ON THE PARAMETERIZATION OF PRIMITIVE IDEALS IN AFFINE PI ALGEBRAS

Edward S. Letzter

Abstract. We consider the following question, concerning associative algebras $R$ over an algebraically closed field $k$: When can the space of (equivalence classes of) finite dimensional irreducible representations of $R$ be topologically embedded into a classical affine space? We provide an affirmative answer for algebraic quantum groups at roots of unity. More generally, we give an affirmative answer for $k$-affine maximal orders satisfying a polynomial identity, when $k$ has characteristic zero. Our approach closely follows the foundational studies by Artin and Procesi on finite dimensional representations. Our results also depend on Procesi’s later study of Cayley-Hamilton identities.

1. Introduction

1.1. Let $k$ be an algebraically closed field, and let $R$ be an associative $k$-algebra with generators $X_1, \ldots, X_s$. In the foundational studies of Artin [1], in 1969, and Procesi [9], in 1974, it was shown that the semisimple $n$-dimensional representations of $R$ (over $k$) were parametrized up to equivalence by a closed subset of $\text{Max} \mathcal{T}(n, s)$, where $\mathcal{T}(n, s)$ is the affine (i.e., finitely generated) commutative $k$-algebra generated by the coefficients of the characteristic polynomials of $s$-many generic $n \times n$ matrices. It was further shown by Artin and Procesi in [1; 9] that $\text{Prim}_n R$, the set of kernels of $n$-dimensional irreducible representations of $R$, is homeomorphic to a locally closed subset of $\text{Max} \mathcal{T}(n, s)$. (Here and throughout, the Jacobson/Zariski topology is employed.) In particular, when the irreducible representations of $R$ all have dimension $n$ (e.g., when $R$ is an Azumaya algebra of rank $n$, by what is now known as the Artin-Procesi theorem [1; 9]), the space $\text{Prim} R$ of kernels of irreducible representations of $R$ is homeomorphic to a locally closed subset of affine space. In this note we examine generalizations of this embedding for more general classes of $k$-affine PI (i.e., polynomial identity) algebras. Our analysis closely follows the above cited work of Artin and Procesi, and also depends on the later study by Procesi of Cayley-Hamilton identities [7].

The author thanks the Department of Mathematics at the University of Pennsylvania for its hospitality; the research for this paper was begun while he was a visitor on sabbatical there in Fall 2004. The author is grateful for support during this period from a Temple University Research and Study Leave Grant. This research was also supported in part by a grant from the National Security Agency.

Typeset by $\text{AMSTeX}$
1.2. Our main result, proved in (5.4):

Theorem. Let $A$ be a prime affine PI algebra over an algebraically closed field $k$ of characteristic zero, and suppose that $A$ is a maximal right (or left) order in a simple artinian ring $Q$. Then Prim $A$ is homeomorphic to a constructible subset of the affine space $k^N$, for a suitable choice of positive integer $N$.

Examples to which the theorem applies include algebraic quantum groups at roots of unity. Recent studies of quantum groups from this general point of view include [4].

1.3. For an arbitrary $k$-affine PI algebra $R$, in arbitrary characteristic, we are able to construct a closed bijection from Prim $R$ onto a constructible subset of $k^N$, again for a suitable choice of $N$. We therefore ask whether the conclusion of the preceding theorem holds for all $k$-affine PI algebras.

1.4. We assume that the reader is familiar with the basic theory of PI algebras; general references include [6, Chapter 13], [8], and [11].

Acknowledgement. The author is happy to acknowledge useful communications with Zinovy Reichstein and Nikolaus Vonessen on the subject matter of this note.

2. Constructing the injection $\Psi$

Our goal in this section is to construct an injection, specified in (2.11), from Prim $R$ into the maximal spectrum of a suitable “trace ring.” The approach is directly adapted from [9], with some added bookkeeping.

2.1 $(R, d, N)$ First Notation and Conventions. The following will remain in effect throughout this paper.

(i) Set

$$R = k\left\langle \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_s \right\rangle / \langle \tilde{f}_1, \tilde{f}_2, \ldots \rangle,$$

the factor of the free associative $k$-algebra in the generators $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_s$ modulo the (not necessarily finitely many) relations $\tilde{f}_1, \tilde{f}_2, \ldots$. Let $X_\ell$ denote the canonical image of $\tilde{X}_\ell$ in $R$, for each $\ell$. Assume that $R$ satisfies a (monic) polynomial identity.

