed to be faithful. By the same reason the natural map \( Z(G) \to U(k[Z(G)])/P \cap k[Z(G)]) = k' \) is an embedding.

In order to prove that \( G/Z(G) \) is torsion free we write \( P = (P \cap k[Z(G)]) k[G] \) (2.2). Hence \( k[G]/P \cong (k[Z(G)])/(P \cap k[Z(G)]) \cong [G/Z(G)] \cong k'[G/Z(G)] \), a twisted group ring of \( G/Z(G) \) over \( k \). The fact that \( k'[G/Z(G)] \) has no zero divisors forces \( G/Z(G) \) to be torsion free. This can be seen by an obvious modification of the familiar argument used in the case of group algebras.

(3.1) Recall that a group $G$ is said to be supersoluble if $G$ has a finite series
$\langle 1 \rangle = G_0 \leq G_1 \leq ... \leq G_n = G$ with $G_i \triangleleft G$ and each
factor $G_{i+1}/G_i$ cyclic. The number of infinite cyclic
factors $G_{i+1}/G_i$ appearing in such a series is an
invariant of the group $G$, called the Hirsch number
$h(G)$ of $G$. The following result generalizes
a zero divisor theorem of Formanek [4].

Proposition. Let $G$ be a group and let $N$ be a
normal subgroup of $G$ such that $G/N$ is torsion free
supersoluble. Let $k$ be a field and let $I_N$ be a
completely prime ideal of $k[N]$ such that $I_N^g = I_N$
for all $g \in G$. Then the ideal $I := I_N k[G]$ of
$k[G]$ is completely prime.

Proof. We write $\overline{G} : = G/N$ and argue by induction
on the Hirsch number $h(\overline{G})$ of $\overline{G}$. The case
$h(\overline{G}) = 0$, i.e. $G = N$, being trivial we assume
that $h(\overline{G}) > 0$ and that for each subgroup $H$ of $G$
such that $N \triangleleft H$ and $h(H) < h(G)$ the ideal
$I_H := I_N k[H]$ of $k[H]$ is completely prime.

It is not hard to see that every infinite
supersoluble group has an infinite cyclic or infinite
dihedral homomorphic image (Formanek [4], Lemma 3).
We treat the two cases separately.

Case 1. \( \overline{G} \) has a normal subgroup \( \overline{K} \) such that \( \overline{G}/\overline{K} \cong \mathbb{Z} \).

Then, if \( K \) denotes the inverse image of \( \overline{K} \) in \( G \), \( h(K) < h(G) \) and, by induction, the ideal \( I_K := I_N k[K] \) of \( k[K] \) is completely prime. Now the familiar degree argument (cf. [9], 13.1.7, 13.1.9) can be used to conclude that \( k[G]/I = k[G]/I_K \) and \( k[G] = (k[K]/I_K) [\mathbb{Z}] \) has no zero divisors. Hence the rest of the proof deals with

Case 2. \( \overline{G} \) has a normal subgroup \( \overline{K} \) such that \( \overline{G}/\overline{K} = \mathbb{Z}_2 \ast \mathbb{Z}_2 \), the infinite dihedral group.

We have an exact sequence

\[ 1 \rightarrow K \rightarrow G \rightarrow A \ast B \rightarrow 1, \]

where \( K \) denotes the inverse image of \( \overline{K} \) in \( G \) and \( A, B \) are isomorphic to \( \mathbb{Z}_2 \). Set \( N_1 := p^{-1}(A), N_2 := p^{-1}(B) \).

Then \( G \) is isomorphic to \( N_1 \ast_{K} N_2 \), the free product of \( N_1 \) and \( N_2 \) amalgamating \( K \). Furthermore obviously \( N \leq K \leq N_1, N_2 \) and \( h(K), h(N_1), h(N_2) < h(G) \).

Using the induction hypothesis we conclude that

\[ R_1 := k[N_1]/I_{N_1}, \quad R_2 := k[N_2]/I_{N_2} \quad \text{and} \quad R := k[K]/I_{K} \]

have no zero divisors. It is easy to
see (cf. the argument in [9], 13.3.5 (ii)) that
\[ S : = R \setminus \{0\} \text{ satisfies the right Ore condition in} \]
\[ R_i \quad (i = 1,2). \quad \text{Hence we may consider the localized} \]
\[ \text{rings } Q_1 : = (R_1)_S \text{ and } Q_2 : = (R_2)_S. \quad Q_1 \text{ and} \]
\[ Q_2 \text{ both contain the division ring } Q : = R_S \text{ and} \]
\[ \text{the both have no zero divisors}. \quad \text{Now Cohn's theorem} \]
([3], Theorem 3.2) implies that the free product
\[ Q_1 * Q_2 \text{ of } Q_1 \text{ and } Q_2 \text{ amalgamating } Q \text{ exists} \]
\[ \text{and has no zero divisors}. \quad \text{Therefore the proof will} \]
\[ \text{be finished as soon as the following assertion will} \]
\[ \text{be established}. \]

