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Calculus in the Complex Plane

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Preface

As is well-known, calculus in the complex plane, more often known as complex analysis, is a rich subject, with thousands of interesting and applicable results, developed over the last two centuries, and with progress continuing at the forefront of research today.

The modest goal of this textbook is to provide both the necessary background, and the core few dozen results, enough to appreciate the richness of the subject.

There are many, many excellent books on this subject, so why another book? In our case, we needed a source that included precalculus material, and also arrived quickly at the core results in as elementary a manner as possible.

Historically, complex numbers first appeared as murky algebraic quantities. As is well-known, these quantities were only fully accepted after they were interpreted as points in the plane.

In chapter 1, we reverse the historical development, and show the arithmetic operations — addition, subtraction, multiplication, and division — can be defined directly on points in the plane. By this, we mean exhibiting the arithmetic operations as the result of synthetic [4] geometric constructions, that lead to the known coordinate formulas. In particular, complex multiplication and division appear as consequences of angle stacking.

Ironically, Descartes [2], who in 1637 coined the term "imaginary", and discovered the correspondence between points in the plane and pairs of real numbers, missed[1] the fact that points within his cartesian plane are quantities possessing an arithmetic, i.e. they *are* numbers. Two hundred years later, complex numbers were firmly embedded in the plane, and in 1831 Gauss ended the confusion, saying [3]

> If one formerly contemplated this subject from a false point of view and therefore found a mysterious darkness, this is in large part attributable to clumsy terminology. Had one not called $+1$, -1, $\sqrt{-1}$ positive, negative, or imaginary (or even impossible) units, but instead, say, direct, inverse, or lateral units, then there could scarcely have been talk of such darkness.

Because of the connection between complex numbers and angle stacking, the basic results of trigonometry follow. This is carried out in Chapter 2, which covers

[1] hindsight is 20/20.

the fundamental theorem of trigonometry, the roots of unity, the binomial theorem, and the core properties of real exponentials and real Taylor series.

Chapter 3 covers complex series and complex elementary functions, leading to Euler's identity.

Chapter 4 covers complex derivatives, the Cauchy-Riemann equation, differentiability of complex power series, and the principal logarithm and the principal square root.

Chapter 5 covers contour integration, winding numbers, and branches.

Chapter 6 covers Cauchy's theorems in the standard sequence: First for rectangles, then for disks, then for arbitrary open sets. Zeros, poles, and other standard applications follow. This material I learned from Ahlfors [1].

Chapter 7 covers the residue theorem and examples of evaluation of real infinite integrals.

Apart from the elementary topology of the real line, and the two-variable real chain rule, the text is self-contained.

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The Complex Plane

In secondary school, one learns that an angle is the "space" between two intersecting lines, and learns to assign a numerical value to each angle. In degrees, the value of an angle varies between 0 and 360, while in radians, the value varies between 0 and 2π. The assignment *angle* $\rightarrow$ *value* is not explained, or explained so badly, that the use of π is seemingly reduced to a choice of units, degrees or radians.

In fact, the angle-to-value assignment problem is so fundamental, it lies at the heart of the meaning of measurement and number. In particular, this underlines the fundamental nature of the absolute constant π. This problem, faced by the greeks, was solved by Archimedes more than 2200 years ago by his bisection method.

The greeks understood how to assign quantities to intervals that are *commensurate*: intervals that are multiples of a single base unit interval. The first indication of the limitations of commensurability was the discovery, by the Pythagoreans, of $\sqrt{2}$ and the incommensurate nature of the length of the diagonal of the square.

Building upon this, it was Archimedes who realized that angle measure involved the measurement of arcs on circles, quantities that are even more incommensurate than $\sqrt{2}$, and that angle stacking reduced to addition of arclength.

Ultimately the key to these issues was recognized in the nineteenth century as the *completeness property*[1] of the real numbers. In effect, the first use of this property was the Babylonian algorithm for $\sqrt{2}$, and the second was the Archimedes bisection method for π.

Angle additivity is our intuitive understanding that the measure of stacked angles is the sum of their measures. As we shall see, making this intuition precise leads to multiplication and division of points in the plane. This exposes a surprising fact: points in the plane possess an arithmetic, they behave like numbers. Because of this, when endowed with this arithmetic, points in the plane are called complex numbers, and the plane becomes the complex plane.

[1] Every increasing bounded sequence of real numbers has a limit.

1.1 Arithmetic of Points

In the plane, each point P has a **shadow**. This is the triangle constructed by dropping the perpendicular from P to a reference line, and joining P to a reference point O on the line, as in Figure 1.1.

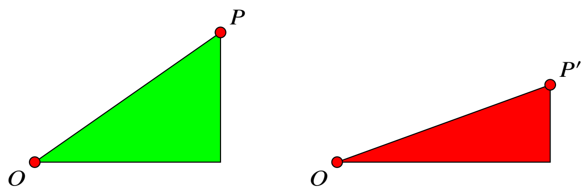

Fig. 1.1 Points P and P' and their shadows in the plane

In the cartesian plane, points $P = (x, y)$ and $P' = (x', y')$ are added by adding their coordinates,

Addition of points

$$P'' = P + P' = (x + x', y + y'). \tag{1.1}$$

This is the same as combining their shadows as in Figure 1.2. Addition of points depends on the choice of reference line and reference point. In the cartesian plane, this is not an issue, because these, the x-axis and the origin, are fixed at the outset.

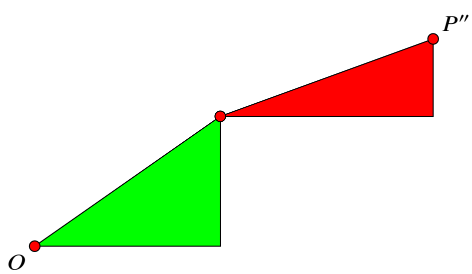

Fig. 1.2 Adding P and P'

For example, $P = (-3, 1)$ and $Q = (-2, -2)$ yields

$$P + Q = (-3, 1) + (2, -2) = (-3 + 2, 1 - 2) = (-1, -1) = R.$$

It is easy to check that this addition satisfies the commutative law

$$P + P' = P' + P$$

and associative law

$$P + (P' + P'') = (P + P') + P''.$$

Distance Formula

If $P = (x, y)$ and $P' = (x', y')$, then the **distance** between P and P' is

$$|P - P'| = \sqrt{(x - x')^2 + (y - y')^2}.$$

The distance of a point $P = (x, y)$ to the origin $O = (0, 0)$ is its **absolute value** or **radius**

$$r = |P| = |P - O| = \sqrt{x^2 + y^2}.$$

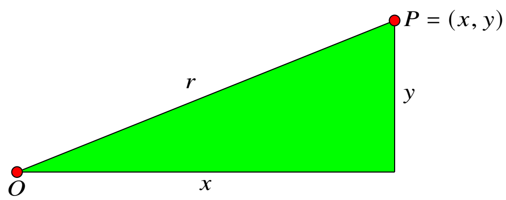

Fig. 1.3 The absolute value $r = |P|$ of P

Thus *the absolute value of P is the length of the hypotenuse of the shadow of P*.

A point $P = (x, y)$ in the plane may be stretched by stretching the shadow as in Figure 1.4. This is **dilation** by t. Note when t is negative, the shadow is also flipped.

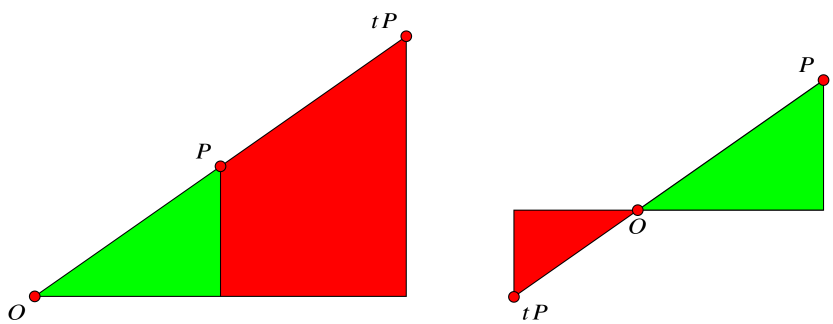

Fig. 1.4 Dilation with $t = 2$ and $t = -2/3$

Given a point P, the dilates of P form a line passing through O (Figure 1.5).

Dilation of points

$$tP = (tx, ty).$$

If t and s are real numbers, it is easy to check

$$t(P + P') = tP + tP' \qquad \text{and} \qquad t(sP) = (ts)P.$$

Thus multiplying P by s, and then multiplying the result by t, has the same effect as multiplying P by ts, in a single step.

We set $-P = (-1)P$, and define subtraction of points by

$$P - Q = P + (-Q).$$

This gives

Subtraction of points

$$P'' = P - P' = (x - x', y - y'). \tag{1.2}$$

The **unit circle** consists of the points which are distance 1 from the origin O. When P is on the unit circle, the line formed by the dilates of P intersects the unit circle at $\pm P$ (Figure 1.5).

The unit circle intersects the horizontal axis at the points $I = (1, 0)$, and $(-1, 0)$, and intersects the vertical axis at the points $(0, 1)$, and $(0, -1)$. These four points are equally spaced on the unit circle (Figure 1.5).

By the distance formula, a point $P = (x, y)$ is on the unit circle when

$$x^2 + y^2 = 1.$$

More generally, any circle with **center** $C = (a, b)$ and **radius** r consists of points $P = (x, y)$ satisfying $|P - C| = r$, or

$$(x - a)^2 + (y - b)^2 = r^2.$$

While we call both $|t|$ and $|P|$ the absolute value, note the former is the absolute value of a number, while the latter is the absolute value of a point in the plane. Even though these quantities refer to different objects, we use the same terminology because $|t|$ and $|P|$ have similar properties.

Given this, it is easy to check

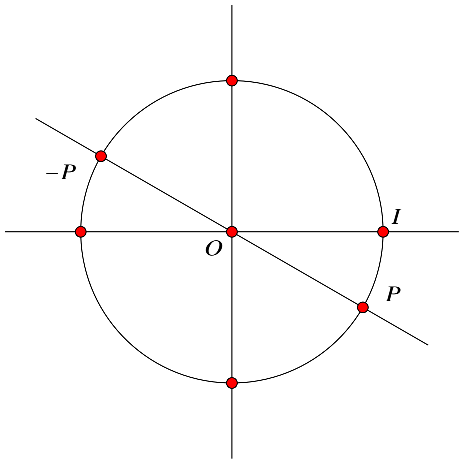

Fig. 1.5 P and its antipode $-P$

$$|tP| = |t|\,|P|$$

for any real number t and point P.

From this, if a point Q is on the unit circle and $r > 0$, then rQ is on the circle of radius r and center O, and $C + rQ$ is on the circle of radius r and center C. Thus P *is on the circle of radius* r *and center* C *iff* P *is of the form* $C + rR$*, for some* R *on the unit circle*.

If P is any point not equal to the origin, then $r = |P|$ is positive, and

$$\left|\frac{1}{r}P\right| = \frac{1}{r}\,|P| = \frac{1}{r}r = 1,$$

so P/r is on the unit circle.

1.2 Angle Stacking

In the cartesian plane, points may be added, subtracted, and dilated. To multiply and divide points, we first need to learn how to stack angles.

An **angle** is an ordered pair of lines ("rays") starting from a common point, which is called the **vertex** of the angle. If the vertex is the origin O, then an angle is determined by the intersection of its rays with the unit circle. In other words, an angle is determined by an ordered pair of points P, P' on the unit circle.

We want to measure angles. Whatever method one takes for the measure θ of the angle, it should be **additive**: When angles are stacked, their measures should add (Figure 1.8).

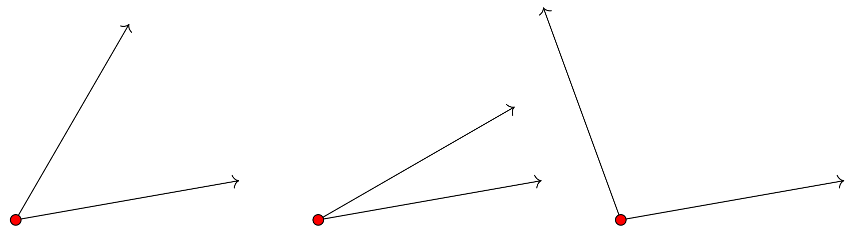

Fig. 1.6 Angles

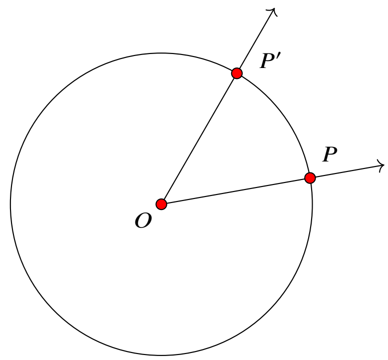

Fig. 1.7 Intersection of angle with unit circle

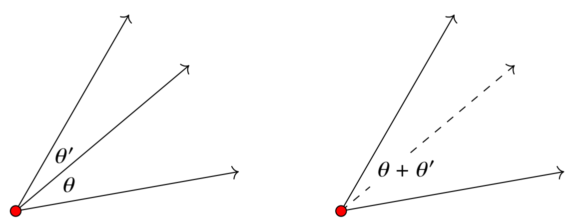

Fig. 1.8 Angle additivity

Additive angle measure θ was introduced by the greek mathematician Archimedes 2200 years ago, as part of his work leading to his famous estimate of the measure of the half-circumference of the unit circle

$$\frac{223}{71} < \pi < \frac{22}{7}. \tag{1.3}$$

By contrast, the greek mathematicians Hipparchus and Ptolemy used chord measure θ_1, which is simpler than θ but not additive, to build their trigonometric tables. Archimedes's method, θ, and θ_1 are in §1.4.

We now explain how to stack angles. We call an angle **anchored** if its first point is $I = (1, 0)$. Then an anchored angle is determined by the second point P on the unit circle.

To stack anchored angles, let $P = (x, y)$ and $P' = (x', y')$ be on the unit circle, and let P'' be obtained by stacking P' atop P, as in Figure 1.9. We seek the formula for P''.

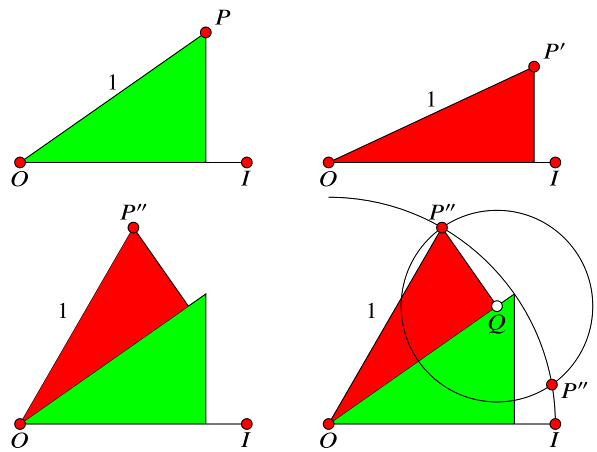

Fig. 1.9 Stacking anchored angles

The above description of stacking is imprecise, because we do not yet know how to measure angles, and in particular do not yet know what a right angle is.

To make precise the stacking construction, assume $P' = (x', y')$ is in the first quadrant $x' > 0$, $y' > 0$, and draw the circle with center $Q = x'P$ and radius y', as in Figure 1.9. Then this circle intersects the unit circle at two points, both denoted P'' (Figure 1.9).

We think of the first point P'' as the result of multiplying P and P', and we write $P'' = PP'$, and we think of the second point P'' as the result of dividing P by P', and we write $P'' = P/P'$. Then we have the

Angle Stacking Formulas

$$\begin{aligned} P'' &= PP' = (xx' - yy', x'y + xy'), \\ P'' &= P/P' = (xx' + yy', x'y - xy'). \end{aligned} \tag{1.4}$$

The angle stacking formulas show multiplication and division of points in the cartesian plane are forced upon us as soon as we attempt to answer the simplest question: How do we make Figure 1.8 precise?

To derive (1.4), let $P^\perp = (-y, x)$ (pronounced "P-perp"). Then

$$x'P + y'P^{\perp} = (x'x, x'y) + (-y'y, y'x) = (xx' - yy', x'y + xy'),$$
$$x'P - y'P^{\perp} = (x'x, x'y) - (-y'y, y'x) = (xx' + yy', x'y - xy'),$$

so the angle stacking formulas are equivalent to $P'' = x'P \pm y'P^{\perp}$.

To establish this, since P'' is on the circle of center Q and radius y', we may write $P'' = Q + y'R$, for some point $R = (a, b)$ on the unit circle (§1.1). Then $P'' = x'P + y'R$ is on the unit circle iff

$$\begin{aligned}1 = |x'P + y'R|^2 &= |(x'x + y'a, x'y + y'b)|^2 \\ &= (x'x + y'a)^2 + (x'y + y'b)^2 \\ &= x'^2(x^2 + y^2) + y'^2(a^2 + b^2) + 2x'y'(ax + by) \\ &= x'^2 + y'^2 + 2x'y'(ax + by) \\ &= 1 + 2x'y'(ax + by).\end{aligned}$$

But this happens iff $ax + by = 0$, which happens iff R is a dilation of $P^{\perp}$. Since R is on the unit circle, this forces $R = \pm P^{\perp}$ (Figure 1.5), establishing (1.4).

Notice the formula $P'' = x'P + y'P^{\perp}$ refers P'' to the axes through P and $P^{\perp}$ as in Figure 1.10.