(ii) All $k$-algebra homomorphisms mentioned will be assumed to be unital. A representation is a $k$-algebra homomorphism into the algebra of linear operators on a $k$-vector space. If $\Gamma$ is a $k$-algebra we will assume that the sets Prim $\Gamma$ of (left) primitive ideals and Max $\Gamma$ of maximal ideals are equipped with the Jacobson/Zariski topology: The closed sets have the form $V_\Gamma(I) = \{P : P \supseteq I\}$ for ideals $I$ of $\Gamma$.

(iii) Recall from Kaplansky’s Theorem and standard PI theory that there exists a positive integer $d$ such that every irreducible representation of $R$ has $(k)$-dimension no greater than $d$. Let $N$ be a common multiple of $1, 2, \ldots, d$. (Note that our choices of $d$ and $N$ remain valid when $R$ is replaced with a homomorphic image.)
2.2 \((\hat{C}_n, x_{ij}^{(\ell,n)}, G(n,s), \mathcal{T}(n,s), \hat{M}_n)\). Let \(n\) be a positive integer. Set
\[
\hat{C}_n = k \left[ x_{ij}^{(\ell,n)} : 1 \leq i, j \leq n, \ell = 1, 2, \ldots \right],
\]
the commutative polynomial \(k\)-algebra in the variables \(x_{ij}^{(\ell,n)}\). Also, set \(\hat{M}_n = M_n(\hat{C}_n)\), the \(k\)-algebra of \(n \times n\) matrices with entries in \(\hat{C}_n\). Identify \(\hat{C}_n\) with the \(\hat{C}_n\)-scalar matrices in \(\hat{M}_n\); in other words, identify \(\hat{C}_n\) with the center \(Z(\hat{M}_n)\) of \(\hat{M}_n\).

Let \(G(n,s)\) denote the \(k\)-subalgebra of \(\hat{M}_n\) generated by the generic matrices
\[
\left( x_{ij}^{(1,n)} \right)_{n \times n}, \ldots, \left( x_{ij}^{(s,n)} \right)_{n \times n}.
\]
The \(k\)-subalgebra of \(\hat{C}_n\) generated by the coefficients of the characteristic polynomials of the elements of \(G(n,s)\) will be denoted \(\mathcal{T}(n,s)\). It is a well known consequence of Shirshov’s Theorem that \(\mathcal{T}(n,s)\) is \(k\)-affine [9, 3.1]. It is also well known that \(\mathcal{T}(n,s)\) is generated, in characteristic zero, by the traces of the elements of \(G(n,s)\).

2.3 (Rel(\(\hat{C}_n\)), Rel(\(\hat{M}_n\))). Now consider the \(k\)-algebra homomorphism
\[
k \left\{ \hat{X}_1, \ldots, \hat{X}_s \right\} \xrightarrow{\bar{\pi}_n} \hat{M}_n,
\]
mapping
\[
\hat{X}_\ell \mapsto \left( x_{ij}^{(\ell,n)} \right)_{n \times n},
\]
for each \(\ell\). Let Rel(\(\hat{C}_n\)) be the ideal of \(\hat{C}_n\) generated by the entries of
\[
\bar{\pi}_n \left( \hat{f}_1 \right), \bar{\pi}_n \left( \hat{f}_2 \right), \ldots,
\]
and let Rel(\(\hat{M}_n\)) be the ideal of \(\hat{M}_n\) generated by Rel(\(\hat{C}_n\)). Then
\[
\text{Rel}(\hat{M}_n) = M_n \left( \text{Rel}(\hat{C}_n) \right), \quad \text{and} \quad \text{Rel}(\hat{C}_n) = \text{Rel}(\hat{M}_n) \cap \hat{C}_n.
\]

2.4 (\(C_n, M_n, x_{ij}^{(\ell,n)}, \pi_n, T_n\)). Set
\[
C_n = \hat{C}_n / \text{Rel}(\hat{C}_n), \quad \text{and} \quad M_n = M_n(C_n) \cong \hat{M}_n / \text{Rel}(\hat{M}_n).
\]
Denote the natural image of each \(x_{ij}^{(\ell,n)}\) in \(C_n\) by \(x_{ij}^{(\ell,n)}\). We obtain a \(k\)-algebra homomorphism
\[
\pi_n : R \xrightarrow{x_{ij}^{(\ell,n)}} M_n.
\]
Note that \(\pi_n(R)\) is a natural image of \(G(n,s)\).

