Note: The audio in this pdf won’t play without additional files.
The Volume of the Ball in $n$ Dimensions
Low Dimensional Balls

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Omar Hijab
University of New Haven

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The volume of \( B^3 \) is \( 4\pi/3 = 4.19 \) and the area of \( S^2 \) is \( 4\pi \).
Low Dimensional Balls, Continued

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S^1 = \{(x, y) : x^2 + y^2 = 1\}
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The area of \(B^2\) is \(\pi\) and the length of \(S^1\) is 2.

\[\text{Omar Hijab University of New Haven} \quad \text{The Volume of the Ball in } n \text{ Dimensions}\]
Low Dimensional Balls, Continued

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The area of \( B^2 \) is \( \pi = 3.14 \) and the length of \( S^1 \) is \( 2\pi \).
Low Dimensional Balls, Continued

$B^1 = \{x : x^2 \leq 1\}$

$S^0 = \{x : x^2 = 1\}$

$\[ -1 \quad 0 \quad 1 \]$
Low Dimensional Balls, Continued

\[ B^1 = \{ x : x^2 \leq 1 \} \]

\[ S^0 = \{ x : x^2 = 1 \} \]

The length of \( B^1 = [-1, 1] \) is 2 and the measure of \( S^0 \) is 2.
Low Dimensional Balls, Continued

\[ B^0 = \{0\} \]
Low Dimensional Balls, Continued

\[ B^0 = \{0\} \]

The measure of \( B^0 \) is 1.
In $n$ dimensions, the ball and the sphere are
The General Formula

In \( n \) dimensions, the ball and the sphere are

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B^n = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\}
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Throughout $|G|$ denotes the size, measure, volume, area, length, etc. of the geometric object $G$. Which it actually is will be clear from the context.
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Throughout $|G|$ denotes the size, measure, volume, area, length, etc. of the geometric object $G$. Which it actually is will be clear from the context. We will show

\[ |B^n| = \frac{\pi^{n/2}}{(n/2)!} \]

\[ |B^n| = \frac{1}{n} |S^{n-1}|. \]
History

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is Archimedes’ formula.
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- **When I was questor in Sicily [in 75 BC, 137 years after the death of Archimedes] I managed to track down his grave. The Syracusians knew nothing about it, and indeed denied that any such thing existed. But there it was, completely surrounded and hidden by bushes of brambles and thorns.**
- **I remembered having heard of some simple lines of verse which had been inscribed on his tomb, referring to a sphere and cylinder modelled in stone on top of the grave. And so I took a good look round all the numerous tombs that stand beside the Agrigentine Gate. Finally I noted a little column just visible above the scrub: it was surmounted by a sphere and a cylinder.**
- **I immediately said to the Syracusans, some of whose leading citizens were with me at the time, that I believed this was the very object I had been looking for. Men were sent in with sickles to clear the site, and when a path to the monument had been opened we walked right up to it. And the verses were still visible, though approximately the second half of each line had been worn away.**
- **So one of the most famous cities in the Greek world, and in former days a great centre of learning as well, would have remained in total ignorance of the tomb of the most brilliant citizen it had ever produced, had a man from Arpinum not come and pointed it out!**
History, Continued
Factorials and $\pi$

$|B^n| = \frac{\pi^{n/2}}{(n/2)!}$

▸ What does $x!$ mean?
Factorials and $\pi$

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- What does $x!$ mean? As Ada puts it ...

Omar Hijab University of New Haven

The Volume of the Ball in $n$ Dimensions
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- Thus we are defining $0!$, $\pi$, and $(1/2)!$ by the general formula for $n = 0$, $n = 2$, and $n = 1$.

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With these definitions, we have

\[ 1! = 1, \quad 2! = 2, \quad 3! = 6, \ldots, \]
\[ (3/2)! = 3\sqrt{\pi}/4, \quad (5/2)! = 15\sqrt{\pi}/8, \quad (7/2)! = 105\sqrt{\pi}/16, \ldots, \]

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Plan of Attack

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- Show the general formula for \(2n + 1\) is a consequence of the general formula for \(2n\).
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The Volume of the Ball in $n$ Dimensions
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So now we have to derive the general formula for $2n$ with $n \geq 2$. 
Even Dimensional Balls

Look at the even case

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For \( z = x + iy \), introduce polar coordinates,

\[ |z|^2 = r^2 = x^2 + y^2, \quad x = r \cos \theta, \quad y = r \sin \theta, \quad z = re^{i\theta} \]

\( r \) is the radius and \( \theta \) is the angle.
Complex Coordinates

To derive the even case,

\[ |B^{2n}| = \frac{\pi^n}{n!}, \]

It's difficult to graph objects in \( \mathbb{C}^n \) for \( n > 1 \). But the map \((z_1, \ldots, z_n) \mapsto (r_1, \ldots, r_n)\) projects \( \mathbb{C}^n \) into \( \mathbb{R}^n \), allowing us a partial visualization, called radial space. The inverse image of a point in radial space is an \( n \)-torus.
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To derive the even case,