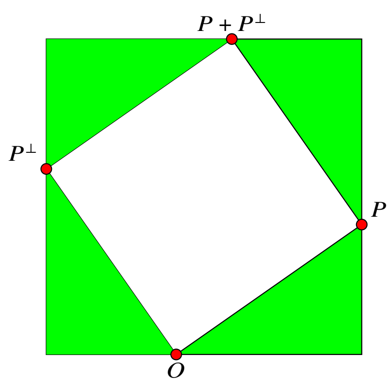

Fig. 1.10 P and $P^{\perp}$

The angle stacking formulas are valid only when P and P' are on the unit circle. To derive the formulas for multiplication and division of points in general, we proceed as follows.

Let $r = |P|$ and $r' = |P'|$. Then (§1.1)

$$P/r = \frac{1}{r}P = \left(\frac{x}{r}, \frac{y}{r}\right) \qquad \text{and} \qquad P'/r' = \frac{1}{r'}P' = \left(\frac{x'}{r'}, \frac{y'}{r'}\right)$$

are on the unit circle.

If we insist multiplication should satisfy $(tP)(sP') = (ts)(PP')$, it is natural to write

$$PP' = rr'(P/r)(P'/r') = rr'\left(\frac{x}{r}\frac{x'}{r'} - \frac{y}{r}\frac{y'}{r'}, \frac{x'}{r'}\frac{y}{r} + \frac{x}{r}\frac{y'}{r'}\right)$$
$$= rr'\frac{1}{rr'}(xx' - yy', x'y + xy') = (xx' - yy', x'y + xy'),$$

Thus the general formula is the same, and stacking of points may be carried out in general[2] (Figure 1.11). In this case, however, the second point P'' is not P/P', it is $P\bar{P}'$, where $\bar{P}' = (x', -y')$.

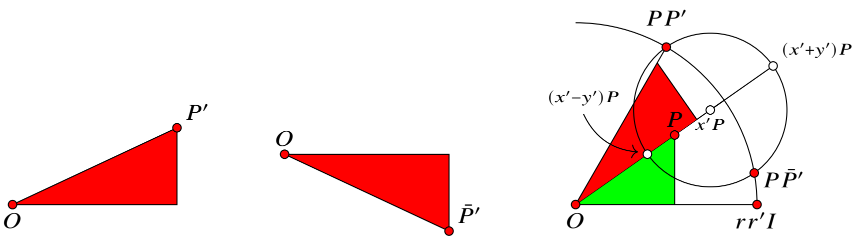

Fig. 1.11 Stacking P and P' in general

Multiplication of Points

$$P'' = PP' = (xx' - yy', x'y + xy'). \tag{1.5}$$

We now turn to division of two points in general. If we insist division should satisfy $(tP)/(sP') = (t/s)(P/P')$, it is natural to write

$$\frac{P}{P'} = \frac{r}{r'} \cdot \frac{P/r}{P'/r'} = \frac{r}{r'}\left(\frac{x}{r}\frac{x'}{r'} + \frac{y}{r}\frac{y'}{r'}, \frac{x'}{r'}\frac{y}{r} - \frac{x}{r}\frac{y'}{r'}\right)$$
$$= \frac{r}{r'} \cdot \frac{1}{rr'}(xx' + yy', x'y - xy') = \frac{1}{r'^2}(xx' + yy', x'y - xy').$$

Thus the general formula is different,

[2] Here the circle with center $Q = x'P$ has radius ry'.

Division of Points

$$P'' = P/P' = \frac{1}{x'^2 + y'^2}\,(xx' + yy', x'y - xy'), \tag{1.6}$$

for $P' \neq O$.

Of course, if P' is on unit circle, then $x'^2 + y'^2 = 1$, so the general formula for division agrees with the angle stacking formula for P/P'.

Even if we ignore the angle stacking justification for (1.5) and (1.6), one can check readily that the four arithmetic operations (1.1), (1.2), (1.5), (1.6) satisfy the usual rules of arithmetic: subtraction is the inverse of addition, O is the additive identity, division is the inverse of multiplication, I is the multiplicative identity, and we have commutativity, associativity, and distributivity.

1.3 Complex Numbers

Points in the cartesian planes may be added, subtracted, multiplied, and divided (1.1),(1.2), (1.5), (1.6). When we consider points with these arithmetic operations, they behave like numbers, so we view them as new kinds of numbers. These are the **complex numbers**. With this viewpoint, the cartesian plane becomes the **complex plane**, which we denote **C**.

We begin our study of complex numbers by introducing the standard notation for them. Instead of writing $(0, 0)$, $(1, 0)$, $(2, 0)$, etc, the horizontal axis is called the **real line** and points on it are written 0, 1, 2, etc.

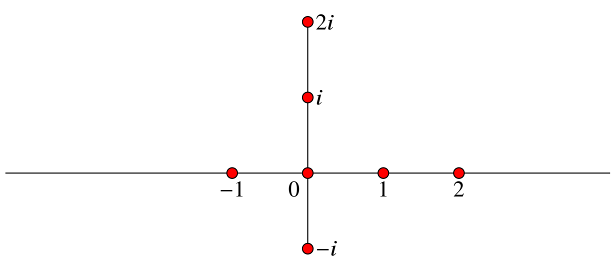

Fig. 1.12 The complex plane

Also, instead of writing $(0, 0)$, $(0, 1)$, $(0, 2)$, etc, the vertical axis is called the **imaginary line** and points on it are written $0i$, i, $2i$, etc. Of course, the origin is the point corresponding to $0 = 01 = 0i$.

Writing

$$(x, y) = (x + 0, 0 + y) = (x, 0) + (0, y) = x(1, 0) + y(0, 1) = x1 + yi = x + iy,$$

we see *every point* (x, y) *may be written* $x + iy$. In the complex plane, points are denoted z, and every complex number is of the form $z = x + iy$.

Points in the plane are complex numbers

$$P = (x, y) \qquad \text{is the same as} \qquad z = x + iy$$

In particular, every real number x is a complex number $x+i0$, and every imaginary number iy is a complex number $0 + iy$.

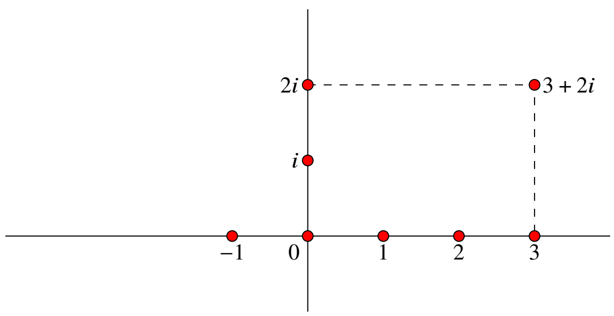

Fig. 1.13 Complex numbers

Adding complex numbers is easy, just add the coordinates: If $z = 3+2i$, $z' = 1-5i$, then

$$z + z' = (3 + 2i) + (1 - 5i) = (3 + 1) + (2 - 5)i = 4 - 2i.$$

Addition of Complex Numbers

If $z = x + iy$ and $z' = x' + iy'$, then

$$z + z' = (x + x') + i(y + y').$$

Of course, this is the same as (1.1) in §1.1. Similarly, (1.2) in §1.1 becomes

Subtraction of Complex Numbers

If $z = x + iy$ and $z' = x' + iy'$, then

$$z - z' = (x - x') + i(y - y').$$

To multiply, it is best to start by computing i^2 following the angle stacking formula. Since the complex number i is the point $(0, 1)$, to compute $i^2 = ii$, we set $P = (x, y) = (0, 1)$ and $P' = (x', y') = (0, 1)$, and use the angle stacking formula

$$i^2 = ii = PP' = (xx' - yy', x'y + xy') = (00 - 11, 01 + 01) = (-1, 0) = -1.$$

We conclude $i^2 = -1$, or i is the square root of -1. Since $(-i)^2 = i^2 = -1$, $-i$ is also the square root of -1, so -1 has two square roots $\pm i$.

Now we multiply complex numbers using standard algebra, then replacing i^2 by -1: If $z = 3 + 2i$, $z' = 1 - 5i$, then

$$\begin{aligned} zz' &= (3 + 2i)(1 - 5i) = 3(1) + 3(-5i) + (2i)1 + (2i)(-5i) \\ &= 3 - 15i + 2i - 10i^2 = 3 - 13i - 10(-1) = 13 - 13i. \end{aligned}$$

In general, if we repeat the same procedure, the product of $z = x + iy$ and $z' = x' + iy'$ is

Multiplication of Complex Numbers

If $z = x + iy$ and $z' = x' + iy'$, then

$$zz' = (x + iy)(x' + iy') = (xx' - yy') + i(x'y + xy').$$

Of course, this is the same as (1.5) in §1.2. Just like before, the absolute value of a complex number $z = x + iy$ is

$$r = |z| = \sqrt{x^2 + y^2},$$

so

$$r^2 = |z|^2 = |x + iy|^2 = x^2 + y^2.$$

Every complex number z has a **real part** $\mathrm{Re}(z) = x$ and an **imaginary part** $\mathrm{Im}(z) = y$. These parts are real numbers: The real part of $z = x + iy$ is x, and the imaginary part is y (not yi). So $\mathrm{Re}(3 + 2i) = 3$ and $\mathrm{Im}(3 + 2i) = 2$ (not $2i$).

Also $\mathrm{Re}(5) = 5$ and $\mathrm{Im}(5) = 0$, since $5 = 5 + 0i$. The real part of i is 0, and the imaginary part of i is 1, since $i = 0 + 1i$.

An important formula is

$$|zz'| = |z|\,|z'|, \qquad \text{or} \qquad |zz'|^2 = |z|^2\,|z'|^2.$$

This says *the absolute value of the product is the product of the absolute values.* This looks less obvious when written out in detail using the angle stacking formula,

$$(xx' - yy')^2 + (x'y + xy')^2 = \left(x^2 + y^2\right)\left(x'^2 + y'^2\right), \tag{1.7}$$

but this is easily checked by multiplying out the terms.

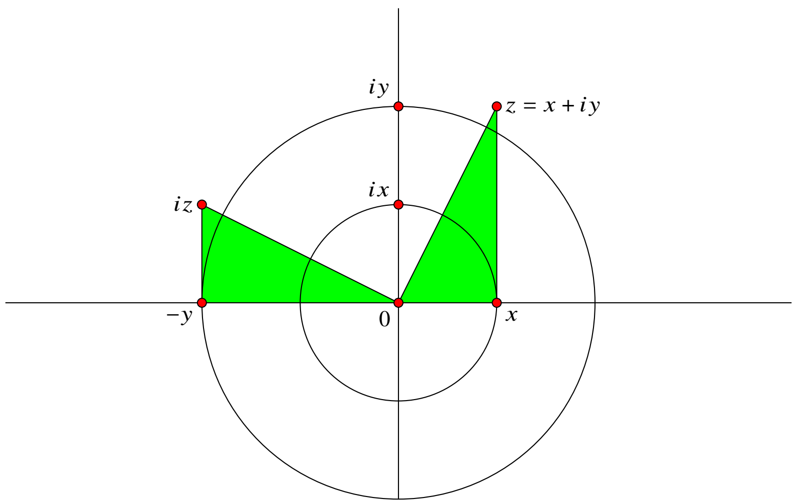

Fig. 1.14 z and iz

Remember (Figure 1.10) if $P = (x, y)$, then $P^{\perp} = (-y, x)$ (pronounced "P-perp"). In terms of complex numbers, this is $z = x + iy$ and

$$iz = i(x + iy) = ix + iy^2 = -y + xi.$$

Thus *if z is the complex number corresponding to P, then iz is the complex number corresponding to $P^{\perp}$* (Figure 1.14).

We interpret the multiplication of complex numbers z and w geometrically. To be specific, we take $w = 2 + 3i$. Multiplying,

$$zw = z(2 + 3i) = 2z + 3iz = 2z + 3(iz),$$

and Figure 1.15 displays the outcome: The product zw is obtained by placing the shadow of w along the line through z, and dilating it by the factor $|z|$.

Above we saw $\pm i$ are the square roots of -1. Now we give a formula for the square root of $z = x + iy$, as long as z is not a negative real number or zero. When $|z| = 1$, z is on the unit circle, and the formula is

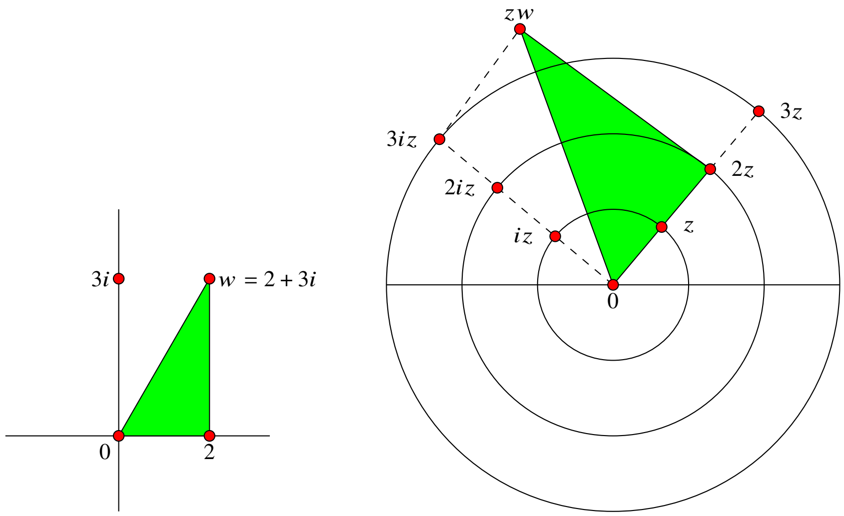

Fig. 1.15 Multiplying z and $w = 2 + 3i$

$$\sqrt{z} = \frac{(x+1) + iy}{\sqrt{2+2x}}. \tag{1.8}$$

In general, the formula is

$$\sqrt{z} = \frac{(x+r) + iy}{\sqrt{2r+2x}}, \tag{1.9}$$

where $r = |z| = \sqrt{x^2 + y^2}$.

Note these formulas for $\sqrt{z}$ are not valid when z is a negative real number or zero, because then the denominator is zero. From (1.9), it follows $\sqrt{z}$ is continuous.[3] Since $(\sqrt{z})^2 = z$, $\sqrt{z}$ is injective.[4]

Since

$$\left(\sqrt{z}\sqrt{z'}\right)^2 = (\sqrt{z})^2 \left(\sqrt{z'}\right)^2 = zz',$$

$\sqrt{z}\sqrt{z'}$ is a square root of zz'. Since zz' has at most two square roots, we must have $\sqrt{zz'} = \pm\sqrt{z}\sqrt{z'}$. In fact, when z, z' are in the unit circle first quadrant, it is easy to check that it is the + sign, so

$$\sqrt{zz'} = \sqrt{z}\sqrt{z'}. \tag{1.10}$$

This says *the product of the square roots is the square root of the product*, when z and z' are in the first quadrant.

If $z = x + iy$, let $\bar{z} = x - yi$. This is the **conjugate** of z. Then

[3] $z_n \to z$ implies $\sqrt{z_n} \to \sqrt{z}$.

[4] $z \neq z'$ implies $\sqrt{z} \neq \sqrt{z'}$.

$$z\bar{z} = (x + iy)(x - yi) = x^2 + y^2 = |z|^2.$$

So, *the product of a complex number and its conjugate equals its absolute value squared*.

We evaluate the reciprocal $1/z$ of z by multiplying by the conjugate $\bar{z}$,

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{1}{x + iy} \cdot \frac{x - iy}{x - iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}.$$

For example,

$$\frac{1}{2 + 3i} = \frac{1}{2 + 3i} \cdot \frac{2 - 3i}{2 - 3i} = \frac{2 - 3i}{2^2 + 3^2} = \frac{2}{13} - \frac{3i}{13}.$$

So the real part of $1/(2 + 3i)$ is $2/13$, and the imaginary part is $-3/13$.

More generally, to divide $z = 1 + 2i$ by $w = 2 + 3i$,

$$\frac{z}{w} = z \cdot \frac{1}{w} = (1 + 2i) \cdot \frac{1}{2 + 3i} = (1 + 2i)\left(\frac{2}{13} - \frac{3i}{13}\right) = \frac{8}{13} + \frac{i}{13}.$$

In general,

Division of Complex Numbers

If $z = x + iy$ and $z' = x' + iy'$ and $z' \neq 0$, then

$$\frac{z}{z'} = \frac{x + iy}{x' + iy'} = \frac{x + iy}{x' + y'i} \cdot \frac{x' - iy'}{x' - iy'} = \frac{(xx' + yy') + i(x'y - xy')}{x'^2 + y'^2}.$$

Of course, this is the same as (1.6) in §1.2.

1.4 Angle Bisection

From Figure 1.16, we see a natural measure of an angle is *the length of the arc along the unit circle*. Let P be a point on the unit circle. How do we measure the length of the arc joining P and $I = (1, 0)$? This measure, which we call θ, tells us how far along the circle the point P is.

Similarly, let z be a complex number on the unit circle. How do we measure the length of the arc joining z and 1? This measure, which we call θ, tells us how far along the circle z is.

These are the same question because, as we saw, there is a natural correspondence between a point $P = (x, y)$ and a complex number $z = x + iy$.