Identify \(C_n\) with \(Z(M_n)\), and let \(T_n = T_n(R)\) denote the \(k\)-subalgebra of \(C_n\) generated by the coefficients of the characteristic polynomials of the elements of \(\pi_n(R)\). Observe that \(T_n\) is a natural image of \(\mathcal{T}(n,s)\).
2.5. Say that a $k$-algebra homomorphism $h: M_n \to M_n(k)$ is matrix unital if $h$ restricts to the identity map on $M_n(k) \subseteq M_n$. Letting $e_{ij}$ denote the $ij$th matrix unit of $M_n(k)$, we see that $h$ is matrix unital if and only if $h(e_{ij}) = e_{ij}$ for all $i$ and $j$.

2.6 ($\tilde{\rho}$). Now let $\rho: R \to M_n(k)$ be a representation. Observe that there is a unique matrix unital $k$-algebra homomorphism $\tilde{\rho}: M_n \to M_n(k)$ such that the following diagram commutes:

$$
\begin{array}{cccccc}
R & \xrightarrow{\pi_n} & M_n & \xleftarrow{\text{inclusion}} & C_n & \xleftarrow{\text{inclusion}} & T_n \\
\downarrow & & \downarrow \tilde{\rho} & & \bar{\rho}|_{C_n} & & \bar{\rho}|_{T_n} \\
R & \xrightarrow{\rho} & M_n(k) & \xleftarrow{\text{inclusion}} & k & \xrightarrow{=} & k
\end{array}
$$

Of course, every $k$-algebra homomorphism $C_n \to k$ produces a representation $R \to M_n(k)$ in an obvious way.

2.7 ($\Theta_n$). Let $\text{Rep}_n R$ denote the set of $n$-dimensional representations of $R$ (without identifying equivalence classes), and let $\text{Alg}(T_n, k)$ denote the set of $k$-algebra homomorphisms from $T_n$ onto $k$. We have a function

$$
\Theta_n: \text{Rep}_n(R, k) \xrightarrow{\rho \mapsto \bar{\rho}|_{T_n}} \text{Alg}(T_n, k) \cong \text{Max } T_n.
$$

For a given representation $\rho: R \to M_n(k)$, let semisimple($\rho$) denote the unique equivalence class of semisimple $n$-dimensional representations corresponding to $\rho$ (i.e., the semisimple representations obtained from the direct sum of the Jordan-Hölder factors of the $R$-module associated to $\rho$). We now recall:

**Theorem.** (Artin [1, §12]; Procesi [9, §4]) (a) $\Theta_n$ is surjective. (b) $\Theta_n(\rho) = \Theta_n(\rho')$ if and only if semisimple($\rho$) = semisimple($\rho'$).

2.8 ($\gamma_P, \Phi_m$). (i) Let $\text{Prim}_m R$ denote the set of (left) primitive ideals of rank $m$ (i.e., the set of kernels of $m$-dimensional irreducible representations of $R$). Note that $1 \leq m \leq d$. Equip $\text{Prim}_m R$ with the relative topology, viewing it as a subspace of $\text{Prim } R$. As noted in [1, §12] and [9, §5], $\text{Prim}_m R$ is a locally closed subset of $\text{Prim } R$.

(ii) Choose $P \in \text{Prim}_m R$. Then $P$ uniquely determines an equivalence class of irreducible $m$-dimensional representations; choose $\rho: R \to M_m(k)$ in this equivalence class. Let $\gamma_P$ denote the $k$-algebra homomorphism $\bar{\rho}|_{T_m}: T_m \to k$. By (2.7), $\gamma_P$ depends only on $P$, and we obtain an injection

$$
\Phi_m: \text{Prim}_m R \xrightarrow{P \mapsto \ker \gamma_P} \text{Max } T_m.
$$

(iii) It follows from [1, §12] and [9, §5] that the image of $\Phi_m$ is an open subset of $\text{Max } T_m$ and that $\Phi_m$ is homeomorphic onto its image.
2.9 \((\rho_N)\). Now choose a positive integer \(m\) no greater than \(d\), and let \(\rho: R \to M_m(k)\) be a representation. We will use \(\rho_N: R \to M_N(k)\) to denote the associated \(N\)-dimensional diagonal representation, mapping

\[
r \mapsto \begin{bmatrix}
\rho(r) \\
\vdots \\
\rho(r)
\end{bmatrix},
\]

for \(r \in R\).

