1. Consider the limit \( \lim_{\|P\| \to 0} \sum_{k=1}^{n} (\sqrt{2c_k + 1} \Delta x_k) \) where \( P \) is a partition of the interval \([0, 4]\).

Express the limit as a definite integral and find its value.

\[
f(x) = \int_{0}^{4} \sqrt{2x + 1} \, dx
\]

Let \( u = 2x + 1 \), so \( du = 2 \, dx \). When \( x = 0, u = 1 \) and when \( x = 4, u = 9 \). In terms of \( u \) the integral becomes

\[
(1/2) \int_{1}^{9} u^{1/2} \, du = (1/2)(2/3u^{3/2})\bigg|_{1}^{9} = (1/3)9^{3/2} - (1/3)1^{3/2} = (27/3) - (1/3) = 26/3.
\]

2. The graph of a function \( f \) consists of a semicircle and line segments as shown below. Let

\[
g(x) = \int_{-2}^{x} f(t) \, dt.
\]

Find

(a) \( g(-2) = \int_{-2}^{-2} f(t) \, dt = 0 \) by a property of integrals which states that \( \int_{a}^{a} f(t) \, dt = 0 \).

\[
g(2) = \int_{-2}^{2} f(t) \, dt
\]

is the negative of the area of the semicircle from \( x = -2 \) to \( x = 2 \) (since the region is under the \( x \)-axis)

\[
= -(1/2)\pi 2^2 = -2\pi
\]

(b) \( g(4) = \int_{-2}^{4} f(t) \, dt = \int_{-2}^{2} f(t) \, dt + \int_{2}^{4} f(t) \, dt\)

\[
= (-2\pi) + \text{(area of triangle from } x = 2 \text{ to } 3) + \text{(area of square from } x = 3 \text{ to } 4)
\]

\[
= -2\pi + 1/2 + 1 = -2\pi + 3/2
\]
3. Let \( y = \int_{0}^{x^2} \cos(t^3) \, dt \). Find \( \frac{dy}{dx} \).

\[
\frac{dy}{dx} = \frac{d}{dx} \int_{0}^{x^2} \cos(t^3) \, dt
\]

\[
= \cos((x^2)^3) \frac{d}{dx}(x^2)
\]

by the Fundamental Theorem and chain rule

\[
= (2x) \cos(x^6)
\]

4. Solve the initial value problem \( \frac{dy}{dx} = \frac{1}{1 - 2x} + \frac{1}{1 + x^2} \), \( y(1) = 0 \).

First,

\[
\int \frac{1}{1 - 2x} + \frac{1}{1 + x^2} \, dx = \int \frac{1}{1 - 2x} \, dx + \int \frac{1}{1 + x^2} \, dx
\]

\[
= -\frac{1}{2} \ln |1 - 2x| + \arctan x + C
\]

(by taking \( u = 1 - 2x \) and \( du = -2dx \) in the first integral)

So \( y = -\frac{1}{2} \ln |1 - 2x| + \arctan x + C \).

Now, \( y(1) = 0 \), implies

\[
-\frac{1}{2} \ln |-1| + \arctan 1 + C = 0 \text{ or } \frac{\pi}{4} + C = 0 \text{ since } \ln 1 = 0 \text{ and } \arctan 1 = \frac{\pi}{4}.
\]

Solving for \( C \), we get \( C = -\frac{\pi}{4} \) and so

\[
y = -\frac{1}{2} \ln |1 - 2x| + \arctan x + -\frac{\pi}{4}.
\]
5. Evaluate the definite integrals.

(a) \( \int_{\pi/4}^{\pi/2} \sin^4(2x) \cos(2x) \, dx \)

Let \( u = \sin(2x) \) and so \( du = 2 \cos(2x) \, dx \), when \( x = \pi/4, u = \sin(\pi/2) = 1 \), and when \( x = \pi/2, u = \sin(\pi) = 0 \). Then in terms of \( u \) the integral becomes

\[
\left( \frac{1}{2} \right) \int_0^0 u^4 \, du = \frac{u^5}{10} \bigg|_1^0 = 0 - \frac{1}{10} = -\frac{1}{10}.
\]

(b) \( \int_0^2 xe^{2x} \, dx \)

Use integration by parts with \( u = x \) and \( dv = e^{2x} \, dx \), so \( du = dx \) and \( v = \frac{1}{2}e^{2x} \). Integration by parts then gives

\[
\int xe^{2x} \, dx = \left( \frac{1}{2} \right) xe^{2x} - \int \left( \frac{1}{2} \right) e^{2x} \, dx = \left( \frac{1}{2} \right) xe^{2x} - \left( \frac{1}{4} \right) e^{2x} + C,
\]

so

\[
\int_0^2 xe^{2x} \, dx = \left( \frac{1}{2} \right) xe^{2x} \bigg|_0^2 - \left( \frac{1}{4} \right) e^{2x} \bigg|_0^2 = \left( \frac{3e^4 + 1}{4} \right).
\]
6. Evaluate the indefinite integrals.