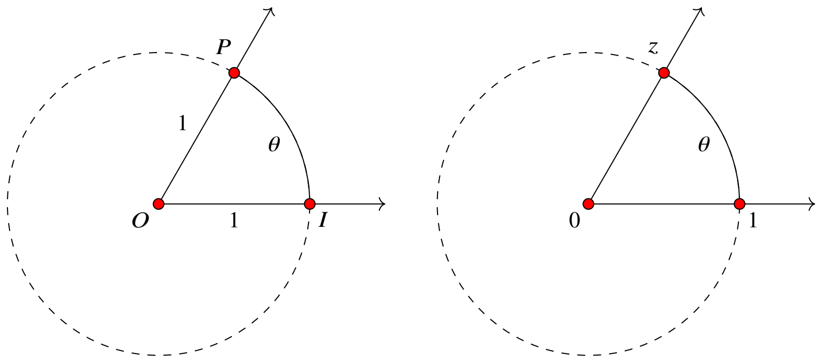

Fig. 1.16 θ is the length of the arc

Archimedes answered this question more than 2200 years ago using his bisection method. This goes as follows.

Let $z = x + iy$ be on the unit circle, and define, as in Figure 1.17,

$$m = \frac{z+1}{2}, \qquad z_1 = \frac{m}{|m|}.$$

When $z = -1$, we are dividing by $|m| = 0$, so we assume $z \neq -1$. Then

$$z_1 = \frac{(x+1) + iy}{\sqrt{2+2x}} = x_1 + iy_1. \tag{1.11}$$

By (1.8), $z_1 = \sqrt{z}$.

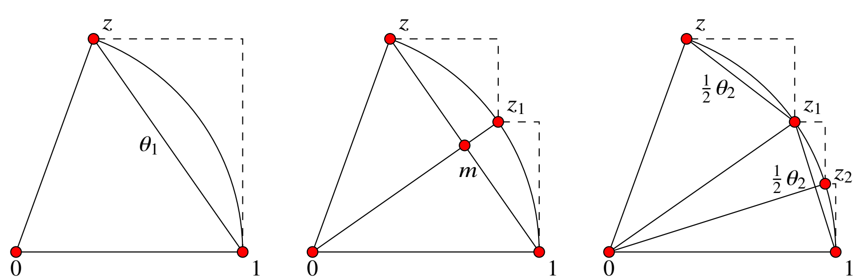

Fig. 1.17 Bisection

With $z_1 = x_1 + iy_1$, $x_1 > 0$, hence $\sqrt{z}$ is in the right-half unit circle. Moreover, the map $\sqrt{z}$ is injective on the **punctured unit circle** $z \neq 1$, and the imaginary parts of z and $\sqrt{z}$ have the same sign.

Let $\theta_1 = \theta_1(z) = |z - 1|$ be as in in Figure 1.17, with z in the unit circle first quadrant. Then θ_1 is **chord measure**, and it is easy to check $\theta_1 = 2y_1$, and

$$y < 2y_1 < \frac{2y_1}{x_1} < \frac{y}{x}. \tag{1.12}$$

Similarly, let $z_2 = x_2 + iy_2 = \sqrt{z_1}$ and let $\theta_2 = \theta_2(z) = 2\theta_1(z_1)$ be the chord-length sum in Figure 1.17. Then $\theta_2 = 4y_2$ and

$$2y_1 < 4y_2 < \frac{4y_2}{x_2} < \frac{2y_1}{x_1}, \tag{1.13}$$

when z is on the upper-half unit circle $y > 0$.

We repeat this process indefinitely as follows. If we define θ_n and $z_n = x_n + iy_n$ recursively by $\theta_{n+1}(z) = 2\theta_n(\sqrt{z})$ and $z_{n+1} = \sqrt{z_n}$, $n \geq 2$, then $\theta_1, \theta_2, \theta_3, \ldots$ are obtained by repeated bisection of the subtended arc, and it is easy to check

$$\theta_1 = 2y_1, \qquad \theta_2 = 4y_2, \qquad \theta_3 = 8y_3, \qquad \theta_4 = 16y_4, \qquad \ldots \tag{1.14}$$

Repeating (1.13) gives $y < 2^n y_n$ and

$$2y_1 < 4y_2 < 8y_3 < 16y_4 < \cdots < \frac{16y_4}{x_4} < \frac{8y_3}{x_3} < \frac{4y_2}{x_2} < \frac{2y_1}{x_1}.$$

Inserting (1.14) gives the **Archimedes sequence**

$$\theta_1 < \theta_2 < \theta_3 < \theta_4 < \cdots < \frac{\theta_4}{x_4} < \frac{\theta_3}{x_3} < \frac{\theta_2}{x_2} < \frac{\theta_1}{x_1}, \tag{1.15}$$

when z is on the upper-half unit circle $y > 0$.

Let z be in the unit circle first quadrant. Then $y - x - 1 < 0$, so

$$y_1 - x_1 = \frac{y - x - 1}{\sqrt{2 + 2x}} < \frac{1}{2}(y - x - 1),$$

leading to

$$2(1 - x_1 + y_1) < 1 - x + y.$$

Repeating this, we obtain

$$y < 2^n y_n < 2^n(1 - x_n + y_n) < 1 - x + y, \tag{1.16}$$

for $n \geq 1$. Hence $y < \theta_n < 1 - x + y$, $n \geq 1$, as suggested by the dashed lines in Figure 1.17.

Since the sequence θ_n is bounded, by (1.14), $y_n \to 0$ as $n \to \infty$. Since, for $0 < x < 1$,

$$x_1 = \sqrt{\frac{1 + x}{2}} > \sqrt{x} > x,$$

the sequence x_n is increasing. It follows $x_n \to 1$ as $n \to \infty$.

By (1.15), the sequence θ_n is increasing and bounded. By the completeness property[5] of the real numbers, this sequence has a limit. By (1.15) again, the sequence θ_n/x_n is decreasing and bounded. By the completeness property of the real numbers again, this sequence has a limit. Since $x_n \to 1$ as $n \to \infty$, these limits coincide, thus the sequences in (1.15) have a common limit $\theta = \theta(z)$. By construction, $\theta(z) = 2\theta(\sqrt{z})$ follows, when z is on the upper-half unit circle $y > 0$.

Appealing to (1.14), and passing to the limit in (1.16), as $n \to \infty$, when z is in the unit circle first quadrant, we obtain

$$y < \theta(z) \leq 1 - x + y, \qquad z = x + iy, \tag{1.17}$$

as suggested by the dashed lines in Figure 1.17.

Extend $\theta(z)$ to the lower-half unit circle $y < 0$ by $\theta(z) = -\theta(1/z)$, and set $\theta(1) = 0$. Then $\theta(1/z) = -\theta(z)$ and $\theta(z) = 2\theta(\sqrt{z})$ for every $z \neq -1$ on the unit circle.

This completes the construction of the Archimedes angle measure $\theta = \theta(z)$, for every z on the unit circle, other than $z = -1$.

Let z be in the unit circle first quadrant. From (1.11), y_1 is an increasing function of y. Similarly, with y_2 playing the role of y_1, y_2 is an increasing function of y_1, hence an increasing function of y. Continuing in this manner, y_n, $n \geq 1$, are increasing functions of y. By (1.14), θ_n, $n \geq 1$, are increasing functions of y. Passing to the limit, it follows $\theta(z)$ is an increasing function of y, when z is in the unit circle first quadrant. Since $\theta(\bar{z}) = \theta(1/z) = -\theta(z)$, $\theta(z)$ is an odd function of y. Thus $\theta(z)$ is an increasing function of y, when z is in the right-half unit circle.

1.5 Angle Additivity

For any nonzero complex number z, let $r = |z|$ be its absolute value. Then

$$\left|\frac{z}{|z|}\right| = \left|\frac{z}{r}\right| = \frac{1}{r} \cdot |z| = 1,$$

so z/r has absolute value 1, so z/r lies on the unit circle (see Figure 1.18).

Extend the definition of $\theta(z)$ to any z that is not a negative real number nor zero, by setting

$$\theta(z) = \theta\left(\frac{z}{r}\right), \qquad r = |z|.$$

Then $r = |z|$ and $\theta = \theta(z)$ are **polar coordinates**: r is the distance of z to 0, and θ is the length of the arc along the unit circle between 1 and z/r.

Since θ is defined for all z except for $z = x \leq 0$, $\theta(i)$ is defined. We call this measure $\pi/2$: $\pi/2$ is defined as the (measure of the) angle of i.

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[5] Every increasing bounded sequence has a limit.

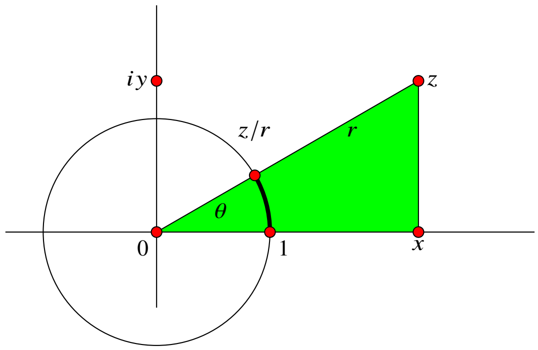

Fig. 1.18 Polar coordinates

Definition of π

$$\pi/2 = \theta(i) \qquad \text{or} \qquad \pi = 2\theta(i).$$

Since $\theta(-i) = \theta(1/i) = -\pi/2$, and $\theta(z)$ is an increasing function of y when z is in the right-half unit circle (§1.4), it follows $\theta(z)$ varies from $-\pi/2$ to $\pi/2$ when z is in the right-half unit circle. Using $\theta(z) = 2\theta(\sqrt{z})$, it follows θ varies between $-\pi$ and π when z is in the punctured unit circle $z \neq -1$.

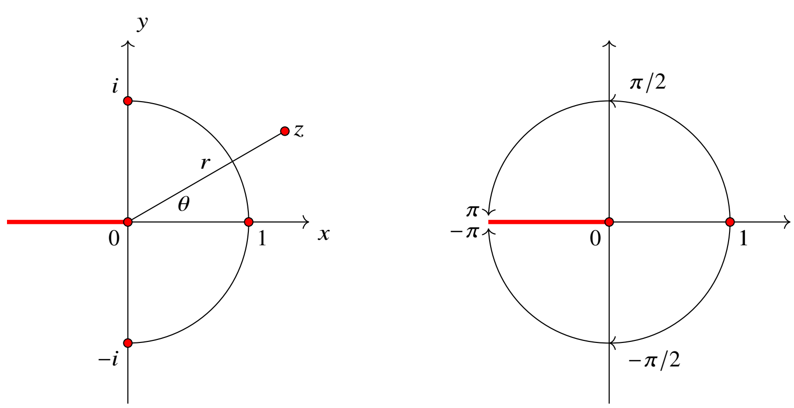

Fig. 1.19 $\theta = \theta(z)$ is defined for all z except for $z = x \leq 0$

The key property of $\theta(z)$ is its angle additivity. By the angle stacking formula, this is the same as additivity as in Figure 1.8.

Angle Additivity

$$\theta(zz') = \theta(z) + \theta(z'), \qquad x > 0, x' > 0. \tag{1.18}$$

Proof (Proof of additivity) Let $z = x+iy$, $z' = x'+iy'$ be in the right-half unit circle, $x > 0$ and $x' > 0$, and let $z'' = zz' = x'' + iy''$. Let $\theta = \theta(z)$, $\theta' = \theta(z')$, and let $\theta'' = \theta(z'')$. Since (1.18) is immediate when $yy'y'' = 0$, we may assume $yy'y'' \neq 0$. There are two cases.

First, if $yy' > 0$, using $\theta(1/z) = -\theta(z)$, we may assume $y > 0$ and $y' > 0$. Let $z'_n = x'_n + iy'_n$, $z''_n = x''_n + iy''_n$, $n \geq 1$, be the Archimedes sequences starting from z', z'' respectively, and let θ'_n, θ''_n, $n \geq 1$, be the corresponding chord-length sums. Then, by (1.10), $z''_n = z_n z'_n$, thus $y''_n = x'_n y_n + x_n y'_n$, hence, by (1.14),

$$\theta''_n = x'_n \theta_n + x_n \theta'_n, \qquad n \geq 1.$$

Now let n approach ∞. Then x_n approaches 1, and x'_n approaches 1, so we obtain $\theta'' = \theta + \theta'$, which is (1.18).

Second, if $yy' < 0$, we have $x'' = xx' - yy' > 0$. Since $yy' < 0$, $y''(-y)$ and $y''(-y')$ have opposite signs. By switching the roles of z and z' if necessary, we may assume $y''(-y) > 0$. Applying the first case to z'' and $1/z = x - iy$,

$$\theta(z') = \theta(z''/z) = \theta(z'') + \theta(1/z) = \theta(zz') - \theta(z),$$

and we obtain (1.18). □

By (1.17), $\theta(z) \neq 0$ when $x > 0$ and $y > 0$. It follows $\theta(z) \neq 0$ when $x > 0$ and $y \neq 0$. Using $\theta(z) = 2\theta(\sqrt{z})$, it follows $\theta(z) \neq 0$ for $z \neq \pm 1$ in the unit circle.

If z and z' are in the right-half unit circle with $z \neq z'$, then $z/z' \neq \pm 1$. By (1.18),

$$\theta(z) - \theta(z') = \theta(z) + \theta(1/z') = \theta(z/z') \neq 0.$$

Thus $\theta(z) \neq \theta(z')$. In short, $\theta(z)$ is injective[6] on the right-half unit circle. Since $\sqrt{z}$ is injective on the punctured circle $z \neq -1$, by $\theta(z) = 2\theta(\sqrt{z})$ again, $\theta(z)$ is injective on the punctured circle $z \neq -1$.

By (1.17),

$$|\theta(z)| \leq 1 - x + |y|, \qquad z = x + iy, x > 0.$$

Hence $z_n = x_n + iy_n \to 1$ implies $\theta(z_n) \to 0$. Thus $\theta(z)$ is continuous at $z = 1$. If $z_n \to z$, then $z_n/z \to 1$. By (1.18),

$$\theta(z_n) - \theta(z) = \theta(z_n) + \theta(1/z) = \theta(z_n/z) \to 0.$$

[6] $z \neq z'$ implies $\theta(z) \neq \theta(z')$.

Thus $\theta(z)$ is continuous[7] on the right-half unit circle. Since $\sqrt{z}$ is itself continuous on the punctured circle $z \neq -1$, by $\theta(z) = 2\theta(\sqrt{z})$, $\theta(z)$ is continuous on the punctured circle $z \neq -1$.

Writing

$$\theta(z) = \theta(x+iy) = \theta\left(\sqrt{1-y^2} + iy\right), \qquad -1 \leq y \leq 1, \tag{1.19}$$

it follows $\theta(z)$ is a continuous strictly increasing function of y on the right-half unit circle $x \geq 0$. By the intermediate value theorem,[8] $\theta(z)$ maps the right-half unit circle $x \geq 0$ onto $[-\pi/2, \pi/2]$. By $\theta(z) = 2\theta(\sqrt{z})$, $\theta(z)$ maps the punctured unit circle $z \neq -1$ onto $(-\pi, \pi)$.

To summarize,

Angle Measure

Angle measure $\theta(z)$ is a one-to-one continuous map of the punctured unit circle $z \neq -1$ onto the interval $(-\pi, \pi)$.

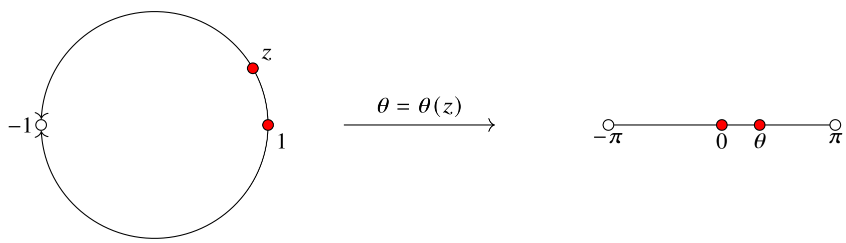

Fig. 1.20 The function $\theta = \theta(z)$

Recall $\pi = 2\theta(i)$. To achieve (1.3) using (1.15), Archimedes effectively calculated

$$2\theta_6(i) = 64\sqrt{2 - \sqrt{\sqrt{\sqrt{\sqrt{\sqrt{2}+2}+2}+2}+2}}. \tag{1.20}$$

In what follows, we refer to $\theta(z)$ as "the angle of z", even though strictly speaking $\theta(z)$ is the measure of the angle of z.

---

[7] $z_n \to z$ implies $\theta(z_n) \to \theta(z)$.

[8] The image of an interval under a continuous function is an interval.

Exercises

Problem 1.1 Let $P = (1, 0)$ and $Q = (3, 2)$ and $R = (0, 1)$. Calculate PQ, P/Q, PR, P/R, QR, Q/R.

Problem 1.2 Label the edge lengths x, y, x', y' in Figure 1.10. What are the areas of the two squares in Figure 1.10, and the areas of the triangles in Figure 1.10? Figure 1.10 is the basis for a classical proof of the Pythagoras theorem.

Problem 1.3 Verify equation (1.7).

Problem 1.4 We say a point P' is the **reciprocal** of a point P if $PP' = I$, where $I = (1, 0)$. Show the reciprocal of (x, y) is

$$\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right).$$

Problem 1.5 Show that there are only two complex numbers $z = x + iy$ satisfying $z^2 = 1$, and they are $z = 1$ and $z = -1$.

Problem 1.6 Show $\sqrt{z}$ given by (1.9) satisfies $(\sqrt{z})^2 = z$.

Problem 1.7 Let w be a complex number not equal to zero. Show there are at most two complex numbers z satisfying $z^2 = w$. Hint: If z_1 and z_2 both satisfy $z_1^2 = w$ and $z_2^2 = w$, then $z = z_1/z_2$ satisfies $z^2 = 1$, so $z = \pm 1$.