2.10 \((C, \pi, M, T, \alpha_{ij}^{(t)})\). In the remainder of this note we mostly will be concerned with the case when \(n = N\), and so we will set \(C = C_N, \pi = \pi_N, M = M_N, T = T_N = T_N(R) = T(R)\), and

\[
(\alpha_{ij}^{(t)}) = (\alpha_{ij}^{(t,N)})_{n \times n}.
\]

2.11 \((\gamma_{N,P}, \Psi)\) The injection. Now let \(P\) be a primitive ideal of \(R\). Proceeding as before, \(P\) uniquely determines an equivalence class of irreducible \(m\)-dimensional representations for some \(1 \leq m \leq d\); choose \(\rho: R \to M_m(k)\) in this equivalence class. Combining (2.6) and (2.9), let \(\gamma_{N,P}\) denote the \(k\)-algebra homomorphism \((\tilde{\rho}_N)|_T: T \to k\). We can now define an injection:

\[
\Psi: \text{Prim } R \xrightarrow{P} \ker \gamma_{N,P} \xrightarrow{\text{Max } T}
\]

In §3 the image of \(\Psi\) will be described. In §4 it will be proved that \(\Psi\) is open (and closed) onto its image. In §5 it will be seen, in certain special cases, that \(\Psi\) is homeomorphic onto its image.

Note now, however, that implicit in the preceding is a natural (and obvious) homeomorphism between \(\text{Prim } R\) and \(\text{Prim } \pi(R)\).

2.12. Choose \(P\), \(m\), and \(\rho\) as in (2.11). Up to equivalence, there is exactly one \(N\)-dimensional representation of \(R\) with kernel \(P\), namely, the representation corresponding to the unique (up to isomorphism) semisimple \(R/P\)-module of length \(N/m\). Therefore, by (2.7), \(\gamma_{N,P}\) depends only on \(P\) and not our specific choice \(\rho_N\) of \(N\)-dimensional representation.

3. The Image of \(\Psi\)

Retain the notation of the preceding section. Throughout this section, \(m\) will denote a positive integer no greater than \(d\). The main result of this section, (3.7), explicitly determines the image of \(\Psi\); in particular, the image is a constructible subset.

3.1. Given an \(N \times N\) matrix, the \((N/m)\)-many \(m \times m\) blocks running consecutively down the main diagonal will form the \(m\)-block diagonal. An \(N \times N\) matrix with only zero entries off the \(m\)-block diagonal will be referred to as an \(m\)-block diagonal matrix.
3.2. Consider the $k$-algebra homomorphism $\hat{C} \rightarrow \hat{C}_m$ mapping the $ij$th entry of $\left(\hat{x}_{ij}^{(\ell)}\right) \in \hat{M}$ to the $ij$th entry of the $m$-block diagonal matrix

\[
\begin{bmatrix}
\left(\hat{x}_{ij}^{(\ell,m)}\right)_{m \times m} \\
\vdots \\
\left(\hat{x}_{ij}^{(\ell,m)}\right)_{m \times m}
\end{bmatrix} \in M_N\left(\hat{C}_m\right).
\]

We obtain a commutative diagram of $k$-algebra homomorphisms:

\[
\begin{array}{ccc}
\hat{C} & \longrightarrow & \hat{C}_m \\
\text{projection} & \downarrow & \text{projection} \\
C & \longrightarrow & C_m \\
\text{inclusion} & \uparrow & \text{inclusion} \\
T & \longrightarrow & T_m
\end{array}
\]

We will refer to the horizontal maps as specializations.

3.3. Note that the preceding maps $\hat{C} \rightarrow \hat{C}_m$ and $C \rightarrow C_m$ are surjective. In characteristic zero, $T$ and $T_m$ are generated, respectively, by the traces of the matrices contained in $\pi(R)$ and $\pi_m(R)$, and it follows in this situation that the specialization $T \rightarrow T_m$ is surjective. In arbitrary characteristic, it is not hard to see that the specialization $T \rightarrow T_m$ is an integral ring homomorphism.

