Let \( g \) be a finite-dimensional solvable Lie algebra over an algebraically closed field \( k \) of characteristic zero and let \( U = U(g) \) denote the enveloping algebra of \( g \). The following problem is still open: given any two primes \( P \subseteq Q \) of \( U \), do all saturated chains of prime ideals \( P = P_0 \subseteq P_1 \subseteq \ldots \subseteq P_s = Q \) have the same length \( s \)? In short, is \( U \) catenary? For abelian Lie algebras, this is of course classical, for in this case \( U \) is just an ordinary polynomial ring in finitely many commuting indeterminates over \( k \). More generally, the answer is known to be positive if \( g \) is nilpotent. In fact, homological techniques yield the result in this case (Malliavin [7], Levasseur [6]) or, alternatively, a proof can be based on the fact that \( U \) is catenary whenever the so-called Dixmier map is bicontinuous for \( g \) (Lorenz and Rentschler: see [7; Proposition 3.2]). Because of the latter, the catenarity problem, in addition to being fascinating in its own right, can also be viewed as a test for the long standing conjecture that the Dixmier map is indeed always bicontinuous (for solvable Lie algebras). We remark that, in general, enveloping algebras of non-solvable Lie algebras are not catenary, as can be seen from the diagrams in [1; p. 39], for example.

It is an easy consequence of Corollary 6 below, originally due to Tauvel [8], that \( U \) is catenary if and only if the following technical condition is satisfied:

\[ (*) \text{ for any two primes } P \subseteq Q \text{ of } U \text{ one has } \]
\[ \text{ht}(Q/P) = d(U/P) - d(U/Q). \]

Here, as usual, \( \text{ht}(Q/P) \) denotes the height of the prime ideal \( Q/P \) of \( U/P \), and \( d(.) \) denotes Gelfand–Kirillov-dimension over \( k \). (See [2] for the definition and basic facts concerning Gelfand–Kirillov-dimension.) The case of \( (*) \) when \( P = 0 \) has been established by Tauvel [6]. In this note, we consider a special class of prime ideals in \( U \) that we shall call extended primes. In Proposition 5, we show that the formula in \( (*) \) holds whenever \( P \) is extended from an ideal \( \mathfrak{rj} \) of \( g \) such that \( U(\mathfrak{rj}) \) is catenary. This does in particular apply to the zero ideal and thus includes Tauvel’s result. Proposition 5 can also be used to establish catenarity for certain almost algebraic Lie algebras (Corollary 8).

The above notation will be kept throughout. In particular, \( g \) will always denote a finite-dimensional solvable Lie algebra over an algebraically closed field \( k \) of characteristic zero. The assumption that \( k \) be algebraically closed is not crucial but should help to clarify the exposition. For a detailed study of field extensions in the present context we refer to [9].

We recall a few general facts that will be used freely in the sequel. Let \( \mathfrak{h} \) be an ideal of \( g \). Then \( U(\mathfrak{h}) \) is a subalgebra of \( U = U(g) \), and \([g, U(\mathfrak{h})] \subseteq U(\mathfrak{h})\). An ideal \( I \) of \( U(\mathfrak{h}) \) is called \( g \)-stable if \([g, I] \subseteq I\). In this case, \( I_1 = IU \) is an ideal of \( U \) with \( I_1 \cap U(\mathfrak{h}) = I \). Moreover, if \( I \) is prime then so is \( I_1 \). Conversely, if \( J \) is an ideal of \( U \),
then $J \cap U(\mathfrak{h})$ is a $g$-stable ideal of $U(\mathfrak{h})$ which is prime if $J$ is prime. All this holds even for non-solvable Lie algebras, and we refer the reader to [4; Chapter 3] for the details.

**Definition.** Let $\mathfrak{h}$ be an ideal of $g$. An ideal $J$ of $U = U(g)$ is said to be extended from $\mathfrak{h}$ if and only if $J = (J \cap U(\mathfrak{h}))U$. Also, $J$ is called extended if $J$ is extended from a suitable proper ideal of $g$.