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we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ via

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For \( n = 1 \), \( B^{2n} = P = D \). For all \( n > 1 \), \( B^{2n} \)
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\begin{tikzpicture}
    % Diagram code here
\end{tikzpicture}
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Let $A_1$ and $A_2$ be regions in euclidean space and let $A_1 \times A_2$ be their cartesian product.
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The Volume of the Ball in $n$ Dimensions
Multiplicativity of Volume

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so we have $|P| = |D^n| = |D \times \cdots \times D| = \pi^n$. But then Sofya would say . . .
Permutations  \( P = D \times \cdots \times D \)

Let \( G \) be the group of permutations \( g \) on \( n \) letters. Then \( |G| = n! \), and there is an action of \( G \) on \( P \): Each \( g \) in \( G \) permutes the coordinates,

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\[ P = D \times \cdots \times D \]
Permutations

Let $G$ be the group of permutations $g$ on $n$ letters. Then $|G| = n!$, and there is an \textit{action} of $G$ on $P$: Each $g$ in $G$ \textit{permutes the coordinates},

$$(z_1, \ldots, z_n) \mapsto (z_{g1}, \ldots, z_{gn}).$$

This is an equivalence relation on $P$. Let $P/G$ be the collection of equivalence classes,
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$D$
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$P/G = D \vee \cdots \vee D$ is the symmetric $n$-th power of $D$.

At this point, Emmy interjects . . .
In fact, $P/G$ is in one-to-one correspondence with

$$P_1 = \{(z_1, \ldots, z_n) \in P : 0 \leq |z_1|^2 \leq |z_2|^2 \leq \cdots \leq |z_n|^2 \leq 1\},$$
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Evariste would say $\vdots$

Thus it remains to find a volume-preserving bijection between $P_1$ and $B^{2n}$. But, before we do that, Pythagoras

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In other words, $G$ tessellates $P$ into cells $P_g = g(P_1)$, $g \in G$. 

\[ \begin{array}{c}
|P| \\
\hline
r_1^2 \leq 1, r_2^2 \leq 1
\end{array} \]

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In other words, $G$ \textit{tessellates} $P$ into cells $P_g = g(P_1)$, $g \in G$. Since the action of $G$ on $P$ is volume-preserving, these $n!$ cells have equal volume, hence $|P/G| = |P_1| = \frac{\pi^n}{n!}$. 
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Since the action of $G$ on $P$ is volume-preserving, these $n!$ cells have equal volume, hence $|P/G| = |P_1| = \frac{\pi^n}{n!}$. \textit{Evariste \bullet} would say \ldots Thus it remains to find a \textit{volume-preserving} bijection between $P_1$ and $B^{2n}$. But, before we do that, \textit{Pythagoras \bullet} complains \ldots
The Pythagorean Tessellation

The Volume of the Ball in $n$ Dimensions
Volume Preserving Maps

Let $z' = (z'_1, \ldots, z'_n)$ denote a point in $B^{2n}$, and $z = (z_1, \ldots, z_n)$ a point in $P_1$. We seek a bijective volume-preserving map between $z$ in $P_1$ and $z'$ in $B^{2n}$. 

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Let $r_1, \ldots, r_n$ and $\theta_1, \ldots, \theta_n$ be the radii and angles of $z_1, \ldots, z_n$. 

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The map \( z \leftrightarrow z' \) we seek is defined by

\[
\theta_1 = \theta'_1, \quad \theta_2 = \theta'_2, \ldots, \quad \theta_n = \theta'_n
\]

and

\[
r_1^2 = r'_1^2 \\
r_2^2 = r'_1^2 + r'_2^2 \\
\vdots \\
r_n^2 = r'_1^2 + r'_2^2 + r'_3^2 + \cdots + r'_n^2.
\]
Let $z' = (z'_1, \ldots, z'_n)$ denote a point in $B^{2n}$, and $z = (z_1, \ldots, z_n)$ a point in $P_1$. We seek a bijective volume-preserving map between $z$ in $P_1$ and $z'$ in $B^{2n}$.

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$$
r_1^2 = r'_1^2
$$

$$
r_2^2 = r'_1^2 + r'_2^2
$$

$$
\ldots
$$

$$
r_n^2 = r'_1^2 + r'_2^2 + r'_3^2 + \cdots + r'_n^2.
$$

This map is certainly a bijection between $z$ in $P_1$ and $z'$ in $B^{2n}$. 
Now a map \( z \rightarrow z' \) in \( C \) is area-preserving if the area elements agree
\[
dxdy = rdrd\theta = r'dr'd\theta' = dx'dy'.
\]
In particular, if \( d\theta' = d\theta \) and \( r'dr' = rdr \), the map is area preserving.