Problem 1.8 Let w be a complex number not equal to zero. Show there are at least two complex numbers z satisfying $z^2 = w$.

Problem 1.9 Given a, b, c complex, with $a \neq 0$, show the solutions of $az^2+bz+c = 0$ satisfy the quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Problem 1.10 Show that $1/z = \bar{z}$ when z is on the unit circle.

Problem 1.11 Assume z and z' are in the first quadrant. Show zz' and $\sqrt{z}\sqrt{z'}$ and $\sqrt{zz'}$ are in the upper-half plane. From this, conclude (1.10).

Problem 1.12 Using $z_1 + iy_1 = \sqrt{z}$ and $z_2 = x_2 + iy_2 = \sqrt{z_1}$, verify (1.12) and (1.13).

Problem 1.13 Verify (1.14).

Problem 1.14 Let $z = x + iy$ be in the first quadrant unit circle. Use (1.11) to show that $y < 2(1 - x_1 + y_1) < 1 - x + y$.

Chapter 2
Real Elementary Functions

In the previous chapter, the measure θ of an angle was defined and its additivity was shown to be intertwined with complex multiplication. Building on this, the usual secondary school definition of $\sin\theta$ as the "opposite over the hypotenuse" now makes sense, paving the way for the study of trigonometry, the circular functions sine and cosine.

The fundamental result of trigonometry states there is only one way to assign points z on the unit circle to real numbers θ in an additive manner, and this assignment is given by the circular functions. Here additivity means the complex product zz' corresponds to the sum $\theta + \theta'$, when z and z' correspond to θ and θ'.

This leads to the n-th roots of unity ω, which play a fundamental role not only in complex analysis, but also in extending the notion of commensurability (greatest common divisor) to other algebraic number systems in the complex plane.

In this chapter we also review, as preparation for the next chapter, additivity of the real exponential function, which is the real law of exponents. This in turn builds on the binomial theorem, and leads to real Taylor series.

2.1 Trigonometry

Let z be a complex number and let $\theta = \theta(z)$ be the angle (§1.4) of z. Then θ is defined for $z = x + iy$ not on the non-positive real half-line $x \leq 0$.

Since $\theta(z)$ is a one-to-one map between the punctured circle $z \neq -1$ and the interval $(-\pi, \pi)$, we may define $\cos\theta$ and $\sin\theta$ by setting them equal to x and y respectively (Figure 2.1).

More precisely, let $z(\theta)$ be the inverse of $\theta(z)$, and let $\cos\theta$ and $\sin\theta$ be the real and imaginary parts of $z(\theta)$, so

$$z(\theta) = \cos\theta + i\sin\theta. \tag{2.1}$$

Then $z(\theta)$, $\cos\theta$, and $\sin\theta$ are defined on $(-\pi, \pi)$.

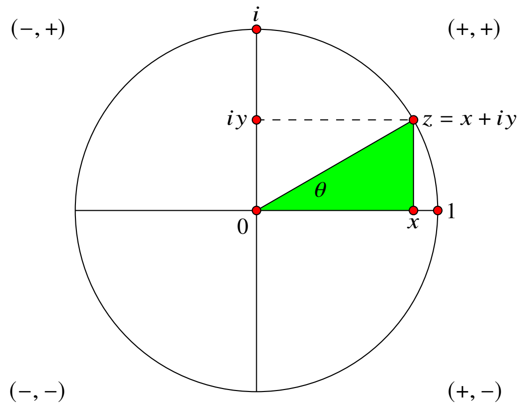

Fig. 2.1 $x = \cos\theta$ and $y = \sin\theta$

Since $z(\theta)$ is in the unit circle,

$$\sin^2\theta + \cos^2\theta = 1,$$

hence the values $\sin\theta$ and $\cos\theta$ range in $[-1, 1]$.

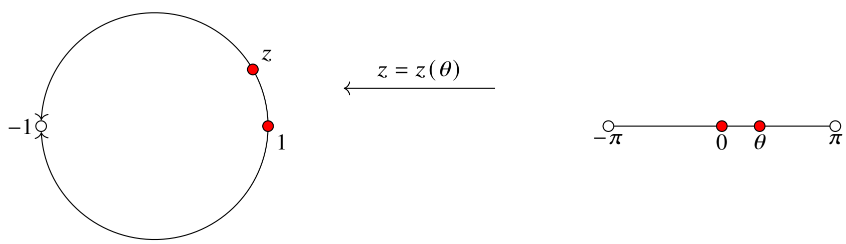

Fig. 2.2 The function $z = z(\theta)$ is the inverse of $\theta = \theta(z)$

From Figure 2.1, when z is in the first quadrant, $\sin\theta$ and $\cos\theta$ are both positive, and when z is in the third quadrant, $\sin\theta$ and $\cos\theta$ are both negative. When z is in the second quadrant, $\sin\theta$ is positive and $\cos\theta$ is negative, and when z is in the fourth quadrant, $\sin\theta$ is negative and $\cos\theta$ is positive.

From $\theta(z) = 2\theta(\sqrt{z})$, we have $\theta(z) = \theta(z^2)/2$. Since $z(\theta)$ is the inverse of $\theta(z)$, this implies

$$z(\theta) = z(\theta/2)^2. \tag{2.2}$$

From this, it follows $\sin\theta$ *and* $\cos\theta$ *are continuous functions on* $(-\pi, \pi)$. In more detail, on the right-half circle, $\theta(z)$ is a continuous strictly increasing function of y (1.19), hence[1] the inverse $y = \sin\theta$ is continuous on $(-\pi/2, \pi/2)$. It follows

[1] The inverse of a continuous bijection on an interval is itself continuous.

$\cos\theta = \sqrt{1-\sin^2\theta}$ is continuous on $(-\pi/2, \pi/2)$, hence $z(\theta)$ is continuous on $(-\pi/2, \pi/2)$. By (2.2), it follows $z(\theta)$, $\sin\theta$, and $\cos\theta$, are continuous on $(-\pi, \pi)$.

The above defines $\sin\theta$ and $\cos\theta$ only on $(-\pi, \pi)$. That we can extend their definition to all real θ, preserving additivity, and in a unique manner, is the content of the

2.1. Fundamental Theorem of Trigonometry

There is a continuous non-constant map $z(\theta)$ of the real line into the unit circle, unique up to rescaling, satisfying

$$z(\theta)z(\theta') = z(\theta+\theta'), \tag{2.3}$$

for all real θ, θ'.

If α is a real number, and a map $z(\theta)$ satisfies (2.3), then the rescaled map $z(\alpha\theta)$ satisfies (2.3). Thus we can only expect uniqueness up to rescaling.

There are two aspects here, *existence* of $z(\theta)$, and *uniqueness* of $z(\theta)$. Let $z(\theta)$ be the inverse of $\theta(z)$. To establish existence, we extend the domain of $z(\theta)$ from $(-\pi, \pi)$ to the whole real line.

To begin, we claim additivity implies the validity of (2.3) for θ, θ' in $(-\pi/2, \pi/2)$. To see this, insert $z = z(\theta)$, $z' = z(\theta')$ into (1.18). Since $\theta(z(\theta)) = \theta$, we obtain $\theta(zz') = \theta + \theta'$, establishing the claim.

To establish existence, we use (2.2) repeatedly to define $z(\theta)$ on successively larger intervals. Define $z(\theta)$ on $(-2\pi, 2\pi)$ by (2.2). This extends the domain of $z(\theta)$ from $(-\pi, \pi)$ to $(-2\pi, 2\pi)$. If θ and θ' are in $(-\pi, \pi)$, then $\theta+\theta'$ is in $(-2\pi, 2\pi)$, and

$$\begin{aligned} z(\theta)z(\theta') &= (z(\theta/2))^2(z(\theta'/2))^2 = (z(\theta/2)z(\theta'/2))^2 \\ &= z((\theta+\theta')/2)^2 = z(\theta+\theta). \end{aligned}$$

Thus the extended map $z(\theta)$ satisfies (2.3) on $(-\pi, \pi)$. Repeating this procedure, use (2.2) to extend $z(\theta)$ to $(-4\pi, 4\pi)$. Then the same argument shows $z(\theta)$ satisfies (2.3) on $(-2\pi, 2\pi)$. Continuing in this manner, we extend $z(\theta)$ to $(-2^n\pi, 2^n\pi)$, and establish (2.3) on $(-2^{n-1}\pi, 2^{n-1}\pi)$, successively for $n = 1, 2, \dots$. Since $z(\theta)$ is continuous on $(-\pi, \pi)$, the extension is continuous on the entire real line.

This establishes the existence of a continuous non-constant map of the real line into the unit circle, satisfying (2.3) for all real θ, θ'. In particular, since $z(\pm\pi/2) = \pm i$, by (2.3), $z(\pm\pi) = -1$. It follows *every point z on the unit circle is of the form $z(\theta)$, with θ in $[-\pi, \pi]$.*

Before we establish uniqueness, we look at the periods of the extended map $z(\theta)$. A **period** of $z(\theta)$ is a *positive* number α such that

$$z(\theta+\alpha) = z(\theta),$$

for all real θ. By (2.3), α is a period iff $z(\alpha) = 1$. Since $z(\pi) = -1$, $z(2\pi) = (-1)^2 = 1$, hence 2π is a period. If α is a period, then $z(\alpha/2) = \pm 1$. Since $z(\theta) \neq \pm 1$ when $0 < \theta < \pi$, *the smallest period is* 2π.

If α is a period, then any positive integer multiple $n\alpha$ is a period. Thus $2n\pi$, $n = 1, 2, 3, \dots$, are periods. Conversely, if α is any period, then $\alpha \geq 2\pi$; let n be the greatest integer $\leq \alpha/2\pi$. Then $\beta = \alpha - n \cdot 2\pi$ satisfies $0 \leq \beta < 2\pi$. If $\beta > 0$, since the difference of two periods is a period, β is a period. But this contradicts the minimality of 2π. It follows $\beta = 0$, hence *the periods of* $z(\theta)$ *are* $2n\pi$, $n = 1, 2, 3, \dots$.

The constant map $z(\theta) \equiv 1$ satisfies (2.3) for all real θ, θ'. Therefore, to rule this uninteresting case out, when establishing uniqueness, we assume $z(\theta)$ is non-constant.

We now establish uniqueness. Let $z_0(\theta) = \cos\theta + i\sin\theta$ be the map constructed above, and let $z(\theta) = x(\theta) + iy(\theta)$ be any continuous non-constant map of the real line into the unit circle, satisfying (2.3) for all real θ, θ'. Then (2.3) implies $z(0)^2 = z(0)$, hence $z(0) = 1$.

Since $z(\theta)$ is non-constant and $z(0) = 1$, there is an $\alpha \neq 0$ with $x(\alpha) < 1$. Since $\cos 0 = 1$, by continuity, there is an $n \geq 1$ such that $x(\alpha) < \cos(2\pi/n) < 1$. By the intermediate value theorem, there is a β with $x(\beta) = \cos(2\pi/n)$, hence $z(\beta) = z_0(\pm 2\pi/n)$. By (2.3), $z(n\beta) = z_0(\pm 2\pi) = 1$, hence $z(\theta)$ has at least one period.

Let θ be a positive number. If α is a period, there is an integer n with $n\alpha \leq \theta < (n+1)\alpha$. Since $n\alpha$ is a period, it follows there is a period within $(\theta - \alpha, \theta + \alpha)$. If $z(\theta)$ has arbitrarily small periods, it follows there are periods that are arbitrarily close to θ. By continuity, this implies θ is a period, or $z(\theta) = 1$. Since θ was arbitrary, this implies $z(\theta)$ is constant, which contradicts our non-constancy assumption. Thus the greatest lower bound α of the set of periods of $z(\theta)$ is positive. By continuity, α is itself a period, hence is a smallest period. This establishes the existence of a smallest period α for $z(\theta)$.

Now rescale $z(\theta)$ to $z_1(\theta) = z(\alpha\theta/2\pi)$. Then $z_1(\theta)$ has smallest period 2π. Thus, without loss of generality, we may assume $\alpha = 2\pi$. With this assumption, $z(\pm\pi) = -1$, $z(\pi/2) = i$ or $z(-\pi/2) = i$, and $z(\theta) \neq \pm 1$ for $0 < |\theta| < \pi$. In particular, $\sqrt{z(\theta)}$ is defined for $-\pi < \theta < \pi$. By (2.2), $z(\theta/2)$ is a square root of $z(\theta)$, hence

$$z(\theta/2) = \sqrt{z(\theta)}, \qquad -\pi < \theta < \pi, \tag{2.4}$$

up to sign.[2] We claim (2.4) is correct as written. This is immediate when $\theta = 0$, so assume $\theta \neq 0$. If the imaginary parts of $z(\theta)$ and $z(\theta/2)$ have opposite signs, by the intermediate value theorem, $z(\theta') = 1$ or $z(\theta') = -1$, for some θ' between θ and $\theta/2$. Since this can't happen, the imaginary parts of $z(\theta)$ and $z(\theta/2)$ must have the same sign. It follows the imaginary parts of $\sqrt{z(\theta)}$ and $z(\theta/2)$ have the same sign, establishing the claim.

By rescaling again $z(\theta)$ to $z(-\theta)$ if necessary, we may also assume $z(\pi/2) = i$. Using (2.3) and (2.4) repeatedly, we conclude $z(\theta\pi/2) = i^\theta$ for all dyadic rationals $\theta = k2^{-n}$. Since the dyadic rationals are dense and $z(\theta)$ is continuous, this

[2] This means $z(\theta/2) = \pm\sqrt{z(\theta)}$.

determines[3] $z(\theta)$ uniquely. This completes the proof of the fundamental theorem of trigonometry.

If we write $z(\theta') = \cos\theta' + i\sin\theta'$, then (2.3) reduces to the

Trigonometric Addition Formulas

For all real θ, θ',

$$\sin(\theta + \theta') = \sin\theta\cos\theta' + \cos\theta\sin\theta',$$

and

$$\cos(\theta + \theta') = \cos\theta\cos\theta' - \sin\theta\sin\theta'.$$

Moreover, replacing θ by 2θ in (2.2), we obtain the

Trigonometric Doubling Formulas

For all real θ,

$$\sin(2\theta) = 2\sin\theta\cos\theta, \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta,$$

and

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}, \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}.$$

To compute the derivatives of $\sin\theta$ and $\cos\theta$, we first establish the following limit

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1. \tag{2.5}$$

For this, it is enough to show

$$1 - \frac{|\sin\theta|}{1+\cos\theta} < \frac{\sin\theta}{\theta} < 1 \tag{2.6}$$

for $0 < |\theta| < \pi/2$, since taking the limit $\theta \to 0$ in (2.6) yields (2.5).

In (2.6), all sides are even functions, so we may assume $0 < \theta < \pi/2$. In this case, we may insert $z = z(\theta)$ into (1.17). Since $z(\theta) = \cos\theta + i\sin\theta$, this is the same as inserting $x = \cos\theta$ and $y = \sin\theta$. Since $z(\theta)$ is the inverse of $\theta(z)$, we obtain

$$\sin\theta < \theta < 1 - \cos\theta + \sin\theta.$$

Dividing the left inequality by θ yields

[3] Continuous functions that agree on a dense set agree everywhere.

$$\frac{\sin\theta}{\theta} < 1.$$

Dividing the right inequality by θ yields

$$1 - \frac{1-\cos\theta}{\theta} < \frac{\sin\theta}{\theta},$$

or

$$\frac{\sin\theta}{\theta} > 1 - \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} = 1 - \frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta} > 1 - \frac{\sin\theta}{1+\cos\theta}.$$

Thus

$$1 - \frac{\sin\theta}{1+\cos\theta} < \frac{\sin\theta}{\theta} < 1,$$

which is (2.6). Taking the limit as $\theta \to 0$, we obtain 2.5.

Since

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta} = 1\cdot\frac{0}{2} = 0,$$

we have also shown

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0. \tag{2.7}$$

By the addition formula,

$$\frac{\sin(\theta+h)-\sin\theta}{h} = \sin\theta\cdot\frac{\cos h - 1}{h} + \cos\theta\cdot\frac{\sin h}{h}.$$

Taking the limit as $h \to 0$ yields

$$(\sin\theta)' = \lim_{h\to 0}\frac{\sin(\theta+h)-\sin\theta}{h} = \sin\theta\cdot 0 + \cos\theta\cdot 1 = \cos\theta.$$

Similarly,

$$\frac{\cos(\theta+h)-\cos\theta}{h} = \cos\theta\cdot\frac{\cos h - 1}{h} - \sin\theta\cdot\frac{\sin h}{h}.$$

Taking the limit as $h \to 0$ yields

$$(\cos\theta)' = \lim_{h\to 0}\frac{\cos(\theta+h)-\cos\theta}{h} = \cos\theta\cdot 0 - \sin\theta\cdot 1 = -\sin\theta.$$

Trigonometric Derivatives

For all θ real,

$$(\sin\theta)' = \cos\theta, \qquad (\cos\theta)' = -\sin\theta,$$

hence

$$(\cos\theta + i\sin\theta)' = i(\cos\theta + i\sin\theta).$$

It follows that $\sin\theta$ and $\cos\theta$ have derivatives of all orders, with

$$(\sin\theta)^{(n)} = \begin{cases} \sin\theta, & n = 0, 4, 8, 12, \dots \\ \cos\theta, & n = 1, 5, 9, 13, \dots \\ -\sin\theta, & n = 2, 6, 10, 14, \dots \\ -\cos\theta, & n = 2, 6, 11, 15, \dots \end{cases}$$

and

$$(\cos\theta)^{(n)} = \begin{cases} \cos\theta, & n = 0, 4, 8, 12, \dots \\ -\sin\theta, & n = 1, 5, 9, 13, \dots \\ -\cos\theta, & n = 2, 6, 10, 14, \dots \\ \sin\theta, & n = 2, 6, 11, 15, \dots \end{cases}$$

These may be combined into

$$\frac{d^n}{d\theta^n}(\cos\theta + i\sin\theta) = i^n(\cos\theta + i\sin\theta), \qquad n \geq 0.$$

2.2 Roots of Unity

Since $\omega = 1$ and $\omega = -1$ both satisfy $\omega^2 = 1$, ω are the square roots of 1. They are the *square roots of unity*. Since $-1 = i^2$ and the angle of i is $\pi/2$ (by definition of π), by angle additivity, the angle of -1 is $\pi/2 + \pi/2 = \pi$.