Without proof, we remark that if $J$ is extended from $\mathfrak{h}_1$ as well as from $\mathfrak{h}_2$, then $J$ is extended from $\mathfrak{h}_1 \cap \mathfrak{h}_2$. In particular, there exists a unique smallest ideal of $g$ from which $J$ is extended. These remarks will not be used in the following, nor will part (ii) of the following lemma, which is included here because it seems interesting in its own right.

**Lemma 1.** Let $P$ be a prime ideal of $U$.

(i) If the semicentre of $A = U/P$ is strictly larger than the centre of $A$, then $P$ is extended. More precisely, let $a$ be a nonzero element of $A$ and $\lambda$ a (nonzero) linear functional on $g$ such that $[x, a] = \lambda(x)a$ for all $x \in g$. Then $P$ is extended from the ideal $\ker(\lambda)$ of $g$.

(ii) If $P$ is primitive, then $P$ is either maximal or extended.

**Proof.** Assertion (i) is [4; 3.3.8]. For (ii), let $P$ be a primitive ideal of $U$, and assume that $P$ is not extended. By part (i), the latter implies that the semicentre of $A = U/P$ is equal to the centre, $Z$, of $A$. The primivity of $P$ implies that $Z = k$. On the other hand, by Lie's theorem, any nonzero ideal of $A$ intersects the semicentre of $A$ nontrivially and hence contains a unit, by the foregoing. Therefore, $A$ is simple, and (ii) follows.

The next lemma essentially rests on the following noncommutative version of Krull's principal ideal theorem, due to Jategaonkar [5]. Let $R$ be a right Noetherian ring with a nonunit $r$ satisfying $rR = Rr$. Then any prime of $R$ minimal over $Rr$ has height at most 1. For a generalization of this result see [3; Theorem 4.6].

**Lemma 2.** Let $\mathfrak{h}$ be an ideal of $g$ and let $P \subseteq Q$ be $g$-stable primes in $U(\mathfrak{h})$. If there exists a prime ideal $T$ with $P \subseteq T \subseteq Q$, then $T$ can be chosen to be $g$-stable.

**Proof.** Since $P$ and $Q$ are $g$-stable, $g$ acts on $Q/P$ via the adjoint action on $U(\mathfrak{h})$. In fact, $Q/P$ is the union of the finite-dimensional $g$-subspaces $Q \cap U_n/P \cap U_r$, where $\{U_n | n \geq 0\}$ denotes the canonical filtration of $U(\mathfrak{h})$. Hence, by Lie's theorem, there exists $0 \neq r \in Q/P$ and $\lambda \in g^*$ such that $[x, r] = \lambda(x)r$ for all $x \in g$, or $(x + P)r = r((x + P) + \lambda(x))$. Since the elements $x + P$, with $x \in g$, generate $R = U(\mathfrak{h})/P$ as a $k$-algebra, we see that $r$ satisfies $rP = rR$. Set $I = rR$. Then $I$ is a $g$-stable ideal of $R$, and $I \subseteq Q/P$. Therefore, $Q/P$ contains a minimal covering prime, $X$, of $I$ in $R$. By [4; 3.3.2], any prime of $R$ minimal over $I$ is $g$-stable, as $I$ is. In particular, $X$ is $g$-stable and, moreover, Jategaonkar's theorem implies that $X$ has height 1 in $R$. Thus, if $\text{ht}(Q/P) > 1$ then $X$ is strictly smaller than $Q/P$, and we can take $T$ to be the inverse image of $X$ in $U(\mathfrak{h})$. This proves the lemma.
The following observation will be used in the proof of our next lemma. Suppose that \( I \) is an ideal of \( g \) of codimension 1, say \( g = I \oplus kx \). Then \( U = \bigoplus_{i \geq 0} U(I)x^i \), and \( U(I) \) is stable under the derivation \( \delta = [x, \cdot] \) of \( U \). Thus \( U \) is isomorphic to the Ore extension \( U(I)[x; \delta] \). Now let \( J \) be an ideal of \( U \) which is extended from \( I \). Then \( J \) can be written as \( J = \bigoplus_{i \geq 0} (J \cap U(I))x^i \), and \( U/J \) is isomorphic to the Ore extension \( (U(I)/J \cap U(I))[x; \delta'] \), where \( \delta' \) denotes the derivation of \( U(I)/J \cap U(I) \) induced by \( \delta \).