3.4 ($H_m$, $I_m$, $J_m$). Let $H_m$ denote the kernel of the specialization $C \rightarrow C_m$. In other words, $H_m$ is the ideal of $C$ generated by the sets

\[
\left\{ x_{ij}^{(\ell)} \mid x_{ij}^{(\ell)} \text{ is not within the } m\text{-block diagonal of } \left( x_{ij}^{(\ell)} \right); \right. \\
1 \leq i, j \leq N; \ell = 1, 2, \ldots
\]

and

\[
\left\{ x_{ij}^{(\ell)} - x_{i'j'}^{(\ell)} \mid x_{ij}^{(\ell)} \text{ and } x_{i'j'}^{(\ell)} \text{ are within the } m\text{-block diagonal of } \left( x_{ij}^{(\ell)} \right); \\
i = i' \text{ (mod } m) \text{ and } j = j' \text{ (mod } m); 1 \leq i, j \leq N; \ell = 1, 2, \ldots
\]

Set $I_m = MH_m = M_N(H_m)$. Let $J_m = H_m \cap T = I_m \cap T$ denote the kernel of the specialization $T \rightarrow T_m$. 
3.5. Now suppose that $P$ is the kernel of an irreducible representation $\rho: R \to M_m(k)$. Recalling the notation of (2.6) and (2.9), we see that the kernel of $\tilde{\rho}_N: M \to M_N(k)$ contains $I_m$, and so the kernel of $\tilde{\rho}|_T$ contains $J_m$. We conclude that $\Psi$ maps $\text{Prim}_m R$ into the set $V_T(J_m)$ of maximal ideals of $T$ containing $J_m$.

3.6 $(E_m)$. We now proceed in a fashion similar to [9, §5], employing the central polynomials of Formanek [5] or Razmyslov [10]. To start (see, e.g., [6, §13.5] or [11, §1.4] for details), we can construct a polynomial $p_m$ in noncommuting indeterminates with the following two properties, holding for all commutative rings $\Lambda$ with identity: First, $p_m(M_m(\Lambda)) \subseteq Z(M_m(\Lambda))$, and second, $p_m(M_m(\Lambda))$ generates $M_m(\Lambda)$ as an additive group. (Here, $p_m(M_m(\Lambda))$ refers to all evaluations of $p_m$ where the indeterminates have been substituted with elements of $M_m(\Lambda)$.) Hence, $\pi(p_m(R)) = p_m(\pi(R))$ is contained, modulo $I_m$, within the center of $M$. Moreover, a representation $R \to M_m(k)$ is irreducible if and only if $p_m(R)$ is not contained in the kernel, if and only if the image of $p_m(R)$ generates as an additive group the full set of scalar matrices in $M_m(k)$.

Next, modulo $I_m$, the characteristic polynomial of $c \in \pi(p_m(R))$ is $(\lambda - c)^N$, and so $c^N \in T + I_m$. We can therefore choose a set $E_m \subseteq T$ of transversals in $M$ for

$$\{ c^N + I_m : c \in \pi(p_m(R)) \},$$

with respect to $I_m$.

Now let $\varphi: R \to M_m(k)$ be a representation. As in (3.5), $I_m \subseteq \ker \tilde{\varphi}_N$ and $J_m \subseteq \ker \tilde{\varphi}|_T$. Observe that

$$\varphi \text{ is irreducible} \iff \varphi(p_m(R)) \neq 0 \iff \tilde{\varphi}_N(\pi(p_m(R))) \neq 0 \iff \tilde{\varphi}_N(E_m) \neq 0 \iff \tilde{\varphi}_N|_T(E_m) \neq 0.$$

3.7. Let $K_m$ be the ideal of $T$ generated by $E_m$ and $J_m$.

**Theorem.** (i) $\Psi$ maps $\text{Prim}_m R$ onto $V_T(J_m) \setminus V_T(K_m)$.

(ii) The image of $\Psi$ is

$$\bigcup_{m=1}^d V_T(J_m) \setminus V_T(K_m).$$

In particular, the image of $\Psi$ is a constructible subset of $\text{Max}T$.

**Proof.** Immediate from (3.5) and (3.6). □

3.8. We ask: Can the image of $\Psi$ be described in a simpler fashion? Is there a simple way to specify how the locally closed subsets in (3.7) fit together?
4. **Ψ is open and closed onto its image**

Retain the notation of §2 and §3. We now begin to consider the topological properties of Ψ. In (4.2) it is shown that Ψ is open and closed onto its image.