Similarly, since

$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \tag{2.8}$$

satisfies $\omega^3 = 1$, we say ω is a *cube root of unity*. Since $\omega^3 = 1$ and the angle of 1 is 2π, by angle additivity, the angle of ω is $2\pi/3$.

To see where the formula for the cube root comes from, write $\omega = x + iy$. Multiplying,

$$1 = \omega^3 = \omega^2\omega = (x + yi)^2(x + yi) = (x^3 - 3xy^2) + i(3x^2y - y^3).$$

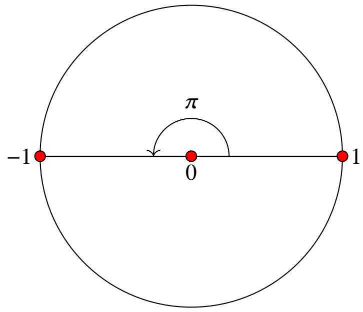

Fig. 2.3 The square roots of unity

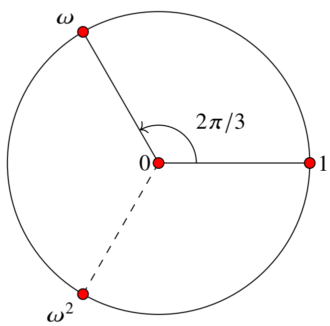

Fig. 2.4 The cube roots of unity

Since 1 has imaginary part 0, we have $3x^2y - y^3 = 0$. Canceling y, we obtain $3x^2 = y^2$. But $x^2 + y^2 = 1$, so $4x^2 = 1$, or $x = \pm 1/2$. Since ω is in the second quadrant, $x = -1/2$. From this, we get $y^2 = 3/4$ or $y = \sqrt{3}/2$. This shows

$$x = \cos(2\pi/3) = -\frac{1}{2} \qquad \text{and} \qquad y = \sin(2\pi/3) = \frac{\sqrt{3}}{2},$$

which justifies (2.8).

Similarly, if ω equals ± 1 or $\pm i$, then $\omega^4 = 1$, so these are the *fourth roots of unity*.

In general, a complex number z satisfying $z^n = 1$ is an n-th **root of unity**. By (2.3), multiplying complex numbers corresponds to adding their angles. From this, the angles of the n-th roots of unity equal the integer multiples of $2\pi/n$.

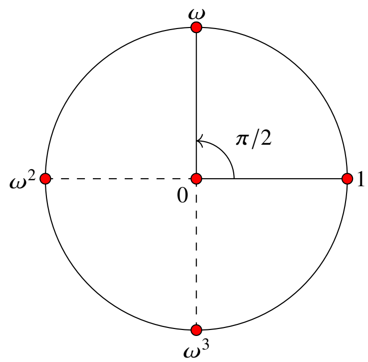

Fig. 2.5 The fourth roots of unity

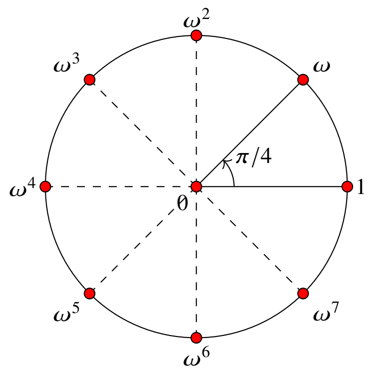

Fig. 2.6 The eighth roots of unity

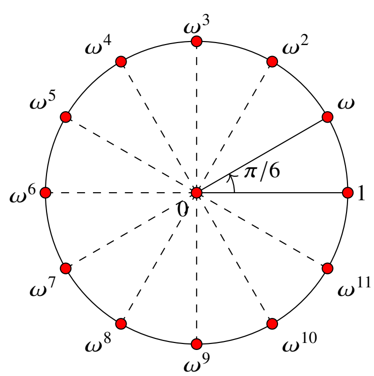

Fig. 2.7 The twelfth roots of unity

The n-th roots of unity

Let $n = 1, 2, 3, \ldots$ be a natural number. An n-th root of unity is a complex number z on the unit circle satisfying

$$z^n = 1.$$

There are n distinct n-th roots of unity, of the form $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where

$$\omega = \cos(2\pi/n) + i \sin(2\pi/n)$$

is the **principal** n-th root of unity.

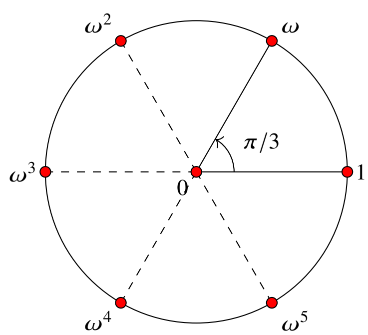

Fig. 2.8 The sixth roots of unity

Thus the angles of the square roots of unity are 0 and π, the angles of the cube roots of unity are 0, $2\pi/3$, $4\pi/3$, the angles of the fourth roots of unity are 0, $\pi/2$, π, $3\pi/2$, the angles of the fifth roots of unity are 0, $2\pi/5$, $4\pi/5$, $6\pi/5$, $8\pi/5$, and the angles of the sixth roots of unity are 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$, $5\pi/3$.

Let us calculate the eighth root of unity $z = \omega = x + iy$, which lies exactly halfway between 1 and i (Figure 2.6), with angle $\theta = \pi/4$. Since $\omega^2 = i$, we have

$$i = \omega^2 = (x + iy)^2 = x^2 - y^2 + i2xy,$$

which implies $x^2 - y^2 = 0$. Since x and y are both positive, we see $x = y$. But $x^2 + y^2 = 1$, so $2x^2 = 1$, or $x = y = 1/\sqrt{2}$. This shows

$$\sin(\pi/4) = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \cos(\pi/4) = \frac{1}{\sqrt{2}}.$$

Since

$$z(\theta + \pi/2) = iz(\theta) = i(x + yi) = -y + ix,$$

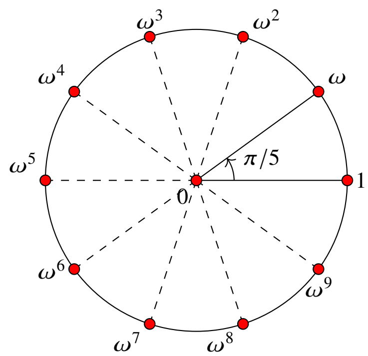

Fig. 2.9 The tenth roots of unity

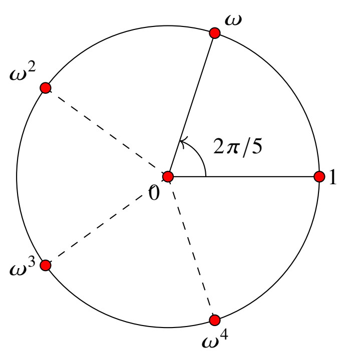

Fig. 2.10 The fifth roots of unity

we have

$$\sin(\theta + \pi/2) = \cos\theta \qquad \text{and} \qquad \cos(\theta + \pi/2) = -\sin\theta.$$

For example,

$$\sin(3\pi/4) = 1/\sqrt{2} \qquad \text{and} \qquad \cos(3\pi/4) = -1/\sqrt{2}.$$

Since the twelfth root unity ω has angle $2\pi/12 = \pi/6$, the angle of $i\omega$ is $\pi/6 + \pi/2 = 2\pi/3$, hence $i\omega = z_3$. From this we obtain

$$x = \cos(\pi/6) = \frac{\sqrt{3}}{2} \qquad \text{and} \qquad y = \sin(\pi/6) = \frac{1}{2}.$$

To derive the formula for the tenth root of unity, write $\omega = x + iy = \cos(\pi/5) + i\sin(\pi/5)$. By the binomial theorem 2.2 with $n = 5$,

$$\omega^5 = (x + iy)^5 = x^5 + 5i\,x^4y - 10\,x^3y^2 - 10i\,x^2y^3 + 5\,xy^4 + i\,y^5.$$

Since $\omega^5 = -1$, the imaginary part vanishes,

$$5x^4y - 10x^2y^3 + y^5 = 0.$$

Canceling y and substituting $x^2 = 1 - y^2$,

$$16y^4 - 20y^2 + 5 = 0.$$

By the quadratic formula,

$$y^2 = \frac{5 \pm \sqrt{5}}{8}.$$

Since $y = \sin(2\pi/10) = \sin(\pi/5) < \sin(\pi/4) = 1/\sqrt{2}$, we take the minus sign. Hence

$$\sin(\pi/5) = y = \sqrt{\frac{5-\sqrt{5}}{8}}, \qquad \cos(\pi/5) = \sqrt{\frac{3+\sqrt{5}}{8}}.$$

Using the trigonometric doubling formulas, the fifth root of unity is

$$\sin(2\pi/5) = y = \sqrt{\frac{5+\sqrt{5}}{8}}, \qquad \cos(2\pi/5) = \sqrt{\frac{3-\sqrt{5}}{8}}.$$

Since $\theta(1) = 0$, we have $\cos 0 = 1$ and $\sin 0 = 0$. Table 2.11 summarizes the results of this section.

θ	degrees	$\cos\theta$	$\sin\theta$	root
0	$0°$	1	0	z_∞
$\pi/6$	$30°$	$\sqrt{3}/2$	$1/2$	z_{12}
$\pi/5$	$36°$	$\sqrt{(3+\sqrt{5})/8}$	$\sqrt{(5-\sqrt{5})/8}$	z_{10}
$\pi/4$	$45°$	$1/\sqrt{2}$	$1/\sqrt{2}$	z_8
$\pi/3$	$60°$	$1/2$	$\sqrt{3}/2$	z_6
$2\pi/5$	$72°$	$\sqrt{(5+\sqrt{5})/8}$	$\sqrt{(3-\sqrt{5})/8}$	z_5
$\pi/2$	$90°$	0	1	z_4
$2\pi/3$	$120°$	$-1/2$	$\sqrt{3}/2$	z_3
π	$180°$	-1	0	z_2
2π	$360°$	1	0	z_1

Fig. 2.11 The standard angles

2.3 Binomial Theorem

Let x and a be variables. A **binomial** is an expression of the form

$$(x + a)^2, \qquad (x + a)^3, \qquad (x + a)^4, \qquad \ldots$$

The **degree** of each of these binomials is 2, 3, and 4.

When binomials are expanded by multiplying out, one obtains a sum of terms. The **binomial theorem** specifies the exact pattern or form of the resulting sum.

Recall that

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd.$$

Similarly,

$$(a + b)(c + d + e) = a(c + d + e) + b(c + d + e) = ac + ad + ae + bc + bd + be.$$

Using this algebra, we can expand each binomial.

Expanding $(x + a)^2$ yields

$$(x + a)^2 = (x + a)(x + a) = x^2 + xa + ax + a^2 = x^2 + 2ax + a^2. \tag{2.9}$$

Similarly, for $(x + a)^3$, we have

$$\begin{aligned}(x + a)^3 &= (x + a)(x + a)^2 = (x + a)(x^2 + 2ax + a^2)\\ &= x^3 + 2x^2a + xa^2 + ax^2 + 2axa + a^3\\ &= x^3 + 3ax^2 + 3a^2x + a^3.\end{aligned} \tag{2.10}$$

For $(x + a)^4$, we have

$$\begin{aligned}(x + a)^4 &= (x + a)(x + a)^3 = (x + a)(x^3 + 3ax^2 + 3a^2x + a^3)\\ &= x^4 + 3ax^3 + 3a^2x^2 + a^3x + ax^3 + 3a^2x^2 + 3a^3x + a^4\\ &= x^4 + 4ax^3 + 6a^2x^2 + 4ax^3 + a^4.\end{aligned} \tag{2.11}$$

Thus

$$\begin{aligned}(x + a)^2 &= x^2 + 2ax + a^2\\ (x + a)^3 &= x^3 + 3ax^2 + 3a^2x + a^3\\ (x + a)^4 &= x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4\\ (x + a)^5 &= \star x^5 + \star ax^4 + \star a^2x^3 + \star a^3x^2 + \star a^4x + \star a^5\end{aligned} \tag{2.12}$$

where $\star$ means we haven't found the coefficient yet.

There is a pattern in (2.12). In the first line, the powers of x are in decreasing order, 2, 1, 0, while the powers of a are in increasing order, 0, 1, 2. In the second line, the powers of x decrease from 3 to 0, while the powers of a increase from 0 to 3. In the third line, the powers of x decrease from 4 to 0, while the powers of a increase from 0 to 4.

This pattern of powers is simple and clear. Now we want to find the pattern for the coefficients in front of each term. In (2.12), these coefficients are $(1, 2, 1)$, $(1, 3, 3, 1)$, $(1, 4, 6, 4, 1)$, and $(\star, \star, \star, \star, \star, \star)$. These coefficients are the **binomial coefficients**.

Before we determine the pattern, we introduce a useful notation for these coefficients by writing

$$\binom{2}{0} = 1, \qquad \binom{2}{1} = 2, \qquad \binom{2}{2} = 1$$

and

$$\binom{3}{0} = 1, \qquad \binom{3}{1} = 3, \qquad \binom{3}{2} = 3, \qquad \binom{3}{3} = 1$$

and

$$\binom{4}{0} = 1, \qquad \binom{4}{1} = 4, \qquad \binom{4}{2} = 6, \qquad \binom{4}{3} = 4, \qquad \binom{4}{4} = 1$$

and

$$\binom{5}{0} = \star, \quad \binom{5}{1} = \star, \quad \binom{5}{2} = \star, \quad \binom{5}{3} = \star, \quad \binom{5}{4} = \star, \quad \binom{5}{5} = \star.$$

With this notation, the number

$$\binom{n}{k} \tag{2.13}$$

is the **coefficient of** $x^{n-k}a^k$ **when you multiply out** $(x + a)^n$**.** This is the binomial coefficient. Here n is the degree of the binomial, and k, which specifies the term in the resulting sum, varies from 0 to n.

It is important to remember that, in this notation, the binomial $(x + a)^2$ expands into the sum of **three** terms x^2, $2ax$, a^2. These are term 0, term 1, and term 2. Alternatively, one says these are the **zeroth term**, the **first term**, and the **second term**. Thus the second term in the expansion of the binomial $(x + a)^4$ is $6a^2x^2$, and the binomial coefficient $\binom{4}{2} = 6$. In general, the binomial $(x + a)^n$ of degree n expands into a sum of $n + 1$ terms.

Since the binomial coefficient $\binom{n}{k}$ is the coefficient of $x^{n-k}a^k$ when you multiply out $(x + a)^n$, we have the binomial theorem.

2.2. Binomial Theorem

The binomial $(x + a)^n$ equals

$$\binom{n}{0}x^n + \binom{n}{1}x^{n-1}a + \binom{n}{2}x^{n-2}a^2 + \cdots + \binom{n}{n-1}xa^{n-1} + \binom{n}{n}a^n. \tag{2.14}$$

It is important to remember that x and a are **variables**, so they can be any numbers whose arithmetic is commutative, associative, and distributive. In particular, x and a may be complex numbers.

The binomial coefficient $\binom{n}{k}$ is called "n-choose-k", because it is the coefficient of the term corresponding to choosing k a's when multiplying the n factors in the product

$$(x+a)^n = (x+a)(x+a)(x+a)\dots(x+a).$$

For example, the term $\binom{4}{2}a^2x^2$ corresponds to choosing two a's, and two x's, when multiplying the four factors in the product

$$(x+a)^4 = (x+a)(x+a)(x+a)(x+a).$$

The binomial coefficients may be arranged in a triangle, **Pascal's triangle** (Figure 2.12). Can you figure out the numbers ⋆ in this triangle before peeking ahead?