Recall that \( d(\cdot) \) denotes Gelfand-Kirillov-dimension over \( k \).

**Lemma 3.** Let \( h = h_0 \subset h_1 \subset \ldots \subset h_t = g \) be a chain of ideals of \( g \) such that \( \dim_k(h_{i+1}) = 1 + \dim_k(h_i) \). Let \( P \) be a prime ideal of \( U = U(g) \) and set \( P_i = (P \cap U(h_i))U \), \( i = 0, 1, \ldots, t \). Then \( P_0 \subset P_1 \subset \ldots \subset P_t = P \) is a chain of prime ideals of \( U \) of length \( t = d(U(h)/P \cap U(h)) - d(U/P) \).

**Proof.** We argue by induction on \( t = \dim_k(g/h) \), the case when \( g = h \) being clear. So assume that \( t \geq 1 \) and write \( I = h_{t-1} \), \( U' = U(I) \) and \( P' = P \cap U' \), a prime ideal of \( U' \). Furthermore, set \( P'_i = (P \cap U(h_i))U' \) for \( i = 0, 1, \ldots, t-1 \) and let \( l' \) denote the length of the chain \( P'_0 \subset P'_1 \subset \ldots \subset P'_{t-1} = P' \) in \( \text{Spec}(U') \). By induction, we conclude that

\[
l' = t - 1 + d(U(h)/P' \cap U(h)) - d(U'/P'),
\]

where, of course, \( P' \cap U(h) = P \cap U(h) \). Now \( P_i = P'_iU \) for \( i = 0, 1, \ldots, t-1 \), and \( P_i = P_{i+1} \) holds if and only if \( P'_i = P'_{i+1} \). Therefore, \( l = l' \) in case \( P = P'U \) and \( l = l' + 1 \) otherwise. In the former case, \( P \) is extended from \( I \) and, by our above remarks, we conclude that \( U/P \cong (U'/P')[x; \delta] \), some Ore extension over \( U'/P' \). Now [2; Lemma 3.1c] implies that \( d(U/P) = d(U'/P') + 1 \), and so our above formula for \( l' \) becomes

\[
l = l' = t - 1 + d(U(h)/P \cap U(h)) - d(U/P) + 1.
\]

Thus we have finished when \( P \) is extended from \( I \). If \( P \not\cong P'U \), then we deduce from [2; Lemma 3.1c, d and Satz 3.4] that

\[
d(U'/P') \leq d(U/P) < d(U/P'U) = d(U'/P') + 1.
\]

Since \( d(U/P) \) is an integer, by [2; Korollar 5.4], we have \( d(U'/P') = d(U/P) \) in this case, and the assertion again follows from our above formula for \( l' \). Thus the assertion holds in either case and the lemma is proved.

**Corollary 4.** Let \( h \) be an ideal of \( g \) and let \( P \) be a prime ideal of \( U \). Then \( P \) is extended from \( h \) if and only if

\[
d(U/P) = d(U(h)/P \cap U(h)) + \dim_k(g/h).
\]

**Proof.** Since \( g \) is solvable, we can choose a chain of ideals \( h = h_0 \subset h_1 \subset \ldots \subset h_t = g \) as in Lemma 3. Clearly, \( P \) is extended from \( h \) if and only if \( P = P_0 \), in the notation of Lemma 3, that is if and only if \( l = 0 \). The assertion follows from this.
We are now ready to prove our extension of Tauvel’s height formula.

