PRIME IDEALS IN FIXED RINGS II

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Recently there has been some interest in so-called "Additivity Principles" [2] which, for a ring extension $S \subseteq R$ and a prime ideal $P$ of $R$, relate the Goldie rank of $R/P$ to the Goldie ranks of $S/Q$, for all primes $Q$ of $S$ which are minimal over $P \cap S$.

In this note, we prove such a theorem for the ring extension $R^G \subseteq R$, where $R^G$ is the fixed subring of a finite group $G$ acting as automorphisms of $R$, such that $|G|^{-1} \subseteq R$. Our result improves the bound on Goldie ranks obtained in [4].

We also include a few additional remarks on prime ideals in fixed rings.

We first require, form [4], some facts about the relationship between prime ideals in $R$ and $R^G$. For $P$ a prime ideal of $R$, let

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Theorem A [4, Proposition 4.2] Let $R$ be a ring, and $G$ a finite group acting as automorphisms on $R$ such that $|G|^{-1} \in R$. Then

1) $\mathfrak{a}$ is a set of semiprime ideals of $R$, and is in one-to-one correspondence with the set of $G$-prime ideals of $R$.

2) Any prime $Q$ of $R^G$ is minimal over some unique $I \in \mathfrak{a}$.

3) There are only finitely many primes of $R^G$ minimal over any $I \in \mathfrak{a}$, say $Q_1, \ldots, Q_m$, where $m \leq |G|$. Moreover, $I = \bigcap_{i=1}^m Q_i$.

Two primes $Q_1, Q_2$ of $R^G$ are said to be equivalent if they are minimal over the same $I \in \mathfrak{a}$. For properties of this equivalence, see [4, Proposition 3.5]; in particular, equivalent primes have the same height.

We also require the additivity principle mentioned at the beginning. For a ring $C$, $\text{rk}(C)$ denotes the Goldie rank of $C$ (also called the Goldie dimension).

Theorem B [2, Lemma 3.8]. Let $A \subseteq B$ be Artinian rings with the same unit element. Let $P$ be a prime ideal of $B$, and let $Q_1, \ldots, Q_r$ be the primes of $A$ which are minimal over $P \cap A$. Then there exist positive integers $z_1, \ldots, z_r$ such that
We are now able to prove our main theorem.

Theorem: Let $R$ be a ring, and $G$ a finite group acting as automorphisms of $R$ with $|G|\cdot R$. Let $P$ be a prime ideal of $R$, and say that $P \cap R^G = Q_1 \cap Q_2 \cdots \cap Q_m$, where the $\{Q_i\}$ are prime ideals of $R^G$. If $R/P$ is a Goldie ring, then:

1) $R^G/Q_i$ is a Goldie ring, all $i = 1, \ldots, n$

2) There exist positive integers $z_1, \ldots, z_m$ such that

$$rk(R/P) = \sum_{i=1}^m z_i \cdot rk(R^G/Q_i)$$

proof: Since $R/P$ is Goldie, $R/P^G$ is also Goldie for each $g \in G$, since $R/P \cong R/p^G$. Thus $R = R/ \cap p^G$ is Goldie since it is a subdirect product of the $\{R/p^G\}$. $R$ has an induced $G$-action, since $\cap p^G$ is $G$-stable, and moreover, $R^G = \bar{R}^G$ since $|G| \cdot R$ (the mapping $\phi(x) = |G|^{-1} \sum_{g \in G} x^g$ is a projection of $R$ onto $R^G$). By passing to $\bar{R}$ we may assume that $R$ is Goldie, $P \cap R^G = (0)$, and $(0) = Q_1 \cap \cdots \cap Q_m$, where $\{Q_i\}$ are the minimal primes of $R^G$.

Now by a theorem of Kharchenko [3], $R$ being Goldie implies that $R^G$ is also Goldie; moreover if $T$ is the set of regular elements of $R^G$, then the elements of $T$ are regular in $R$ and $Q(R) = RT^{-1} [1]$, where $Q(R)$ is the semi-simple Artinian quotient ring of $R$.

Since $R^G$ is Goldie and $Q_i$ is a minimal prime, $R^G/Q_i$ is also Goldie, proving 1).
Let $A = Q(R^G)$ and $B = Q(R)$. Since $P \cap T = \emptyset$, $PT^{-1}$ is prime in $B$ and $\text{rk}(B/PT^{-1}) = \text{rk}(R/P)$. Also, $Q_1T^{-1}, \ldots, Q_mT^{-1}$ are precisely the primes of $A = R^{G_1T^{-1}}$, and $\text{rk}(A/Q_1T^{-1}) = \text{rk}(R^G/Q_1)$. The proof is now finished by applying Theorem B to the Artinian rings $A \subseteq B$. 

We first give an example to show that some hypothesis about $|G|$ is required.

**Example 1:** A PI ring $R$ of characteristic $p \neq 0$, and $G \subseteq \text{Aut}(R)$ of order $p$, and a prime ideal $P$ of $R$ so that $R/P$ is Goldie of rank 3 but $P \cap R^G = Q$ is a prime ideal of $R^G$ of rank 2. Thus the additivity principle does not hold.