0:	1
1:	1 1
2:	1 2 1
3:	1 3 3 1
4:	1 4 6 4 1
5:	1 5 10 10 5 1
6:	⋆ 6 15 20 15 6 ⋆
7:	1 ⋆ 21 35 35 21 ⋆ 1
8:	1 8 ⋆ 56 70 56 ⋆ 8 1
9:	1 9 36 ⋆ 126 126 ⋆ 36 9 1
10:	1 10 45 120 ⋆ 252 ⋆ 120 45 10 1

Fig. 2.12 Pascal's triangle

In Pascal's triangle, the very top row has one number in it: This is the **zeroth row** corresponding to $n = 0$ and the binomial expansion of $(x+a)^0 = 1$. The **first row** corresponds to $n = 1$; it contains the numbers $(1, 1)$, which correspond to the binomial expansion of $(x+a)^1 = 1x + 1a$. We say the **zeroth entry** ($k = 0$) in the **first row** ($n = 1$) is 1 and the **first entry** ($k = 1$) in the first row is 1. Similarly, the **zeroth entry** ($k = 0$) in the **second row** ($n = 2$) is 1, and the **second entry** ($k = 2$) in the **second row** ($n = 2$) is 1. The **second entry** ($k = 2$) in the **fourth row** ($n = 4$) is 6. For every row, the entries are counted starting from $k = 0$, and end with $k = n$, so there are $n + 1$ entries in row n. With this understood, the k-th entry in the n-th row is the binomial coefficient n-choose-k. So 10-choose-2 is

$$\binom{10}{2} = 45.$$

We can learn a lot about the binomial coefficients from this triangle. First, we have 1's all along the left edge. Next, we have 1's all along the right edge. Similarly, one step in from the left or right edge, we have the row number. Thus we have

$$\binom{n}{0} = 1 = \binom{n}{n}, \qquad \binom{n}{1} = n = \binom{n}{n-1}, \qquad n \geq 1.$$

Note also Pascal's triangle has a left-to-right symmetry: If you read off the coefficients in a particular row, you can't tell if you're reading them from left to right, or from right to left. It's the same either way: The fifth row is $(1, 5, 10, 10, 5, 1)$. In terms of our notation, this is written

$$\binom{n}{k} = \binom{n}{n-k}, \qquad 0 \leq k \leq n;$$

the binomial coefficients remain unchanged when k is replaced by $n - k$.

The key step in finding a formula for n-choose-k is to notice

$$(x + a)^{n+1} = (x + a)(x + a)^n.$$

Let's work this out when $n = 3$. Then the left side is $(x + a)^4$. From (2.12), we get

$$\begin{aligned}
\binom{4}{0}x^4 &+ \binom{4}{1}x^3a + \binom{4}{2}x^2a^2 + \binom{4}{3}xa^3 + \binom{4}{4}a^4 \\
&= (x+a)\left(\binom{3}{0}x^3 + \binom{3}{1}x^2a + \binom{3}{2}xa^2 + \binom{3}{3}a^3\right) \\
&= \binom{3}{0}x^4 + \binom{3}{1}x^3a + \binom{3}{2}x^2a^2 + \binom{3}{3}xa^3 \\
&\qquad + \binom{3}{0}x^3a + \binom{3}{1}x^2a^2 + \binom{3}{2}xa^3 + \binom{3}{3}a^4 \\
&= \binom{3}{0}x^4 + \left(\binom{3}{1} + \binom{3}{0}\right)x^3a + \left(\binom{3}{2} + \binom{3}{1}\right)x^2a^2 \\
&\qquad + \left(\binom{3}{3} + \binom{3}{2}\right)xa^3 + \binom{3}{3}a^4.
\end{aligned}$$

Equating corresponding coefficients of x, we get,

$$\binom{4}{1} = \binom{3}{1} + \binom{3}{0}, \qquad \binom{4}{2} = \binom{3}{2} + \binom{3}{1}, \qquad \binom{4}{3} = \binom{3}{3} + \binom{3}{2}.$$

In general, a similar calculation establishes

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \qquad 1 \le k \le n. \tag{2.15}$$

This allows us to build Pascal's triangle (Figure 2.12), where, apart from the ones on either end, each term ("the child") in a given row is the sum of the two terms ("the parents") located directly above in the previous row.

Insert $x = 1$ and $a = 1$ in the binomial theorem to get

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}. \tag{2.16}$$

We conclude *the sum of the binomial coefficients along the n-th row of Pascal's triangle is* 2^n (remember n starts from 0).

Now insert $x = 1$ and $a = -1$. You get

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots \pm \binom{n}{n-1} \pm \binom{n}{n}.$$

Hence: *the alternating*[4] *sum of the binomial coefficients along the n-th row of Pascal's triangle is zero*.

Here is a formula for n-choose-k:

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot \cdots \cdot (n-k+1)}{1 \cdot 2 \cdot \cdots \cdot k}, \qquad 1 \le k \le n, \tag{2.17}$$

so

$$\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35 = \binom{7}{4} \qquad \text{and} \qquad \binom{10}{2} = \frac{10 \cdot 9}{1 \cdot 2} = 45 = \binom{10}{8}.$$

The formula (2.17) is easy to remember: There are k terms in the numerator as well as the denominator, the factors in the denominator increase starting from 1, and the factors in the numerator decrease starting from n.

Now we express the binomial coefficients in terms of factorials. Given a positive integer n, **n-factorial** is the product

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 4 \cdot 3 \cdot 2, \qquad n \ge 2. \tag{2.18}$$

So $2! = 2$, $3! = 6$, $4! = 24$, and so on.

We defined $n!$ for $n \ge 2$. If we set $0! = 1! = 1$, then the binomial coefficient formula (2.17) may be rewritten

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \qquad 0 \le k \le n,$$

as may be verified by cancellation of common factors from the numerator and denominator. For example,

[4] *Alternating* means the plus-minus pattern $+ - + - + - \ldots$.

$$\frac{7!}{3!(7-3)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \binom{7}{3}.$$

It is important to know roughly the size of $n!$ for large n. By knowing the size of $n!$, we mean being able to compare $n!$ with more familiar quantities.

Bounds for $n!$

$$3\left(\frac{n}{3}\right)^n \leq n! \leq 2\left(\frac{n}{2}\right)^n, \qquad n \geq 1 \tag{2.19}$$

This is derived in a series of steps.

Step 1 Use the binomial theorem to derive

$$\left(1 + \frac{1}{n}\right)^n \geq 2, \qquad n \geq 1.$$

Step 2 Let $s_1 = 1$, $s_2 = 3/2$, $s_3 = 7/4$, $s_4 = 15/8$, and, in general,

$$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}}, \qquad n \geq 1.$$

Show s_n never exceeds 2. (Multiply s_n by 2 to obtain

$$2s_n = 2 + s_n - \frac{1}{2^{n-1}},$$

then solve for s_n.)

Step 3 Use the binomial theorem and the previous step to derive

$$\left(1 + \frac{1}{n}\right)^n \leq 1 + s_n \leq 3, \qquad n \geq 1.$$

(Insert $x = 1$ and $a = 1/n$ in (2.14).)

Step 4 Combining the previous steps, we get

$$2 \leq \left(1 + \frac{1}{n}\right)^n \leq 3, \qquad n \geq 1 \tag{2.20}$$

Step 5 Let $a_n = n!$ and $b_n = 2(n/2)^n$. Use the left side of (2.20) to derive

$$\frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n}.$$

Since $a_1 = b_1$, we obtain

$$1 = \frac{a_1}{b_1} \geq \frac{a_2}{b_2} \geq \frac{a_3}{b_3} \geq \dots$$

Thus $a_n \le b_n$ for all $n \ge 1$, which is the right side of (2.19).

Step 6 Let $a_n = n!$ and $b_n = 3(n/3)^n$. Use the right side of (2.20) to derive

$$\frac{a_n}{b_n} \le \frac{a_{n+1}}{b_{n+1}}.$$

Since $a_1 = b_1$, we obtain

$$1 = \frac{a_1}{b_1} \le \frac{a_2}{b_2} \le \frac{a_3}{b_3} \le \dots$$

Thus $a_n \ge b_n$ for all $n \ge 1$, which is the left side of (2.19).

2.4 Real Exponential

Exponential Function

The exponential function is the following limit,

$$e^x = \exp x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n, \qquad (2.21)$$

valid for all x real.

The exponential function $\exp x$ is not a power, like 3^n, nor a root, like $3^{1/n}$. However, because $\exp x$ satisfies the law of exponents (below), we denote $\exp x = e^x$.

Since $(1 + x/n)^n$ increases when x increases, e^x is an increasing function, as soon as the limit exists. To establish the existence of the limit, we use the binomial theorem, and the geometric sum identity

$$1 - x^n = (1 - x)(1 + x + x^2 + \dots + x^{n-1}), \qquad n \ge 1, \qquad (2.22)$$

which can be easily verified by multiplying out the factors.

Since $e^0 = 1$, we assume first $x > 0$. We use the binomial theorem to show the sequence in (2.21) is increasing and bounded, hence has a limit e^x.

For k fixed,

$$\frac{1}{n^k}\binom{n}{k} = \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{k-1}{n}\right)$$

is an increasing function of n. By the binomial theorem,

$$\begin{aligned}\left(1+\frac{x}{n}\right)^n &= \sum_{k=0}^{n} \frac{1}{n^k}\binom{n}{k}x^k \\ &\le \sum_{k=0}^{n} \frac{1}{(n+1)^k}\binom{n+1}{k}x^k \\ &\le \sum_{k=0}^{n+1} \frac{1}{(n+1)^k}\binom{n+1}{k}x^k \\ &= \left(1+\frac{x}{n+1}\right)^{n+1}.\end{aligned}$$

Thus[5] the sequence in (2.21) increases to a possibly infinite limit.

When $0 < x \le 1$, by (2.20), the sequence in (2.21) is bounded by 3. Thus the limit e^x in (2.21) exists for $0 < x \le 1$. In particular, $e = e^1$ is between 2 and 3. When[6] $0 < x \le N$,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{x}{nN}\right)^{nN} = (e^{x/N})^N.$$

Thus the limit e^x in (2.21) exists for $x > 0$, and $e^x = (e^{x/N})^N$.

When $0 < x < 1$, the identity (2.22) implies

$$1 - x^n < n(1-x).$$

Replacing x by $1 - x/n^2$ results in

$$1 > \left(1-\frac{x}{n^2}\right)^n > 1 - \frac{x}{n},$$

when $0 < x < n^2$.

As a consequence,

$$\lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n = \lim_{n\to\infty}\frac{\left(1-\frac{x^2}{n^2}\right)^n}{\left(1+\frac{x}{n}\right)^n} = \frac{1}{e^x}.$$

This shows the limit in (2.21) exists for $x < 0$, and

$$e^{-x} = \frac{1}{e^x}, \qquad x > 0.$$

Thus the limit e^x exists and is positive for all real x.

Above we derived $(e^{x/N})^N = e^x$ when $x > 0$. This is also true when $x < 0$, since

$$(e^{x/N})^N = 1/(e^{-x/N})^N = 1/e^{-x} = e^x.$$

[5] An increasing sequence always has a (possibly infinite) limit.

[6] If a sequence converges to a limit, then any subsequence converges to the same limit.

Replacing x by Nx, $(e^x)^N = e^{Nx}$ follows. We conclude

$$(e^x)^n = e^{nx}, \qquad (e^x)^{1/m} = e^{x/m}, \qquad n, m \geq 1,$$

when x real. As a consequence, *when $x = n/m$ is a rational, $e^x = e^{n/m}$ is the n-th power of the m-th root of e.*

Since $a^{1/n} \to 1$ as $n \to \infty$ for any $a > 0$,

$$\lim_{n\to\infty} e^{x/n} = \lim_{n\to\infty} (e^x)^{1/n} = 1. \tag{2.23}$$

For $x > 0$ and $y > 0$ and $n \geq N$,

$$\left(1 + \frac{x+y}{n}\right) \leq \left(1 + \frac{x}{n}\right)\left(1 + \frac{y}{n}\right) \leq \left(1 + \frac{x+y}{n}\right)\left(1 + \frac{xy}{nN}\right).$$

Raising to the n-th power and passing to the limit $n \to \infty$,

$$e^{x+y} \leq e^x e^y \leq e^{x+y} e^{xy/N}, \qquad x > 0, y > 0, N \geq 1. \tag{2.24}$$

Taking the limit as $N \to \infty$ in (2.24) yields the

Real Law of Exponents

For all x and y real,

$$e^x e^y = e^{x+y}.$$

We established this when $x > 0$ and $y > 0$. To establish the case $x < 0$ and $y < 0$,

$$e^x e^y = \frac{1}{e^{-x}e^{-y}} = \frac{1}{e^{(-x)+(-y)}} = \frac{1}{e^{-(x+y)}} = e^{x+y}.$$

When x and y have opposite signs, since the law of exponents is symmetric, we may assume $x > 0$ and $y < 0$. Here there are two sub-cases. If $x + y > 0$, then $e^{x+y}/e^y = e^{x+y}e^{-y} = e^x$. If $x + y < 0$, $e^x/e^{x+y} = e^x e^{-(x+y)} = e^{-y} = 1/e^y$. This establishes the law.

For $1 \leq k \leq n$,

$$\begin{aligned}\frac{1}{n^k}\binom{n}{k} &= \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{k-2}{n}\right)\left(1 - \frac{k-1}{n}\right)\\ &\leq \frac{1}{(k-1)!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{k-2}{n}\right)\\ &= \frac{1}{n^{k-1}}\binom{n}{k-1},\end{aligned}$$

so $\binom{n}{k}/n^k$ is decreasing in k, for n fixed. By the binomial theorem, for $x > 0$,

$$\begin{aligned}\left(1+\frac{x}{n}\right)^n - 1 - x &= \sum_{k=2}^{n} \frac{1}{n^k}\binom{n}{k} x^k \\ &\le \sum_{k=2}^{n} \frac{1}{n^{k-2}}\binom{n}{k-2} x^k \\ &= x^2 \sum_{k=0}^{n-2} \frac{1}{n^k}\binom{n}{k} x^k \\ &< x^2 \sum_{k=0}^{n} \frac{1}{n^k}\binom{n}{k} x^k \\ &= x^2\left(1+\frac{x}{n}\right)^n .\end{aligned}$$

Passing to the limit $n \to \infty$,

$$0 < e^x - 1 - x \le x^2 e^x, \qquad x > 0. \tag{2.25}$$

We claim

$$|e^x - 1 - x| \le x^2 e^{|x|} \tag{2.26}$$

for x real. When $x = 0$, this is clear. For $x > 0$, this is (2.25). For $x < 0$, insert $-x$ for x in (2.25) and multiply by e^x, to obtain

$$0 < 1 - e^x + xe^x \le x^2.$$

Since $e^x < 1$ and $x < 0$, this implies

$$0 > e^x - 1 - x \ge xe^x - x - x^2 > -x^2 \ge -x^2 e^{|x|},$$

which is (2.26) for $x < 0$. Dividing by x then passing to the limit $x \to 0$ in (2.26), we obtain

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1,$$

hence the derivative of e^x at $x = 0$ exists and equals 1. By the law of exponents,

$$\lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h} = e^x \cdot \lim_{h\to 0} \frac{e^h - 1}{h} = e^x,$$

for all x real. We have derived

Exponential Derivative

For all x real,

$$(e^x)' = e^x.$$

It follows that e^x has derivatives of all orders, with

$$\frac{d^n}{dx^n} e^x = e^x, \qquad n \geq 0.$$

2.5 Real Taylor Series

Let $f(x)$ be a real function of a real variable x, and suppose we have derivatives of all orders $f', f'', f''', \dots$. We are most interested in the cases $f(x) = e^x$, $f(x) = \sin x$, $f(x) = \cos x$.

For such a function, we can write the polynomial

$$f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots + f^{(n)}(0)\frac{x^n}{n!}.$$

This is the **Taylor polynomial** $T_n(x)$ of degree n. The simplest example is $f(x) = x^n/n!$. In this case, check that $T_n(x) = f(x)$.

Taylor Polynomial

For each $n \geq 0$, we have

$$f(x) = T_n(x) + R_n(x),$$

where the remainder is

$$R_n(x) = \int_0^x \int_0^{x_1} \cdots \int_0^{x_n} f^{(n+1)}(x_{n+1})\, dx_{n+1} dx_n \dots dx_1. \qquad (2.27)$$

This is valid whether x is positive or negative.

Proof Apply the real fundamental theorem of calculus

$$f(b) - f(a) = \int_a^b f'(x)\, dx \qquad (2.28)$$

repeatedly. For example, with $b = x$, $a = 0$,

$$f(x) = f(0) + \int_0^x f'(x_1)\, dx_1.$$

Now repeat this with f' playing the role of f, and with $b = x_1$ and $a = 0$, to get

$$f'(x_1) = f'(0) + \int_0^{x_1} f''(x_2)\, dx_2.$$

Plugging the last equation into the previous one yields

$$f(x) = f(0) + f'(0)x + \int_0^x \int_0^{x_1} f''(x_2)\, dx_2 dx_1.$$

If we repeat this with f'' playing the role of f, and with $b = x_2$ and $a = 0$, we get

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + \int_0^x \int_0^{x_1} \int_0^{x_2} f''(x_3)\, dx_3 dx_2 dx_1$$

and so on. □

The simplest example is $f(x) = x^{n+1}/(n+1)!$. In this case, check that $T_n(x) = 0$ which implies $R_n(x) = f(x)$. Since for this $f(x)$, $f^{(n+1)}(x) \equiv 1$, this is saying

$$\int_0^x \int_0^{x_1} \cdots \int_0^{x_n} 1\, dx_{n+1} dx_n \ldots dx_1 = \frac{x^{n+1}}{(n+1)!}. \tag{2.29}$$

The Taylor polynomial is the n-th partial sum of the **real Taylor series**

$$\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots$$

As a consequence,

Real Taylor Series

Let $R_n(x)$ be as in (2.27). If

$$\lim_{n\to\infty} R_n(x) = 0, \tag{2.30}$$

then the Taylor series converges to $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots.$$

The Taylor polynomials of $\cos x$ and $\sin x$ are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + R_n(x)$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + R_n(x).$$

Now write the Taylor polynomial for e^x,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_n(x).$$

If $x > 0$, then $R_n(x)$ is positive [look at its formula in detail: if a function is positive, the area under the graph is positive]. If you remove $R_n(x)$ from this last equation, you get

$$e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \tag{2.31}$$

for x positive and any $n \geq 0$. Let s_n be the sum in (2.31). When x is positive, the sequence (s_n) is increasing and bounded by e^x. By the completeness property of real numbers, (s_n) has a limit. Since s_n is the n-th partial sum of the series

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \tag{2.32}$$

this means the series (2.32) converges. This for $x > 0$.

