PH.D. COMPREHENSIVE EXAMINATION
ABSTRACT ALGEBRA SECTION

January 1996

Part I. Do three (3) of these problems.

I.1. a) Let \( G \) be a group with a subgroup \( H \) and a normal subgroup \( N \). Prove that \( H \cap N \) is a normal subgroup of \( H \) and that \( H/(H \cap N) \cong HN/N \).

b) Use the result in part (a) to prove that a subgroup \( H \) of a solvable group is solvable.
(Note: A group \( G \) is solvable if there exists a subnormal series \( \{1\} = N_k \triangleleft N_{k-1} \triangleleft \cdots \triangleleft N_2 \triangleleft N_1 \triangleleft N_0 \triangleleft G \), where, for each \( i = 1, \ldots, k \), \( N_{i-1}/N_i \) is abelian.)

I.2. Let \( R \) be an Euclidean domain with unity.

a) Prove that \( R \) is a principal ideal domain.

For \( r \) and \( s \) in \( R \), let \( \langle r, s \rangle = \{ rm + sn \mid m, n \in R \} \)

b) Prove that \( \langle r, s \rangle \) is an ideal of \( R \).

c) Suppose that \( R = Q[x] \) (where \( Q \) represents the field of rational numbers), and that \( r = x^2 - 3x + 2 \) and \( s = x^3 - 9x^2 + 23x - 15 \). Then, by parts (a) and (b), \( \langle r, s \rangle = \langle p \rangle \), for some \( p \in Q[X] \). Find \( p \). (Show all work and justify your answer.)

I.3. Let \( F \) be a field. Then \( F[x] \) is a commutative ring with unity. (You may accept this without proof.)

a) Show that \( F[x] \) is an integral domain but that it cannot be a field.

Let \( E \) be an extension field of \( F \). For \( \alpha \in E \), let \( \phi : F[x] \to E \) be a homomorphism which fixes the elements of \( F \) and which maps \( x \) into \( \alpha \).

b) Describe the kernel of \( \phi \).

c) Prove that if \( \alpha \) is algebraic then the image of \( \phi \) is a subfield of \( E \).

d) For \( F = Q \) (the rationals) and for \( \alpha = \sqrt{2} \), describe the image of \( \phi \) as a subfield of \( R \) (the reals). (I.e., find a basis and a unique representation for each element.)

I.4. Let \( V \) be a finite dimensional vector space with an inner product \( \langle \cdot, \cdot \rangle \). Fix a non-zero vector \( u \) in \( V \) and define a mapping \( T : V \to V \) by \( T(v) = \langle u, v \rangle u \).

a) Show that \( T \) is a linear transformation.

b) Find the characteristic polynomial, eigenvalues, minimal polynomial, and Jordan canonical form of \( T \).

c) If, instead of \( T \), we consider the mapping \( F : V \to V \) given by \( F(v) = \langle u, v \rangle v \), explain why the analogues of the questions asked in part b) are not meaningful for \( F \).
Part II. Do two (2) of these problems.

II.1.  a) Find all groups of order 325.
       b) Find all groups of order 22.

II.2. Let $A$ be an $m \times n$ matrix over a field $F$.
      a) Show that the rank of $A$ is equal to the smallest integer $r$ such that $A$ can be
          factored as $A = BC$ for suitable matrices $B$ and $C$ of sizes $m \times r$ and $r \times n$
          respectively.
      b) Use part (a) to deduce the familiar fact that “row rank = column rank”.

II.3. Let $F$ be a finite field with $q$ elements.
      a) Prove that the product of the non-zero elements of $F$ is $-1$.
      b) Prove that if $q$ is even then every element of $F$ is a square; and that if $q$ is odd, then
          the set of non-zero squares of $F$ is a subgroup of index 2 of the group of non-zero
          elements of $F$. 