**Proposition 5.** Let \( \mathfrak{h} \) be an ideal of \( g \) and assume that \((*)\) holds in \( U(\mathfrak{h}) \). Let \( P \subset Q \) be prime ideals in \( U = U(g) \) such that \( P \) is extended from \( \mathfrak{h} \). Then

\[
\text{ht}(Q/P) = d(U/P) - d(U/Q).
\]

**Proof.** Write \( U' = U(\mathfrak{h}) \), \( P' = P \cap U' \), \( Q' = Q \cap U' \), and set \( s = d(U'/P') - d(U'/Q') \). Then \( P' \subset Q' \) are \( g \)-stable primes in \( U' \), and we claim that there exists a chain

\[
P' = P'_0 \subset P'_1 \subset ... \subset P'_s = Q'
\]

of length \( s \) consisting of \( g \)-stable prime ideals of \( U' \). Indeed, choose any maximal chain of \( g \)-stable primes in \( U' \) connecting \( P' \) to \( Q' \). Then Lemma 2 implies that this chain is in fact saturated in \( \text{Spec}(U') \), and hence it must have length \( s \), since \((*)\) holds in \( U(\mathfrak{h}) \).

For each \( i = 0, 1, ..., s \) set \( P_i = P'_i U \). Furthermore, choose a chain of ideals \( \mathfrak{h} = \mathfrak{h}_0 \subset \mathfrak{h}_1 \subset ... \subset \mathfrak{h}_s = g \) as in Lemma 3 and set \( Q_i = (Q \cap U(\mathfrak{h}_i)) U \). Then

\[
P = P_0 \subset P_1 \subset ... \subset P_s = Q' U = Q_0 \subset Q_1 \subset ... \subset Q_s = Q
\]
is a chain of prime ideals in \( U \) of length

\[
l = s + t + d(U'/Q') - d(U/Q).
\]

Using the definition of \( s \), we can rewrite this as \( l = t + d(U'/P') - d(U/Q) \). Since \( P \) is extended from \( \mathfrak{h} \), Corollary 4 yields that \( d(U/P) = d(U'/P') + t \), and so we get

\[
l = d(U/P) - d(U/Q).
\]

Finally, we clearly have \( l \leq \text{ht}(Q/P) \), and \( \text{ht}(Q/P) \leq d(U/P) - d(U/Q) \) holds quite generally, as a consequence of [2; Satz 3.4]. Therefore, we obtain the desired equality, \( \text{ht}(Q/P) = d(U/P) - d(U/Q) \), and the proof is complete.

We remark that the crucial assumption above is of course that \( P \) should be extended from \( \mathfrak{h} \), whereas, proceeding by induction on \( \dim_k(g) \), one can usually assume that \((*)\) holds in \( U(\mathfrak{h}) \).

Note that if \( \mathfrak{h} \) is the zero ideal of \( g \), then \( U(\mathfrak{h}) = k \) and this surely satisfies \((*)\). Moreover, the zero ideal of \( U \) is clearly extended from the zero ideal of \( g \). Thus if we take \( P = 0 \) in Proposition 5 and observe that \( d(U) = \dim_k(g) \), by Corollary 4, for example, then we get Tauvel’s height formula.

**Corollary 6 (Tauvel [8]).** Let \( Q \) be a prime ideal of \( U = U(g) \). Then \( \text{ht}(Q) = d(U) - \text{dim}_k(g) - d(U/Q) \).

Let \( \mathfrak{h} \) be an ideal of \( g \) and let \( Q \) be a \( g \)-stable prime ideal of \( U(\mathfrak{h}) \). Define \( \text{ht}_g(Q) \) to be the maximum possible length of all chains \( 0 = Q_0 \subsetneq Q_1 \subsetneq ... \subsetneq Q_s = Q \) consisting of \( g \)-stable primes in \( U(\mathfrak{h}) \). Then, clearly, \( \text{ht}_g(Q) \leq \text{ht}(Q) \). Our next corollary shows that we have in fact equality here.
Corollary 7. Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$ and let $Q$ be a $\mathfrak{g}$-stable prime ideal of $U(\mathfrak{h})$. Then $ht_\mathfrak{g}(Q) = ht(Q)$.