By the n-th term test (Theorem 3.1), it follows

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0, \qquad x > 0. \tag{2.33}$$

Let x be a positive real. Look at the formula for $R_n(x)$ for e^x. In this formula,

$$x_{n+1} < x_n < x_{n-1} < \cdots < x_1 < x,$$

so $e^{x_{n+1}} < e^x$. Therefore by (2.29),

$$0 < R_n(x) < e^x \int_0^x \int_0^{x_1} \cdots \int_0^{x_n} 1 \, dx_{n+1} dx_n \ldots dx_1 = e^x \cdot \frac{x^{n+1}}{(n+1)!}.$$

Now we deal with the case of x negative. Instead of writing $R_n(x)$ for e^x for x negative, we write $R_n(x)$ for e^{-x} for x positive. This will make the calculation clearer. Then

$$R_n(x) = (-1)^{n+1} \int_0^x \int_0^{x_1} \cdots \int_0^{x_n} e^{-x_{n+1}} \, dx_{n+1} dx_n \ldots dx_1,$$

so, since $e^{-x} < 1$, by (2.29) again,

$$|R_n(x)| = \int_0^x \int_0^{x_1} \cdots \int_0^{x_n} e^{-x_{n+1}} \, dx_{n+1} dx_n \ldots dx_1 < \frac{x^{n+1}}{(n+1)!}.$$

Combining these results, for any real x, positive or negative, we have

$$|R_n(x)| < e^{|x|} \cdot \frac{|x|^{n+1}}{(n+1)!}, \qquad n = 0, 1, 2, \ldots \tag{2.34}$$

Of course, when $x = 0$, $R_n(0) = 0$, so there is nothing to check there.

For $f(x) = \sin x$ and $f(x) = \cos x$, since $|f^{(n)}(x)| \leq 1$ for any real x and any $n \geq 0$, by (2.29), we have

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}, \qquad n = 0, 1, 2, \dots$$

By (2.33), in each of the three cases e^x, $\sin x$, $\cos x$, (2.30) is valid, for any real x.

From this we conclude the everywhere convergent Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{2.35}$$

for all x real and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

for all x real and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

for all x real.

Exercises

Problem 2.1 Verify $\omega^3 = 1$.

Problem 2.2 Show the square of a twelfth root of unity is a sixth root of unity. Use this to compute the principal sixth root of unity.

Problem 2.3 (2.35) shows

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

Show $2.5 < e < 3$.

Problem 2.4 Let ω be the n-th root of unity. Show

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0.$$

Chapter 3
Euler's Identity

The goal of this chapter is to define the complex exponential and to derive

Euler's Identity

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{3.1}$$

This implies

$$e^{i\pi} + 1 = 0,$$

which relates the five numbers 0, 1, i, e, π in one equation.

Using his identity, Euler makes the fundamental theorem of trigonometry transparent, by reducing it to the complex law of exponents,

$$e^{i\theta}e^{i\theta'} = e^{i(\theta+\theta')}.$$

Ultimately, this law reflects the symmetry of the unit circle, each point z of which corresponds to a rotation θ of the plane.

It is natural to search for laws of exponents that reflect other symmetries, such as rotations in three dimensional space. Here, rotations e^{iz} are matrices, and the law of exponents fails. Correction terms must be inserted,

$$e^{iz}e^{iz'} = e^{i(z+z'+[z,z']+\dots)},$$

the first term being the so-called *bracket* $[z, z']$ of z and z'. Because symmetry lies at the basis of many phenomena, this and other generalized laws are widely applicable. Of course, in this text, we derive only the law of exponents in the complex plane.

3.1 Triangle Inequality

Let $z = x + iy$ and $w = u + iv$ be complex numbers. Let $\bar{w} = u - iv$ be the conjugate of w, let $\text{Re}(w)$ be the real part of w, and let $|z| = \sqrt{x^2 + y^2}$ be the absolute value of z. Then $w + \bar{w} = 2\text{Re}(w)$ and $z\bar{z} = |z|^2$.

Since

$$z\bar{w} = (x + iy)(u - iv) = (xu + yv) + i(yu - xv),$$

we have

$$\text{Re}(z\bar{w}) = xu + yv,$$

which is the standard dot product of two vectors (x, y) and (u, v) in the plane.

Expanding $|z + w|^2$ yields

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z + w}) \\ &= (z + w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\ &= |z|^2 + 2\text{Re}(z\bar{w}) + |w|^2. \end{aligned}$$

Cauchy-Schwarz inequality

For any complex numbers z, w, we have

$$|\text{Re}(z\bar{w})| \leq |z|\,|w|. \tag{3.2}$$

Proof Let t be a real number. If we replace w by tw in the last equation, we get

$$|z + tw|^2 = |z|^2 + 2t\text{Re}(z\bar{w}) + t^2|w|^2.$$

Since this is a quadratic in t which is always nonnegative, it has at most one root. Hence its discriminant must be ≤ 0. But the discriminant is

$$4\left(\text{Re}(z\bar{w})\right)^2 - 4|z|^2|w|^2,$$

so we get (3.2). □

Triangle Inequality

For any complex numbers z, w, we have

$$|z + w| \leq |z| + |w|. \tag{3.3}$$

Proof By the Cauchy-Schwarz inequality,

$$|z+w|^2 = |z|^2 + 2\mathrm{Re}(z\bar{w}) + |w|^2 \le |z|^2 + 2|z|\,|w| + |w|^2 = (|z|+|w|)^2 .$$

Taking square roots yields (3.3). □

In general, if $z_1, z_2, \dots, z_n$ are complex numbers, then the triangle inequality is

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|.$$

This is derived by applying (3.3) repeatedly.

A **sequence** is an infinite list of complex numbers $z_0, z_1, z_2, \dots$. Sequences are denoted (z_n). We say a sequence (z_n) **approaches** a complex number z, or **converges** to z, or the **limit** of (z_n) is z, if

$$\lim_{n\to\infty} |z_n - z| = 0.$$

The sum of finitely many convergent sequences is convergent, with limit equal to the sum of their limits. The same result is valid for the difference, the product, and the quotient, under the usual conditions. We use these properties in what follows without comment.

We say a sequence (z_n) is **Cauchy** if the terms approach each other, in the sense

$$\lim_{n,m\to\infty} |z_n - z_m| = 0.$$

If a sequence (z_n) converges to a limit z, then it's clear the sequence is Cauchy, because, by the triangle inequality,

$$\lim_{n,m\to\infty} |z_n - z_m| \le \lim_{n,m\to\infty} (|z_n - z| + |z - z_m|) = 0 + 0 = 0.$$

The converse is a basic property of complex numbers.

Completeness Property

Let (z_n) be a sequence of complex numbers. If the sequence is Cauchy, then the sequence has a limit.

Informally speaking, if the terms of the sequence approach each other, then the terms of the sequence approach something.

Proof Let $z_n = x_n + iy_n$. Since

$$|x_n - x_m| \le |z_n - z_m|, \qquad |y_n - y_m| \le |z_n - z_m|,$$

the real sequences (x_n) and (y_n) are Cauchy. By the completeness property of the real numbers, there are reals x, y with $x_n \to x$ and $y_n \to y$. Let $z = x + iy$. Since

$$|z_n - z| \le |x_n - x| + |y_n - y|,$$

$z_n \to z$ follows. □

3.2 Polynomials

A complex function $w = f(z)$ can be thought as two real functions $w = u + iv$, each a function of two real variables $z = x + iy$,

$$w = u + iv = f(z) = f(x + iy).$$

Thus $w = f(z)$ can be thought of as two real functions (u, v) of two real variables (x, y).

deg n poly has at most n roots. p and q deg n and p and q have same roots implies $p = q$.

$$z^n - 1 = \prod_{k=0}^{n-1}(z - \omega^k).$$

$$1 + z + z^2 + \cdots + z^{n-1} = \frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1}(z - \omega^k).$$

Not complete ...

3.3 Series

If (z_n) is a sequence, for each $n \geq 0$, let

$$s_n = z_0 + z_1 + z_2 + \cdots + z_n.$$

Then s_n is called the **n-th partial sum**. So

$$\begin{aligned} s_0 &= z_0 \\ s_1 &= z_0 + z_1 \\ s_2 &= z_0 + z_1 + z_2 \\ s_3 &= z_0 + z_1 + z_2 + z_3 \\ &\dots \end{aligned} \tag{3.4}$$

This way, from the sequence (z_n), we obtain another sequence (s_n).

A **series** $\sum z_n$ is an expression of the form

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + z_3 + \dots$$

Let (s_n) be the sequence of n-th partial sums of the sequence (z_n), and let z be a complex number. We say the series $\sum z_n$ **sums** to z, or **converges** to z, or has **limit** z, if

$$\lim_{n\to\infty} s_n = z.$$

In this case, we write

$$z = \sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + z_3 + \dots$$

The term-by-term sum of finitely many convergent series is convergent, with limit equal to the sum of their limits. We use this property in what follows without comment.

Let (s_n) be the sequence of n-th partial sums of the sequence (z_n), and suppose the series $\sum z_n$ converges to z. The n-th **tail of the series** is

$$t_n = \sum_{k=n+1}^{\infty} z_k.$$

Since $t_n = z - s_n$, t_n vanishes as $n \to \infty$,

$$t_n = \sum_{k=n+1}^{\infty} z_k = z - s_n \to 0, \qquad n \to \infty.$$

The most basic seriesis the **geometric series**

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots, \qquad |z| < 1. \tag{3.5}$$

To see this, by cross-multiplying, check

$$s_n = 1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1-z}, \tag{3.6}$$

hence

$$\left|\frac{1}{1-z} - (1 + z + z^2 + \cdots + z^n)\right| = \frac{|z|^{n+1}}{|1-z|}.$$

When $|z| < 1$, the limit of $|z|^{n+1}$ as $n \to \infty$ is 0. Thus the n-th partial sum s_n of the series (3.5) converges to $1/(1-z)$ for $|z| < 1$.

From (3.6), the n-th tail of the geometric series is

$$t_n = \frac{1}{1-z} - \left(1 + z + z^2 + \cdots + z^n\right) = \frac{z^{n+1}}{1-z}. \tag{3.7}$$

A basic fact is the

3.1. *n*-th Term Test

If a series $\sum z_n$ converges, then

$$\lim_{n\to\infty} z_n = 0.$$

Proof Let (s_n) be the partial sums of the series. Since the series converges to some limit z, we have $s_n \to z$ as $n \to \infty$. Also we have $s_{n-1} \to z$ as $n \to \infty$. Hence

$$z_n = s_n - s_{n-1} \to 0$$

as $n \to \infty$. □

Informally, this says if you stack infinitely many boxes on top of each other, and the top box never reaches the ceiling, then the limit of the box heights is zero. Of course the converse of the n-th term test is not true.

3.2. Absolute Convergence

Let $z_0, z_1, z_2, \ldots$ be complex numbers. If the positive series

$$\sum_{n=0}^{\infty} |z_n|$$

is finite, then the complex series

$$\sum_{n=0}^{\infty} z_n$$

converges, and

$$\left|\sum_{n=0}^{\infty} z_n\right| \le \sum_{n=0}^{\infty} |z_n|.$$

Proof Let $r_n = |z_n|$ and let (s_n) be the sequence of the partial sums of $\sum z_n$. Since the series $\sum r_n$ converges, the sequence of its partial sums is Cauchy, so

$$\sum_{k=m+1}^{n} r_k \to 0, \qquad m, n \to \infty.$$

Then, for $n > m$,

$$s_n - s_m = z_{m+1} + \cdots + z_n = \sum_{k=m+1}^{n} z_k.$$

By the triangle inequality,

$$|s_n - s_m| = \left|\sum_{k=m+1}^{n} z_k\right| \le \sum_{k=m+1}^{n} |z_k| = \sum_{k=m+1}^{n} r_k.$$

Hence

$$|s_n - s_m| \to 0, \qquad m, n \to \infty..$$

Thus (s_n) is Cauchy. By the completeness property, (s_n) is convergent, hence $\sum z_n$ is convergent. □

When the positive series $\sum |z_n|$ is finite, we say the series $\sum z_n$ **converges absolutely**.

Next we discuss **multiplication** of series. Suppose we have two sequences z_0, z_1, $z_2, \dots$ and z'_0, z'_1, $z'_2, \dots$. How do we multiply the series

$$(z_0 + z_1 + z_2 + z_3 + \dots)(z'_0 + z'_1 + z'_2 + z'_3 + \dots)?$$

The simplest way to multiply is term-by-term to get

$$z''_0 + z''_1 + z''_2 + z''_3 + \dots,$$

where

$$\begin{aligned} z''_0 &= z_0 z'_0, \\ z''_1 &= z_0 z'_1 + z_1 z'_0, \\ z''_2 &= z_0 z'_2 + z_1 z'_1 + z_2 z'_0, \\ z''_3 &= z_0 z'_3 + z_1 z'_2 + z_2 z'_1 + z_3 z'_0, \end{aligned}$$

and in general

$$z''_k = z_0 z'_k + z_1 z'_{k-1} + \dots + z_{k-1} z'_1 + z_k z'_0.$$

Thus, to obtain z''_k, we are grouping together all terms $z_n z'_m$ with $n + m = k$.

Therefore we obtain a third series $\sum z''_k$, the **product series**. When does the product series converge, and when does it equal the product of $\sum z_n$ and $\sum z'_m$?

Assume first the terms z_n, z'_n, $n \ge 0$, are positive, let s_n, s'_n be the n-th partial sums of the series $\sum z_n$, $\sum z'_n$, and let s_∞, s'_∞ be the sums of the series. Let z''_k be as above, let s''_n and s''_∞ be the n-th partial sum and the sum of the series $\sum z''_n$ respectively. Then (s_n), (s'_n), (s''_n) are increasing sequences with limits s_∞, s'_∞, s''_∞ respectively.

Let $N \ge 0$. Multiplying out the terms in the product $s_N s'_N$, every term in s''_N appears as a term in the expansion of $s_N s'_N$. In more detail, if $n+m \le N$, then $n \le N$ and $m \le N$. Since the terms are positive, $s_N s'_N \ge s''_N$ follows. Passing to the limit $N \to \infty$, we obtain $s_\infty s'_\infty \ge s''_\infty$, or

$$\left(\sum_{n=0}^{\infty} z_n\right)\left(\sum_{m=0}^{\infty} z'_m\right) \ge \sum_{k=0}^{\infty}\left(\sum_{n+m=k} z_n z'_m\right).$$

Conversely, if $n \le N$ and $m \le N$, then $n+m \le 2N$. Thus every term in the expansion of $s_N s'_N$ appears as a term in s''_{2N}, hence $s_N s'_N \le s''_{2N}$. Passing to the limit $N \to \infty$, we obtain $s_\infty s'_\infty \le s''_\infty$, or

$$\left(\sum_{n=0}^{\infty} z_n\right)\left(\sum_{m=0}^{\infty} z'_m\right) \le \sum_{k=0}^{\infty}\left(\sum_{n+m=k} z_n z'_m\right).$$

This establishes (3.8) below, when the series are positive. For complex series, when the series converge absolutely, we can derive the same result.

3.3. Products of Series

Let (z_n), (z'_n) be sequences of complex numbers. If these sequences are positive, then

$$\left(\sum_{n=0}^{\infty} z_n\right)\left(\sum_{m=0}^{\infty} z'_m\right) = \sum_{k=0}^{\infty}\left(\sum_{n+m=k} z_n z'_m\right). \tag{3.8}$$

If the sequences are complex, and both series on the left of (3.8) converge absolutely, then the series on the right of (3.8) converges absolutely, and the equality (3.8) holds.

When the series are positive, as we have seen above, (3.8) holds even if one side is infinite, in which case so is the other side.

Proof Let $r_n = |z_n|$, $r'_n = |z'_n|$, $n \ge 0$. By assumption, the n-th partial sums of the series $\sum r_n$, $\sum r'_n$ converge, so the tails vanish

$$\sum_{n=N+1}^{\infty} r_n \to 0, \qquad \sum_{m=N+1}^{\infty} r'_m \to 0, \qquad N \to \infty. \tag{3.9}$$

By the triangle inequality and (3.8) in the positive case,

$$\sum_{k=0}^{\infty}\left|\sum_{n+m=k} z_n z'_m\right| \le \sum_{k=0}^{\infty}\left(\sum_{n+m=k} r_n r'_m\right) = \left(\sum_{n=0}^{\infty} r_n\right)\left(\sum_{m=0}^{\infty} r'_m\right) < \infty.$$

By Theorem 3.2, it follows the series $\sum z''_k$ converge absolutely.

