Part I. Do three (3) of these problems.

I.1. Show that the alternating group $A_6$ has no subgroup of prime index.

I.2. Let $R$ be a commutative domain in which every element $x$ satisfies $x^n = x$ for some $n > 1$ (depending on $x$). Show that $R$ is a field of positive characteristic. Is $R$ necessarily finite?

I.3. Let $V$ be a finite-dimensional vector space and let $\phi : V \to V$ be an endomorphism. Suppose that, for some $v \in V$ and $k \geq 1$, $\phi^k(v) = 0$ but $\phi^{k-1}(v) \neq 0$. Prove:

   (1) The subspace $W$ of $V$ that is generated by $\{v, \phi(v), \ldots, \phi^{k-1}(v)\}$ is $\phi$-invariant (i.e., $\phi(W) \subseteq W$) and satisfies $\dim(W) = k$.

   (2) The minimal polynomial $m(X)$ of $\phi$ is divisible by $X^k$.

I.4. Let $F$ be a field of characteristic $p > 0$ and let $f(X) \in F[X]$ be an irreducible polynomial which is not separable (i.e., $f(X)$ has repeated roots). Show that $f(X) = g(X^p)$ for some irreducible polynomial $g(X) \in F[X]$.

Part II. Do two (2) of these problems.

II.1. Show that there is no simple group of order 56 (without quoting Burnside’s $p^aq^b$-Theorem or special cases thereof).

II.2. Let $M \neq 0$ be a finitely generated torsion module over a commutative PID $R$.

   (1) Show that $M$ is indecomposable (i.e., $M$ is not the direct sum of two nonzero submodules) if and only if $M \cong R/p^nR$ for some irreducible element $p$ of $R$ and some $n > 0$.

   (2) Show that $M$ is irreducible (i.e., $M$ has no submodules other than 0 and $M$) if and only if $M \cong R/pR$ for some irreducible element $p$ of $R$.

II.3. Let $F$ be a finite field and $n$ a positive integer. Prove that there exists an irreducible polynomial over $F$ of degree $n$.