PART I: Do three of the following problems.

1. An abelian group is uniform if every intersection of two nontrivial subgroups is also nontrivial.
   (a) Let $A$ be a nontrivial finite uniform abelian group. Prove that $A \cong \mathbb{Z}/\mathbb{Z}_{p^n}$ for some positive integer $n$ and some prime number $p$.
   (b) Suppose that $A$ is a finitely generated infinite abelian group. Prove that $A$ is uniform if and only if $A$ is cyclic.
   (c) Give a complete description, up to isomorphism, of the finitely generated uniform abelian groups.

2. Let $F = \mathbb{F}_3$ be the field with 3 elements and let $G = \text{GL}_2(F)/F^*$ denote the group of invertible $2 \times 2$-matrices over $F$ modulo the scalar matrices.
   (a) Show that $|G| = 24$.
   (b) Show that $G$ acts on the set of all 1-dimensional subspaces of the vector space $V = F^2$, and only the identity element of $G$ fixes all 1-dimensional subspaces.
   (c) Conclude that $G$ is isomorphic to the symmetric group $S_4$.

3. Set $R = \mathbb{Z}[x]$, the polynomial ring in the single variable $x$, with integer coefficients. Prove that $R$ is not a principal ideal domain.

4. Let $F$ be a field and let $\alpha, \beta$ be distinct elements of $F$. Let $F[x]$ denote the polynomial algebra over $F$ and define $R = \{f/g \mid f, g \in F[x], g(\alpha)g(\beta) \neq 0\}$.
   (a) Show that $R$ is a subring of $F(x)$, the field of rational functions over $F$.
   (b) Determine the units of $R$.
   (c) Determine the maximal ideals of $R$. 
Part II: Do two of the following problems.

1. Let $G$ be a group of order 231.

   (a) Prove that $G$ contains a unique Sylow 11-subgroup $P$.
   (b) Determine $\text{Aut}(P)$, the group of automorphisms of $P$.
   (c) Prove that there is a group homomorphism from $G$ into $\text{Aut}(P)$.
   (d) Prove that $P$ is contained in the center of $G$.

2. Let $n$ be a positive integer, and let $k$ be an algebraically closed field of characteristic 0. Let $X$ and $Y$ be $n \times n$ matrices over $k$ such that $XY - YX = Y$. Prove that $X$ and $Y$ have a common eigenvector. (Hint: First prove, if $v \in k^n$ is an eigenvector for $X$, that $v, Yv, Y^2v, \ldots$ span a vector subspace of $k^n$ invariant under both $X$ and $Y$.)

3. Let $K/F$ be a finite Galois extension of fields, with Galois group $\Gamma = \text{Gal}(K/F)$. For $\alpha \in K$, define $\text{Tr}_{K/F}(\alpha) = \sum_{\gamma \in \Gamma} \gamma(\alpha)$.

   (a) Prove that $\text{Tr}_{K/F}(\alpha) \in F$.
   (b) Let $x^m + cx^{m-1} + \cdots$ be the minimal polynomial of $\alpha$ over $F$. Show that $m$ divides the degree $[K:F]$ and that $\text{Tr}_{K/F}(\alpha) = -\frac{[K:F]}{m} \cdot c$. 