Proof. The result is clear when $\mathfrak{h} = 0$. So assume that $\mathfrak{h}$ is nonzero and choose an ideal $\mathfrak{h}'$ of $\mathfrak{g}$ of codimension 1 in $\mathfrak{h}$. Set $U' = U(\mathfrak{h}')$ and $Q' = Q \cap U'$, a $\mathfrak{g}$-stable prime ideal of $U'$. By induction on $\dim_\mathbb{k}(\mathfrak{h})$, we may assume that $ht_\mathfrak{g}(Q') = ht(Q')$. Since for any $\mathfrak{g}$-stable prime $X$ of $U'$ the extended ideal $XU(\mathfrak{h})$ is a $\mathfrak{g}$-stable prime of $U(\mathfrak{h})$, we conclude immediately that

$$ht(Q') = ht_\mathfrak{g}(Q') \leq ht_\mathfrak{g}(Q'U(\mathfrak{h})) \leq ht_\mathfrak{g}(Q).$$

If $Q$ is extended from $\mathfrak{h}'$, then we deduce from Corollaries 4 and 6 that $ht(Q) = ht(Q')$, whence $ht_\mathfrak{g}(Q) \geq ht(Q)$, as claimed. If $Q$ is not extended from $\mathfrak{h}'$, then Lemma 3 and Corollary 6 similarly yield that $ht_\mathfrak{g}(Q) \geq ht_\mathfrak{g}(Q'U(\mathfrak{h}))+1$. Again, the desired inequality $ht_\mathfrak{g}(Q) \geq ht(Q)$ follows, since now $ht_\mathfrak{g}(Q) \geq ht_\mathfrak{g}(Q'U(\mathfrak{h}))+1$. This proves the corollary.

A solvable Lie algebra $\mathfrak{g}$ is said to be almost algebraic if $\mathfrak{g}$ has the form $\mathfrak{g} = n + s$, where $n$ is a nilpotent ideal of $\mathfrak{g}$ and $s$ is an abelian subalgebra of $\mathfrak{g}$ acting semisimply on $n$. By enlarging $n$ and shrinking $s$, if necessary, one can always assume that $n$ is the nilpotent radical of $\mathfrak{g}$ and that the sum $n + s$ is direct. As our last application of Proposition 5 we show that at least in the special case where $n$ is abelian $U(\mathfrak{g})$ is in fact catenary.

Corollary 8. Assume that $\mathfrak{g} = a + s$, where $a$ is an abelian ideal of $\mathfrak{g}$ and $s$ is a commutative subalgebra of $\mathfrak{g}$ acting semisimply on $a$. Then $(\ast)$ holds in $U(\mathfrak{g})$.

Proof. Let $P \subseteq Q$ be primes in $U = U(\mathfrak{g})$. Note that homomorphic images of $\mathfrak{g}$ clearly inherit the structure of $\mathfrak{g}$ as described in the hypotheses. Thus, upon factoring out the ideal $\mathfrak{g} \cap P$ from $\mathfrak{g}$ if necessary and arguing by induction on $\dim_\mathbb{k}(\mathfrak{g})$, we may assume that $P$ is faithful, that is $\mathfrak{g} \cap P = 0$. By the assumption on $\mathfrak{g}$, we can write $a = \sum_i a^i$, where $\lambda_i$ are suitable linear functionals on $\mathfrak{g}$ vanishing on $a$ such that

$$a^i = \{a \in a \mid [x, a] = \lambda_i(x)a \text{ for all } x \in \mathfrak{g}\}$$

is nonzero. Note that the image of $a^i$ in $U/P$ belongs to the semicentre of $U/P$ and is nonzero, as $P$ is faithful. Therefore, if $\lambda_i \neq 0$ for some $i$, then we conclude from Lemma 1(i) that $P$ is extended from $\mathfrak{h} = \ker(\lambda_i)$, and $\mathfrak{h} = a + (s \cap \mathfrak{h})$ has the same structure as $\mathfrak{g}$ but smaller dimension. Thus, by induction, we may assume that $(\ast)$ holds in $U(\mathfrak{h})$. It now suffices to quote Proposition 5 to finish the proof in this case.

Finally, if $\lambda_i = 0$ for all $i$, then $\mathfrak{g}$ is abelian and the result is classical.

References


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