4.1. Let $R \to R'$ be a $k$-algebra homomorphism. As described in [9, pp. 177–178], the construction in (2.11) is functorial in the following sense.

(i) To start, we have a commutative diagram:

\[
\begin{array}{ccc}
R & \stackrel{\pi}{\longrightarrow} & M \\
\downarrow & & \downarrow \\
R' & \stackrel{\pi'}{\longrightarrow} & M'
\end{array}
\]

Moreover, if $R \to R'$ is surjective then so too is $T(R) \to T(R')$.

(ii) Next, assuming that $I$ is an ideal of $R$, that $R' = R/I$, and that $R \to R'$ is the natural projection, we obtain a commutative diagram:

\[
\begin{array}{ccc}
Prim R & \overset{\Psi}{\longrightarrow} & Max T \\
\downarrow P/I \longrightarrow P & \leftarrow \uparrow \frac{m}{(M\pi(I)M) \cap T} & \leftarrow \downarrow m \\
Prim R' & \overset{\Psi'}{\longrightarrow} & Max T'
\end{array}
\]

Each arrow represents an injection, and each vertical arrow represents a topological embedding onto a closed subset.

The kernel of the homomorphism $T \to T'$ is $(M\pi(I)M) \cap T$.

4.2. Now let $I$ be an arbitrary ideal of $R$, with corresponding closed subset $V_R(I)$ of $Prim R$. Set

\[J = (M\pi(I)M) \cap T.\]

**Proposition.** Ψ is open and closed onto its image. In particular,

\[\Psi(V_R(I)) = \text{Image } \Psi \cap (V_T(J)), \quad \text{and } \Psi(W_R(I)) = \text{Image } \Psi \cap (W_T(J)),\]

where $W_R(I)$ denotes the complement of $V_R(I)$.

**Proof.** This follows from (4.1ii) and the injectivity of Ψ. □
4.3. Combining (2.8) and (3.2), we have a commutative diagram:

\[
\begin{array}{ccc}
\text{Prim } R & \xrightarrow{\Psi} & \text{Max } T \\
\text{inclusion} & & \uparrow m \mapsto \text{specialization}^{-1}(m) \\
\text{Prim}_m R & \xrightarrow{\Phi_m} & \text{Max } T_m
\end{array}
\]

We conclude from (2.8iii) that the restriction of \( \Psi \) to \( \text{Prim}_m R \) is continuous. Recalling from (2.8iii) that \( \text{Prim}_m R \) is a locally closed subset of \( \text{Prim } R \), we may view this last conclusion as an assertion that \( \Psi \) is “piecewise continuous.” Also, we can conclude that the preimage under \( \Psi \) of a constructible subset of \( \text{Max } T \) is constructible.

4.4. We ask: Is \( \Psi \) necessarily continuous? More generally, is \( \text{Prim } R \) homeomorphic to a subspace of affine \( n \)-space, for sufficiently large \( n \)? A partial answer is given in §5.

5. Applications to Algebras Satisfying Cayley-Hamilton Identities

We retain the notation of the previous sections, but assume in this section that \( k \) has characteristic zero. Our approach closely follows [7].

5.1 [7, §2] Formal Traces and Cayley-Hamilton Identities. Let \( \Gamma \) be a \( k \)-algebra.

(i) [7, 2.3] Say that \( \Gamma \) is equipped with a (formal) trace (over \( k \)) provided there exists a \( k \)-linear function \( \text{tr}: \Gamma \to \Gamma \) such that for all \( a, b \in \Gamma \),

\[
\text{tr}(ab) = \text{tr}(ba),\quad \text{tr}(ab) = \text{tr}(ba),\quad \text{and}\quad \text{tr}(\text{tr}(a)b) = \text{tr}(a)\text{tr}(b).
\]

(ii) [7, 2.4] Suppose that \( \Gamma \) is equipped with a trace \( \text{tr} \). For each \( r \in \Gamma \), set

\[
\chi_r^{(n)}(t) = \prod_{i=1}^{n} (t - t_{r,i}),
\]

where the \( t_{r,i} \) are “formal eigenvectors” for \( r \) satisfying

\[
\sum_{i=1}^{n} \theta_{r,i}^j = \text{tr}(r^j),
\]

for all non-negative integers \( j \). Say that \( \Gamma \) satisfies the \( n \)-th Cayley-Hamilton identity if \( \chi_r^{(n)}(r) = 0 \) for all \( r \in \Gamma \).