(a) \( \int e^{1 + \tan t} \sec^2 t \, dt \)

Let \( u = 1 + \tan t \), so \( du = \sec^2 t \, dt \) and the integral becomes
\[
\int e^u \, du = e^u + C = e^{1 + \tan t} + C.
\]

(b) \( \int \sin^{-1} x \, dx \) \hspace{1cm} (NOTE: \( \sin^{-1} x = \arcsin x \))

Use integration by parts with \( u = \arcsin x \) and \( dv = dx \), so \( du = \frac{1}{\sqrt{1 - x^2}} \, dx \) and \( v = x \). Integration by parts then gives
\[
\int \sin^{-1} x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} \, dx.
\]

Now for the new integral, make a u-substitution with \( u = 1 - x^2 \) and \( du = -2x \, dx \) and the integral becomes \((-1/2) \int u^{-1/2} \, du = -u^{1/2} + C = -\sqrt{1 - x^2} + C\).

Putting it all together gives
\[
\int \sin^{-1} x \, dx = x \arcsin x + \sqrt{1 - x^2} + C.
\]
(c) \[ \int \sqrt{x} \left( x^2 + \frac{1}{x} \right) \, dx \]

\[ \int \sqrt{x} \left( x^2 + \frac{1}{x} \right) \, dx = \int (x^{5/2} + x^{-1/2}) \, dx \] by distributing the \( \sqrt{x} \) factor.

Now \[ \int (x^{5/2} + x^{-1/2}) \, dx = \left( \frac{2}{7} \right)x^{7/2} + 2x^{1/2} + C. \]

(d) \[ \int \frac{1}{x(\ln x)^3} \, dx \]

Let \( u = \ln x \) and so \( du = \frac{1}{x} \, dx \) and the integral becomes

\[ \int \frac{1}{u^3} \, du = \int u^{-3} \, du = \frac{u^{-2}}{-2} + C = \frac{-1}{2u^2} + C = \frac{-1}{2(\ln x)^2} + C. \]

10 points 7. Find the area of the region bounded by \( y = e^x, \ y = e^{-x}, \ x = -1 \) and \( x = 1 \).

By symmetry, the area, \( A \), of the bounded region from \( x = -1 \) to \( x = 1 \) is twice the area of the region from \( x = 0 \) to \( x = 1 \). So, since \( y = e^x \) is above \( y = e^{-x} \) on the interval \((0, 1)\), we get

\[ A = 2 \int_{0}^{1} (e^x - e^{-x}) \, dx = 2(e^x + e^{-x})\bigg|_{0}^{1} = 2[(e + e^{-1}) - (e^0 + e^0)] = 2(e + 1/e - 2). \]
8. Consider the region bounded by the curves \( y = x^2 + 1 \) and \( y = 2x + 1 \).

(a) Find the volume of the solid generated by revolving the region about the \( x \)-axis.

Since the region is being revolved about a horizontal axis, the integral will be with respect to the variable \( x \). To find the limits of integration, find the intersection points of the curves by solving \( x^2 + 1 = 2x + 1 \) or \( x^2 - 2x = 0 \) which gives \( x(x - 2) = 0 \) and so \( x = 0 \) and \( x = 2 \) are the \( x \)-coordinates of the intersection points.

Since revolving an arbitrary vertical strip results in a 'washer' for a cross-section, the volume, \( V \), will have the form \( V = \pi \int_0^2 (R^2 - r^2) \, dx \) where 

\[
R = 2x + 1 \quad \text{and} \quad r = x^2 + 1.
\]

Simplifying, we get

\[
R^2 - r^2 = (2x + 1)^2 - (x^2 + 1)^2 = (4x^2 + 4x + 1) - (x^4 + 2x^2 + 1) = 2x^2 + 4x - x^4
\]

and thus

\[
V = \pi \int_0^2 (2x^2 + 4x - x^4) \, dx = \pi \left( \frac{2}{3}x^3 + 2x^2 - \frac{1}{5}x^5 \right) \bigg|_0^2
\]

\[
= \left( \frac{16}{3} + 8 - \frac{32}{5} \right) = \frac{90 + 120 - 96}{15} = \frac{114}{15}.
\]

(b) Set up, but DO NOT EVALUATE, an integral to find the volume of the solid generated by revolving the region about the \( y \)-axis.

Now the region is being revolved about a vertical axis, so we need to integrate with respect to the variable \( y \). The limits are now the \( y \)-coordinates of the points of intersection: when \( x = 0 \), \( y = 1 \) and when \( x = 2 \), \( y = 5 \). And once again we need 'washers' and so the volume, \( V \), has the form \( V = \pi \int_1^5 (R^2 - r^2) \, dy \).

Now \( R \) is the \( x \)-coordinate along the curve \( y = x^2 + 1 \); solving for \( x^2 \) we get \( R^2 = y - 1 \) (or \( R = \sqrt{y - 1} \)) and \( r \) is the \( x \)-coordinate along the line \( y = 2x + 1 \); solving for \( x \), we get \( r = \frac{y - 1}{2} \). So

\[
V = \pi \int_1^5 ((y - 1) - ((y - 1)/2)^2) \, dy.
\]