Similarly, a map \( z \rightarrow z' \) in \( C^n \) is volume-preserving if
\[
d\theta_1 \wedge \ldots \wedge d\theta_n = r_1 dr_1 \wedge \ldots \wedge r_n dr_n.
\]

Let \( r_1^2 + r_2^2 \leq 1 \) be the volume of the ball in \( n \) dimensions.
Now a map \( z \mapsto z' \) in \( \mathbb{C} \) is area-preserving if the area elements agree

\[
dxdy = r
drd\theta = r'
dr'\ d\theta' = dx'\ dy'.
\]
Volume Preserving Maps, Continued

Now a map $z \mapsto z'$ in $\mathbb{C}$ is area-preserving if the area elements agree

$$dxdy = r drd\theta = r' dr' d\theta' = dx' dy'.$$

In particular, if $d\theta' = d\theta$ and $r' dr' = r dr$, the map is area preserving.
Now a map $z \mapsto z'$ in $\mathbb{C}$ is area-preserving if the area elements agree

$$\text{d}x\text{d}y = r \text{d}r \text{d}\theta = r' \text{d}r' \text{d}\theta' = \text{d}x' \text{d}y'.$$

In particular, if $\text{d}\theta' = \text{d}\theta$ and $r' \text{d}r' = r \text{d}r$, the map is area preserving. Similarly, a map $z \mapsto z'$ in $\mathbb{C}^n$ is volume-preserving if

$$\text{d}\theta'_1 \text{d}\theta'_2 \ldots \text{d}\theta'_n = \text{d}\theta_1 \text{d}\theta_2 \ldots \text{d}\theta_n,$$

$$r'_1 \text{d}r'_1 r'_2 \text{d}r'_2 \ldots r'_n \text{d}r'_n = r_1 \text{d}r_1 r_2 \text{d}r_2 \ldots r_n \text{d}r_n.$$
But our map preserves coordinate angles and satisfies

\[ r_1^2 = r'_1^2 \]
\[ r_2^2 = r'_1^2 + r'_2^2 \]
\[ r_3^2 = r'_1^2 + r'_2^2 + r'_3^2, \]
\[ \ldots \]
\[ r_n^2 = r'_1^2 + r'_2^2 + r'_3^2 + \cdots + r'_n^2. \]
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Differentiating yields

\[ r_1dr_1 = r'_1dr'_1 \]
\[ r_2dr_2 = r'_1dr'_1 + r'_2dr'_2 \]
\[ r_3dr_3 = r'_1dr'_1 + r'_2dr'_2 + r'_3dr'_3, \]
\[ \ldots \]
\[ r_ndr_n = r'_1dr'_1 + r'_2dr'_2 + r'_3dr'_3 + \cdots + r'_ndr'_n. \]
Volume Preserving Maps, Continued

Since this is a sequence of (infinitesimal) shears,

\[ r'_1 \, dr'_1 \, r'_2 \, dr'_2 \, \ldots \, r'_n \, dr'_n = r_1 \, dr_1 \, r_2 \, dr_2 \, \ldots \, r_n \, dr_n, \]
Since this is a sequence of (infinitesimal) shears,
\[ r_1' \, dr_1' \, r_2' \, dr_2' \, \ldots \, r_n' \, dr_n' = r_1 \, dr_1 \, r_2 \, dr_2 \, \ldots \, r_n \, dr_n, \]
and we’re done.
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and we’re done. Setting \( t_1 = r_1'^2/2, \ldots, t_n = r_n'^2/2 \), all of the above is the same as saying

\[ B^{2n} = \begin{array}{c}
\end{array} \times \begin{array}{c}
\end{array} \quad (n = 3) \]
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\[ B^{2n} = \frac{\pi^n}{n!} \]
Odd Dimensional Spheres

The sphere $S^{2n+1}$ consists of all points
$$(z_1, \ldots, z_n, z_{n+1}) = (z, z_{n+1})$$
in $\mathbb{C}^{n+1}$ satisfying
$$r^2 = r_1^2 + \cdots + r_n^2 + r_{n+1}^2 = 1.$$
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$$r^2 = r_1^2 + \cdots + r_n^2 + r_{n+1}^2 = 1.$$  

Using $r_1 dr_1 + \cdots r_n dr_n + r_{n+1} dr_{n+1} = 0$, the map

$$(z, \theta) \rightarrow \left( z, \theta, \sqrt{1 - (r_1^2 + \cdots + r_n^2)} \right)$$

is a volume-preserving bijection

$$B^{2n} \times S^1 \rightarrow S^{2n+1}.$$
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is a volume-preserving bijection 
\[ B^{2n} \times S^1 \rightarrow S^{2n+1}. \]

Hence $|S^{2n+1}| = |S^1| |B^{2n}|$. 
Rephrasing the previous slide, the circle action on $S^{2n+1}$

$$(z_1, \ldots, z_n, z_{n+1}) \rightarrow (z_1, \ldots, z_n, e^{i\theta} z_{n+1})$$

has orbit space $B^{2n}$. 
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has orbit space $\mathbb{C}P^n$, so we have projections
\[ S^{2n+1} \xrightarrow{S^1} B^{2n} \xleftarrow{S^1} \mathbb{C}P^n \]
Rephrasing the previous slide, the circle action on $S^{2n+1}$

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