Claim. \( k[G]/I \) is isomorphically embedded into
\[ Q_1 Q Q_2. \]

Pf. Consider the following commutative diagram:

\[ k[K] \quad \longrightarrow \quad Q \]
\[ k[N_1] \quad \quad \quad \quad \quad k[N_2] \quad \longrightarrow \quad Q_2 \quad Q_1 \]
\[ k[G] = k[N_1 \ast N_2] \quad \longrightarrow \quad Q_1 \ast Q_2 \]

Here all nonspecified maps are the natural \( k \)-algebra
homomorphisms (f.i. \( k[K] \rightarrow Q \) is the composite of
by the universal property of \( k[N_1 * N_2] \) as a push out. We show that \( I \subseteq \text{Ker} \phi \): An element \( a \in I \) can be written in the form \( a = \sum a_i g_i \), where \( a_i \in k[K] \) and \( g_i \in G \) are in different cosets modulo \( K \).

The equality \( I = (I \cap k[K]) k[G] \) shows that \( a_i \in I \).

Therefore \( \phi(a) = \sum \phi(a_i) \phi(g_i) = 0 \), since \( \phi | I \cap k[K] = 0 \).

It remains to show that the induced map

\[
\psi: k[G]/I \rightarrow Q_1 \otimes Q_2
\]

is injective: Fix a transversal \( \varphi_i U \{1\} \) for \( K \) in \( N_i \) \((i = 1, 2)\). Then a transversal \( \varphi \) for \( K \) in \( G \) is given by \( 1 \) together with the products \( s_{i_1} s_{i_2} \cdots s_{i_m} \) such that \( s_{i_j}, s_{i_{j+1}} \) are in different \( \varphi_i \). The elements \( x + I \ (x \in \varphi) \) generate \( k[G]/I \) as a left \( (k[K]/I \cap k[K]) \)-module.

We regard \( Q_1 \otimes Q_2 \) as a left vector space over \( Q \).

In order to prove the injectivity of \( \psi \) it suffices to check that the elements \( \psi(x + I) \ (x \in \varphi) \) of \( Q_1 \otimes Q_2 \) are linearly independent over \( Q \). But since \( I_{N_1} = (I \cap k[K]) k[N_1] \), the elements \( \psi(x + I_{N_1}) = x + I_{N_1} \ (x \in \varphi U \{1\}) \) are easily seen
to be linearly independent over $\mathbb{Q}$ in $Q_i$ ($i = 1, 2$).

Extend \{\{x + I_{N_i} : x \in \mathfrak{O}_i \cup \{1\}\} to a basis $B_i \cup \{1\}$ of $Q_i$ as a left vector space over $\mathbb{Q}$. Then the set of monomials on the alphabet $B_1 \cup B_2$ with consecutive letters in different factors forms, together with 1, a $\mathbb{Q}$-basis for $Q_1 \ast Q_2$. In particular the elements $\psi(x + I)$ ($x \in \mathfrak{O}$) are linearly independent over $\mathbb{Q}$, thus proving the injectivity of $\psi$.

(3.2) In order to formulate the next result we need some more notations: Let $N \triangleleft G$. We write $\text{Priv}_N(k[G]) = \{I \in \text{Priv}(k[G]): G \cap (1+I) = N\}$. $\text{Privc}_N(k[G])$ and $\text{Spec}_N(k[G])$ are defined analogously. $\phi_N$ denotes the canonical homomorphism $\phi_N : k[G] \rightarrow k[G/N]$. Let $f \in \text{Hom}(G, k')$. Then $k[f] : k[G] \rightarrow k$ is the $k$-algebra-homomorphism given by $k[f](\Sigma k_1 g_1) = \Sigma k_1 f(g_1)$.