**proof:** Let $k$ be a field of characteristic $p \neq 0$ and let $k[x,y]$ be the commutative polynomial ring over $k$. Define $\phi : k[x,y] \rightarrow k[x,y]$ by $\phi(y) = x$, $\phi(x) = 0$. Let $B = M_3(k[x,y])$ and let

$$A = \left\{ \begin{pmatrix} a & b \\ xc & d \\ 0 & 0 \end{pmatrix} \bigg| a, b, c, d \in k[x,y] \right\}.$$  

Note that $A \subseteq B$ with the same unit element, and that $A$ is prime with $\text{rk}(A) = 2$.

Now let $R = \left\{ \begin{pmatrix} a & b_1 \\ 0 & b_2 \end{pmatrix} \bigg| a \in A, b_1, b_2 \in B \right\}$. Let $g \in \text{Aut}(R)$ be given by conjugation by

$$S = \begin{pmatrix} I_3 & I_3 \\ 0 & I_3 \end{pmatrix} \in R; \text{ since } S^p = I, g \text{ has order } p. \text{ Let } G = \langle g \rangle;$$
then \( R^G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, a \in A, b \in B \right\} \). Now let \( P = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, a \in A, b \in B \right\} \), a prime ideal of \( R \). \( R/P \simeq B \), which is Goldie of rank 3. However 
\( P \cap R^G = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, b \in B \right\} = Q \), a prime, and \( R^G/Q \simeq A \), which is Goldie of rank 2.

Our next example shows that the conclusions of the theorem do not hold if one begins with a prime \( Q \) of \( R^G \). Moreover, it provides an example of two equivalent primes of \( R^G \) which have different depths (as was noted above, equivalent primes always have the same height).

**Example 2:** A prime ring \( R \) which is not Goldie, and \( G \subseteq \text{Aut}(R) \) with \( |G|^{-1} \in R \) such that \( R^G \) has two minimal primes \( Q_1, Q_2 \), with \( Q_1 \cap Q_2 = (0) \), such that \( R^G/Q_1 \) is Goldie but \( R^G/Q_2 \) is not. Moreover, \( Q_1 \) is maximal but \( Q_2 \) is not.

**proof:** Let \( k \) be a field containing a primitive \( n \)th root of 1, say \( \alpha \), for some \( n > 1 \). Let \( V \) be a countable dimensional vector space over \( k \), and let \( R = \text{Hom}_k(V, V) \). Choose a basis \( \{v_1, v_2, \ldots, v_n, \ldots\} \) for \( V \), and define \( T \in R \) by \( Tv_1 = \alpha v_1 \) and \( Tv_i = v_{i-1} \), \( i \geq 2 \). Let \( g \in \text{Aut}(R) \) be defined to be conjugation by \( T \). Then \( G = \langle g \rangle \) has order \( n \), and \( R^G = k v_1 \oplus R' \), where \( R' = \text{Hom}_k(V', V') \) and \( V' \) is the subspace with basis \( \{v_i, i \geq 2\} \).

Let \( Q_1 = (0, R') \) and \( Q_2 = (kv_1, 0) \). Since \( R^G/Q_1 \cong k, Q_1 \) is maximal and \( R^G/Q_1 \) is Goldie. However, \( R^G/Q_2 \cong R' \), which is not Goldie. Finally, let \( S \) denote the socle of \( R' \). Since \( R'/S \) is simple, \( Q = (kv_1, S) \) is a maximal ideal of \( R^G \). Thus \( Q_2 \) is not maximal. \( \square \)
We remark, however, that if $|G|^{-1} \in R$ and $Q$ is a prime ideal of $R^G$ such that for every prime $Q_i$ equivalent to $Q$, $R^G/Q_i$ is Goldie, then $R/P$ is Goldie for any prime $P$ with $P \cap R^G = \bigcap_{g \in G} P = \bigcap_{C_1 \subseteq Q} Q_i$, and the conclusion of the Theorem holds. For in that case, one may pass to $\tilde{R} = R/\cap P^G$ and $\tilde{R}^G = R^G/\cap Q_i$ as before. Since $\tilde{R}^G$ is Goldie, $\tilde{R}$ is Goldie by [3], and so $\tilde{R}/\tilde{P} \cong R/P$ is Goldie since $\tilde{P}$ is a minimal prime of $\tilde{R}$.

We close with one final example to illustrate the pathology which can occur when a finite group acts on a non-commutative ring and $|G|^{-1} \notin R$.

**Example 3:** A prime ring $R$, and a finite group $G \subseteq \text{Aut}(R)$, such that $R$ has an infinite number of primes $P$, and $P \cap R^G = (0)$ for all $P$.

**Proof:** We use an example of G. Bergman. Let $k$ be a field of characteristic $p \neq 0$, which contains a primitive $n$th root of 1, say $a$, for some $n > 1$. Let $k[x,y]$ denote the free algebra in $x$ and $y$, and let $R = M_2(k[x,y])$. Let $G$ be the group generated by conjugation by $(1 \ x), (1 \ y), \text{and } (a \ 0, 0 \ 1)$. Then $|G| = np^2$, and $R^G = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \mid a \in k \right\}$. Certainly $R$ has infinitely many primes, as $k[x,y]$ does, and any prime satisfies $P \cap R^G = (0)$.

The situation in the above example cannot occur if either $R$ is commutative or $|G|^{-1} \notin R$, for in those cases $G$ is transitive on the set of primes which have a common intersection with $R^G$ (that is, $P_1 \cap R^G = P_2 \cap R^G$ implies $P_2 = P_1^g$, for some $g \in G$).
References


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