PH.D. COMPREHENSIVE EXAMINATION
ALGEBRA SECTION

January 1995

Part I. Do three (3) of these problems.

I.1. If a subgroup $G$ of the symmetric group $S_n$ contains an odd permutation, then $|G|$ is even and exactly half the elements of $G$ are odd permutations.

I.2. Let $R$ be a commutative ring with no nonzero nilpotent elements (that is, $a^n = 0$ implies $a = 0$). If the polynomial $f(X) = a_0 + a_1 X + \ldots + a_m X^m$ in $R[X]$ is a zero-divisor (that is, $g(X)f(X) = 0$ for some nonzero polynomial $g(X) \in R[X]$), prove that there is an element $b \neq 0$ in $R$ such that $ba_0 = ba_1 = \ldots ba_m = 0$.

I.3. Let $V$ be a finite-dimensional vector space over a field $F$. An endomorphism $\phi$ of $V$ is called a pseudoreflection if $\phi - 1$ has rank at most 1. Prove:

a) $\phi$ is a pseudoreflection precisely if there exists a basis of $V$ such that the matrix of $\phi$ has the form

\[
\begin{bmatrix}
* & * & * & \ldots & * \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix}
\]

b) Show that the Jordan canonical form of a pseudoreflection $\phi$ is

\[
\begin{bmatrix}
1 & 1 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
* & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}
\]

I.4. Let $F \supseteq K$ be an algebraic extension of fields and let $R$ be a subring of $F$ with $R \supseteq K$. Show that $R$ is a field.
Part II. Do two (2) of these problems.

II.1. Let $G$ be a finite group and let $H$ be a proper subgroup of $G$. Show that $G$ is not the set-theoretic union of all conjugates of $H$.

II.2. Let $K$ be the splitting field over the rationals $\mathbb{Q}$ for the polynomial $f(x)$. For each of the following examples, find the degree $[K : \mathbb{Q}]$, determine the structure of the Galois group $G(K/\mathbb{Q})$, describe its action on the roots of $f(x)$ and identify the group.

a) $f(x) = x^4 - 3$

b) $f(x) = x^4 + x^2 - 6$

II.3. Let $G$ be a group of order $165 = 3 \cdot 5 \cdot 11$. Prove:

a) $G$ has a normal Sylow 11-subgroup, say $C$.

b) $G/C$ is cyclic. (HINT: Show that every group of order 15 is cyclic.)

c) $G$ has normal subgroups of orders 33 and 55.