Theorem. Let $G$ be a supersoluble group and let $k$ be an algebraically closed field. 

a) Let $N \triangleleft G$. Then $\text{Privc}_N(k[G]) \neq \emptyset$ if and only if each of the following conditions holds:
(1) $\Delta(\bar{G}) = Z(\bar{G})$ \quad (\bar{G} : = G/N)

(2) $\bar{G}/\bar{Z}$ is torsion free \quad ($\bar{Z} : = Z(\bar{G})$)

(3) $\mathbb{Z}^{*}_{\text{inj}} = \{ f \in \text{Hom}(\bar{Z}, k') : f \text{ is injective} \} \neq \emptyset$

b) Let $\text{Priv}_{N}(k[G]) \neq \emptyset$. Then all prime ideals in $\text{Spec}_{N}(k[G])$ are completely prime. Furthermore there is a one-to-one correspondence

$$
\mathcal{F}_{N}: \mathbb{Z}^{*}_{\text{inj}} \longrightarrow \text{Priv}_{N}(k[G])
$$

$$
f \longrightarrow \bar{z}^{-1}(\ker k[f], k[G])
$$

**Proof.** The necessity of the conditions (1), (2), (3) has been settled in (2.5a). So, conversely, assume that (1), (2), (3) do hold. Choose $f \in \mathbb{Z}^{*}_{\text{inj}}$. We claim that the ideal $\bar{F} : = \ker k[f], k[\bar{G}]$ of $k[\bar{G}]$ is faithful, completely prime and maximal.

$G(\bar{F}) = \bar{G} \cap (1 + \bar{F}) = \langle 1 \rangle$: In case $G(\bar{F}) = \langle 1 \rangle$ it would follow that $G(\bar{F}) \cap \Delta(\bar{G}) = \langle 1 \rangle$. Hence, by (1),

$$
\langle 1 \rangle \neq \bar{Z} \cap G(\bar{F}) = \{ z \in \bar{Z} : z-1 \in \bar{F} \} = \{ z \in \bar{Z} : z-1 \in \ker k[f] \} = \ker f = \langle 1 \rangle
$$

a contradiction.

$\bar{F}$ is completely prime: This follows immediately from (3.1), together with condition (2).

$\bar{F}$ is maximal: Let $\bar{M}$ be a maximal ideal of $k[\bar{G}]$ containing $\bar{F}$. Then $\bar{M} \cap k[\bar{Z}] = \bar{F} \cap k[\bar{Z}] = \ker k[f]$. Therefore, using condition (1), one concludes that $G(\bar{M}) = \langle 1 \rangle$. Corollary (2.2) yields the equality $\bar{M} = (\bar{M} \cap k[\bar{Z}])k[\bar{G}]$ and hence $\bar{M} = \bar{F}$. 
The ideal $\mathcal{I}_N(f) = \phi_N^{-1}(\overline{f})$ is completely prime and maximal, since $k[G]/\mathcal{I}_N(f) \cong k[\overline{G}]/\overline{f}$.

Furthermore, the equality $G(\overline{f}) = <1>$ implies that $G(\mathcal{I}_N(f)) = N$. Hence $\mathcal{I}_N(f) \in \text{Priv}_{N}(k[G])$, and (1), (2), (3) are shown to be sufficient. We have also shown that $\text{Im} \mathcal{I}_N \subseteq \text{Priv}_{N}(k[G])$. Since $\mathcal{I}_N$ is obviously injective we proceed to prove the surjectivity: Let $P \in \text{Priv}_{N}(k[G])$. Then $\omega(k[N])k[G] \subseteq P$ and $\overline{P} = P/\omega(k[N])k[G] \in \text{Priv}_{<1>}(k[\overline{G}])$.

Since $k$ is algebraically closed, Lemma (2.4a) yields an isomorphism $k[\overline{Z}]/\overline{P} \cong k$. The induced group homomorphism $P : \overline{Z} \rightarrow U(k[Z]/\overline{P} \cap k[Z]) = k'$ is injective, for $G(\overline{f}) = <1>$. Furthermore, by (2.2) together with condition (1), we conclude that $\overline{P} = (\overline{P} \cap k[Z])k[G] = \text{Ker} k[f].k[G]$. Finally $P = \phi_N^{-1}(\overline{f}) = \mathcal{I}_N(f)$.

It remains to show that all prime ideals in $\text{Spec}_{N}(k[G])$ are completely prime: Let $P \in \text{Spec}_{N}(k[G])$.

Then $\overline{P} = P/\omega(k[N])k[G] \in \text{Spec}_{<1>}(k[\overline{G}])$ and, by (2.2) and condition (1), $\overline{P} = (\overline{P} \cap k[Z])k[\overline{G}]$.

It follows from (3.1) and condition (2) that $\overline{P}$, and hence $P$, is completely prime.

(3.3) Remarks. In case $k'$ is locally finite, conditions (1), (2), (3) are easily seen to be
equivalent to the requirement that \( G = G/N \) be finite cyclic. If \( k' \) is not locally finite then it contains free abelian subgroups of arbitrarily large rank. Hence (3) holds if and only if the torsion subgroup of \( \bar{G} \) is finite cyclic.

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