(iii) Suppose that \( \Gamma \) is equipped with a trace \( \text{tr} \). By [7, Theorem], there exists a commutative \( k \)-algebra \( \Lambda \), and a trace compatible \( k \)-algebra embedding of \( \Gamma \) into \( M_n(\Lambda) \), if and only if \( \Gamma \) satisfies the \( n \)-th Cayley-Hamilton identity.

(iv) (Cf. [4, 3.10].) Let \( p \) be a positive integer, let \( \Lambda \) be a commutative \( k \)-algebra, and suppose that there is a trace compatible \( k \)-algebra embedding \( \Gamma \to M_n(\Lambda) \). The block diagonal embedding of \( M_n(\Lambda) \) into \( M_{pn}(\Lambda) \) then provides a trace compatible embedding of \( \Gamma \) into \( M_{pn}(\Lambda) \).

(v) Suppose that \( \Gamma \) is equipped with a trace. We conclude from (iii) and (iv) that if \( \Gamma \) satisfies the \( n \)-th Cayley-Hamilton identity then \( \Gamma \) satisfies the Cayley-Hamilton identity for all positive multiples of \( n \).
5.2. Returning to the setting of the previous sections (but with $k$ now having characteristic zero), suppose that $R$, as in (2.1), satisfies the $N$-th Cayley-Hamilton identity. It then follows directly from [7, 2.6] (the main theorem in [7]) that $T$ is contained in $\pi(R)$, the image of $\pi: R \to M$. Since $T$ must be central in $\pi(R)$, and since every irreducible representation of $R$ is finite dimensional over $k$, it follows from well known arguments that the function

$$\text{Prim } \pi(R) \xrightarrow{P \mapsto P \cap T} \text{Max } T$$

is continuous. Furthermore, as noted in (2.11), $\pi$ produces a natural homeomorphism between $\text{Prim } R$ and $\text{Prim } \pi(R)$. However, the composition

$$\text{Prim } R \xrightarrow{\text{natural homeomorphism}} \text{Prim } \pi(R) \xrightarrow{P \mapsto P \cap T} \text{Max } T$$

is precisely the function $\Psi$ of (2.11), which must therefore be continuous.

We obtain:

5.3 Proposition. (Recall that $k$ has characteristic zero.) Suppose that $R$ satisfies the $N$-th Cayley-Hamilton identity. Then $\Psi: \text{Prim } R \to \text{Max } T$ is homeomorphic onto its image.

Proof. That $\Psi$ is closed onto its image follows from (4.2). That $\Psi$ is continuous onto its image follows from (5.2). □

Combining (5.3) with our previous analysis produces our main result:

5.4 Theorem. Let $A$ be a prime affine PI algebra over an algebraically closed field $k$ of characteristic zero, and suppose that $A$ is a maximal right (or left) order in a simple artinian ring $Q$. Further suppose that $Q$ has rank $d$, that $A$ is a maximal order in $Q$, and that $N$ is a common multiple of $1, 2, \ldots, d$. Then $\text{Prim } A$ is homeomorphic to a constructible subset of the affine space $k^N$.

Proof. To start, the irreducible representations of $A$ all have dimension no greater than $d$. Next, $A$ is equipped with both a trace and a trace compatible embedding into $d \times d$ matrices over a commutative ring, since $A$ is a maximal order in $Q$; see (e.g.) [6, §13.9]. Hence, by (5.3ii), $A$ satisfies the $d$-th Cayley-Hamilton identity, and so, by (5.1iv), $A$ satisfies the $N$-th Cayley-Hamilton identity. The theorem now follows from (3.7) and (5.3). □

5.5 Quantum groups. For suitable complex roots of unity $\epsilon$, the quantum enveloping algebras $U_\epsilon$ and quantum function algebras $F_\epsilon$ are prime affine PI $C$-algebras and are maximal orders; see [2] and (e.g.) [3]. In particular, (5.4) applies to these algebras.
AFFINE PI ALGEBRAS

References


Department of Mathematics, Temple University, Philadelphia, PA 19122

E-mail address: letzter@math.temple.edu