Let $s_N, s'_N, s''_N, s_\infty, s'_\infty, s''_\infty$ be as above. The goal is to establish $s_\infty s'_\infty = s''_\infty$. Since every term in the expansion of $s_N s'_N$ appears in s''_{2N}, after canceling, the remaining terms $z_n z'_m$ in the difference

$$s''_{2N} - s_N s'_N$$

satisfy $n > N$ or $m > N$. These terms appear in the expansion of $(s_{2N} - s_N)s'_{2N}$ or in the expansion of $s_{2N}(s'_{2N} - s'_N)$. By the triangle inequality,

$$|s''_{2N} - s_N s'_N| \le \left(\sum_{n=N+1}^{2N} r_n\right)\left(\sum_{m=0}^{2N} r'_m\right) + \left(\sum_{n=0}^{2N} r_n\right)\left(\sum_{m=N+1}^{2N} r'_m\right).$$

Now let $N \to \infty$. By (3.9), we obtain

$$|s''_\infty - s_\infty s'_\infty| \le 0 \cdot \left(\sum_{n=0}^{\infty} r'_n\right) + \left(\sum_{n=0}^{\infty} r_n\right) \cdot 0 = 0.$$

This establishes (3.8). □

The simplest application of the series product formula is the square of the geometric series. Since

$$\sum_{n+m=k} z^n z^m = \sum_{n=0}^{k} z^k = (k+1)z^k,$$

by (3.8), we have

$$\frac{1}{(1-z)^2} = \left(\sum_{n=0}^{\infty} z^n\right)^2 = \sum_{k=0}^{\infty}(k+1)z^k = 1 + 2z + 3z^2 + 4z^3 + \dots, \tag{3.10}$$

valid for $|z| < 1$. By the n-th term test, this implies

$$\lim_{n\to\infty} nz^n = 0, \qquad |z| < 1. \tag{3.11}$$

3.4 Complex Elementary Functions

By (2.35), the positive series

$$e^{|z|} = 1 + |z| + \frac{|z|^2}{2!} + \frac{|z|^3}{3!} + \dots$$

converges for every complex z. By Theorem 3.2, it follows the complex series

$$\exp z = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \tag{3.12}$$

and

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \tag{3.13}$$

and

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \tag{3.14}$$

all converge for any complex z.

We **define** $\exp z = e^z$, $\sin z$, $\cos z$ by **these** series, for z complex. Then, by the results in §2.5, when $z = x$ is real, the complex functions e^z, $\sin z$, $\cos z$ equal the real functions e^x, $\sin x$, $\cos x$ discussed in Chapter 2. We now derive the law of exponents for complex exponentials.

Complex Law of Exponents

For every z and w complex,

$$e^z e^w = e^{z+w}.$$

Proof By the binomial theorem,

$$(z+w)^k = \sum_{n=0}^{k} \binom{k}{n} z^n w^{k-n} = \sum_{n+m=k} \binom{k}{n} z^n w^m,$$

where

$$\binom{k}{n} = \frac{k!}{n!(k-n)!} = \frac{k!}{n!m!}$$

is the binomial coefficient. Since the exponential series converges absolutely, by (3.8),

$$\begin{aligned} e^z e^w &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{w^m}{m!}\right) = \sum_{k=0}^{\infty}\left(\sum_{n+m=k} \frac{z^n}{n!}\frac{w^m}{m!}\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{n+m=k} \binom{k}{n} z^n w^m\right) = \sum_{k=0}^{\infty} \frac{1}{k!}(z+w)^k = e^{z+w}. \end{aligned}$$

Let $e_n(z)$, $s_n(z)$, and $c_n(z)$ be the n-th partial sums of the series e^z, $\sin z$, and $\cos z$ respectively. Then

$$e_n(iz) = c_n(z) + i s_n(z).$$

Passing to the limit $n \to \infty$ yields

Euler's Identity

For z complex,

$$e^{iz} = \cos z + i \sin z.$$

When $z = \theta$ is real, this is (3.1). From Euler's identity, we have

Exponential Form of $\sin$ **and** $\cos$

For z complex,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

We also have

Polar form

Every complex number z except 0 may be written in the form

$$z = x + iy = re^{i\theta} = r\cos\theta + ir\sin\theta,$$

where $r = |z|$ and θ is real, determined uniquely up to an integer multiple of 2π. In particular, z is on the unit circle iff

$$z = e^{i\theta}.$$

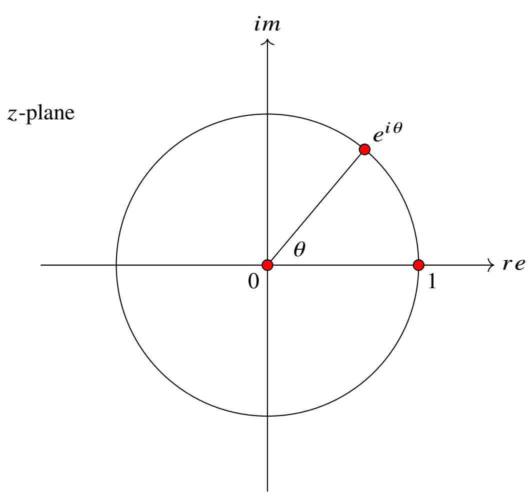

Fig. 3.1 A point on the unit circle

Recalling the roots of unity from §2.2, we obtain

Roots of Unity

Fix $n \geq 1$. Then

$$\omega = e^{2\pi i/n} \tag{3.15}$$

is the principal n-th root of unity, and the n complex numbers

$$1, \omega, \omega^2, \omega^3, \dots, \omega^{n-1}$$

are the roots of the degree n polynomial equation

$$z^n = 1.$$

Also, we can write

$$w = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Therefore we can write

$$w = e^z \quad \Longleftrightarrow \quad w = re^{i\theta} \quad \text{where} \quad r = e^x, \theta = y. \tag{3.16}$$

Use this to show e^z is periodic with period $2\pi i$:

$$e^z = e^w \quad \Longleftrightarrow \quad z - w \text{ is an integer multiple of } 2\pi i.$$

Because of this, the natural domain for $w = e^z$ is the region

$$G = \{z = x + iy : -\pi < y < \pi, -\infty < x < \infty\}.$$

Then the image e^G is the region (Figure 3.2)

$$e^G = \{w = re^{i\theta} : -\pi < \theta < \pi, r > 0\}.$$

Notice e^z is never zero, $e^z \neq 0$, so $w = 0$ is not in the range of e^z. Also the red lines $y = \pm\pi i$ are mapped to the red negative real axis.

3.5 Complex Integrals

Let $f(t)$ be a real-valued continuous function on a closed interval $[t_1, t_2]$. Then, from real calculus, the integral

$$\int_{t_1}^{t_2} f(t)\, dt \tag{3.17}$$

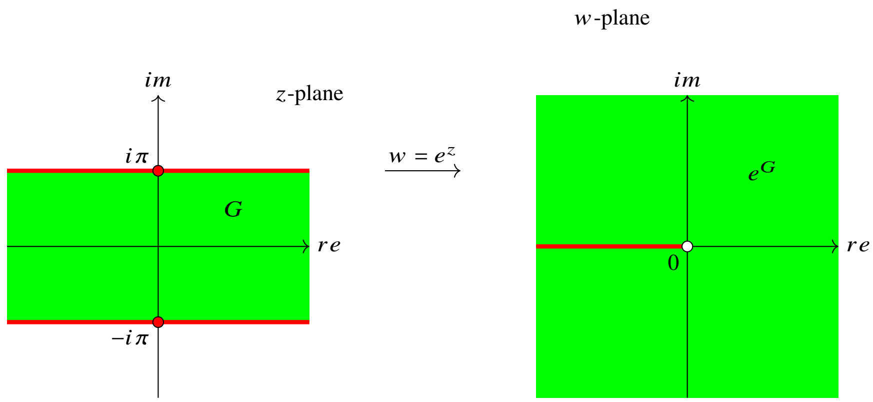

Fig. 3.2 The map $w = e^z$ from G to e^G

is well-defined. When $f(t) = u(t) + iv(t)$ is complex-valued, we define the **integral** by

$$\int_{t_1}^{t_2} f(t)\,dt = \int_{t_1}^{t_2} u(t)\,dt + i\int_{t_1}^{t_2} v(t)\,dt.$$

For example,

$$\begin{aligned}\int_0^{\pi/2} e^{it}\,dt &= \int_0^{\pi/2} (\cos t + i\sin t)\,dt \\ &= \int_0^{\pi/2} \cos t\,dt + i\int_0^{\pi/2} \sin t\,dt \\ &= \sin t\Big|_0^{\pi/2} + i\,(-\cos t)\Big|_0^{\pi/2} = 1 + i.\end{aligned}$$

Then *the real part of the integral is the integral of the real part*, and *the imaginary part of the integral is the integral of the imaginary part*,

$$\mathrm{Re}\left(\int_{t_1}^{t_2} f(t)\,dt\right) = \int_{t_1}^{t_2} \mathrm{Re}(f(t))\,dt, \qquad \mathrm{Im}\left(\int_{t_1}^{t_2} f(t)\,dt\right) = \int_{t_1}^{t_2} \mathrm{Im}(f(t))\,dt.$$

Just like the real integral, the complex integral is linear: For any complex constants a and b, we have

$$\int_{t_1}^{t_2} (af(t) + bg(t))\,dt = a\int_{t_1}^{t_2} f(t)\,dt + b\int_{t_1}^{t_2} g(t)\,dt.$$

We already know the following for real integrals.

3.4. Triangle Inequality for Complex Integrals

Let $f(t)$ be a complex-valued continuous function on a closed interval $[t_1, t_2]$. Then

$$\left|\int_{t_1}^{t_2} f(t)\,dt\right| \le \int_{t_1}^{t_2} |f(t)|\,dt.$$

Proof Let I be the integral (3.17). Since I is a complex number, we can write $I = re^{i\theta}$. Then $r = e^{-i\theta} I$. Taking the real parts of both sides, $|I| = r = \mathrm{Re}(e^{-i\theta} I)$. Since $e^{-i\theta}$ is a constant,

$$|I| = e^{-i\theta}\int_{t_1}^{t_2} f(t)\,dt = \int_{t_1}^{t_2} e^{-i\theta} f(t)\,dt.$$

Taking the real part of both sides,

$$|I| = \mathrm{Re}\left(\int_{t_1}^{t_2} e^{-i\theta} f(t)\,dt\right) = \int_{t_1}^{t_2} \mathrm{Re}\left(e^{-i\theta} f(t)\right)\,dt.$$

But for any complex number a, $\mathrm{Re}(a) \le |a|$. Since $|e^{-i\theta}| = 1$, we obtain

$$|I| = \int_{t_1}^{t_2} \mathrm{Re}\left(e^{-i\theta} f(t)\right)\,dt \le \int_{t_1}^{t_2} |e^{-i\theta} f(t)|\,dt = \int_{t_1}^{t_2} |f(t)|\,dt.$$

3.5. Complex Fundamental Theorem of Calculus

If $f(t)$ is a continuous function and $F'(t) = f(t)$ on a closed interval $[t_1, t_2]$, then

$$\int_{t_1}^{t_2} f(t)\,dt = F(t_2) - F(t_1).$$

Proof This is an immediate consequence of the real fundamental theorem of calculus (2.28). Let $f(t) = u(t) + iv(t)$ and $F(t) = U(t) + iV(t)$. Then $U'(t) = u(t)$ and $V'(t) = v(t)$, so

$$\int_{t_1}^{t_2} f(t)\,dt = \int_{t_1}^{t_2} u(t)\,dt + i\int_{t_1}^{t_2} v(t)\,dt = (U(t_2) - U(t_1)) + i(V(t_2) - V(t_1)),$$

which equals $F(t_2) - F(t_1)$. □

3.6. Substitution Under the Integral

If $f(t)$ is a continuous function on a closed interval $[t_1, t_2]$, and t in $[t_1, t_2]$ is a continuously differentiable function $t(s)$ of s in $[s_1, s_2]$, then

$$\int_{t_1}^{t_2} f(t)\,dt = \int_{s_1}^{s_2} f(t(s))\,t'(s)\,ds.$$

Proof Let $F(t)$ be an anti-derivative for $f(t)$, $F'(t) = f(t)$. By the chain rule,

$$\frac{d}{ds}F(t(s)) = F'(t(s))t'(s),$$

so $F(t(s))$ is anti-derivative for $f(t(s))t'(s)$. Now $t(s_1) = t_1$ and $t(s_2) = t_2$, hence

$$\int_{t_1}^{t_2} f(t)\,dt = F(t_2) - F(t_1) = F(t(s_2)) - F(t(s_1)) = \int_{s_1}^{s_2} F'(t(s))t'(s)\,ds.$$

3.7. Switching the Order of Integration

Let $f(t, s)$ be a continuous function of t in $[t_1, t_2]$ and s in $[s_1, s_2]$. Then

$$\int_{t_1}^{t_2} \left(\int_{s_1}^{s_2} f(u, v)\,dv\right) du = \int_{s_1}^{s_2} \left(\int_{t_1}^{t_2} f(u, v)\,du\right) dv.$$

Proof Define

$$F(s) = \int_{t_1}^{t_2} \left(\int_{s_1}^{s} f(u, v)\,dv\right) du.$$

Then

$$F(s_2) - F(s_1) = \int_{t_1}^{t_2} \left(\int_{s_1}^{s_2} f(u, v)\,dv\right) du,$$

and

$$\begin{aligned} F(s+h) - F(s) - h\int_{t_1}^{t_2} f(u, s)\,du \\ = \int_{t_1}^{t_2} \left(\int_{s}^{s+h} [f(u, v) - f(u, s)]\,dv\right) du. \end{aligned}$$

Let $\epsilon(h)$ be the maximum error

$$\epsilon(h) = \max |f(u, v) - f(u, s)|$$

over u in $[t_1, t_2]$, s and v in $[s_1, s_2]$, and $|v - s| \leq h$. Then, by continuity, $\epsilon(h) \to 0$ as $h \to 0$.

By the triangle inequality for integrals,

$$\left| F(s+h) - F(s) - h \int_{t_1}^{t_2} f(u,s)\, du \right| \leq h(T - t_0)\epsilon(h).$$

Dividing by h and passing to the limit as $h \to 0$, we conclude

$$F'(s) = \int_{t_1}^{t_2} f(u,s)\, du.$$

By the fundamental theorem of calculus,

$$F(s_2) - F(s_1) = \int_{s_1}^{s_2} F'(v)\, dv = \int_{s_1}^{s_2} \left(\int_{t_1}^{t_2} f(u,v)\, du \right) dv,$$

establishing the result. □

Exercises

Problem 3.1 Join n equally spaced points on the unit circle consecutively by line segments to obtain a regular polygon P. Use

$$1 + z + z^2 + \cdots + z^{n-1} = \frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1} (z - \omega^k).$$

to show the product of the lengths of the lines joining the vertices of P to a given vertex equals n.

Chapter 4
Complex Derivatives

4.1 Contours

Let $[t_1, t_2]$ denote the closed interval $t_1 \le t \le t_2$. A **connected contour** is a function

$$C: \qquad z(t) = x(t) + iy(t), \qquad t_1 \le t \le t_2, \tag{4.1}$$

that is continuously differentiable on $[t_1, t_2]$. Continuous differentiability means $z(t)$ has a derivative

$$z'(t) = x'(t) + iy'(t), \qquad t_1 \le t \le t_2,$$

and $z'(t)$ is itself a continuous function on $[t_1, t_2]$. So a connected contour is a function of a real variable t whose domain is an interval $[t_1, t_2]$ and is complex-valued.

The **length** of a connected contour C is

$$|C| = \int_{t_1}^{t_2} |z'(t)|\,dt = \int_{t_1}^{t_2} \sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

We say the connected contour (4.1) **starts** at $z(t_1)$ and **ends** at $z(t_2)$. Let a and b be complex numbers. The simplest connected contour C starting at a and ending at b is the line segment $z(t) = (1-t)a + tb$, $0 \le t \le 1$. Then $z'(t) = b - a$ so the length is

$$|C| = \int_0^1 |z'(t)|\,dt = \int_0^1 |b-a|\,dt = |b-a|.$$

This contour is denoted $[a, b]$.

Let a be a real number. If C_1 is a connected contour $z_1(t)$, $t_1 \le t \le t_2$, and C_2 is the connected contour given by $z_2(s) = z_1(s+a)$, $t_1 - a \le s \le t_2 - a$, we say C_1 and C_2 are equivalent. In this case, it is easy to see C_1 and C_2 have the same length.

Similarly, if a is a positive real number, and C_3 is given by $z_3(s) = z_1(as)$, $t_1/a \le s \le t_2/a$, we say C_1 and C_3 are equivalent. In this case, C_1 and C_3 have the same length.

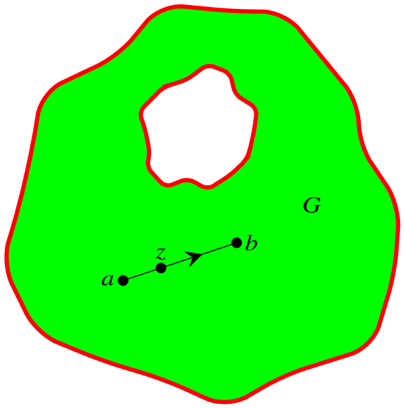

Fig. 4.1 The contour $[a, b]$

It is important that a be positive in this last example. When a is negative, we do not consider C_1 and C_3 to be equivalent, as C_1 starts where C_3 ends, and C_3 starts where C_1 ends. In this case, we write $-C$ for the connected contour C_3. Nevertheless, we still have $|-C| = |C|$.

The connected contour given by

$$z(\theta) = c + re^{i\theta} = c + r\cos\theta + ir\sin\theta, \qquad 0 \le \theta \le 2\pi,$$

is the circle $C(c, r)$ with center c and radius r traversed in the counter-clockwise direction. Then $-C(c, r)$ is the circle traversed in the clockwise direction.

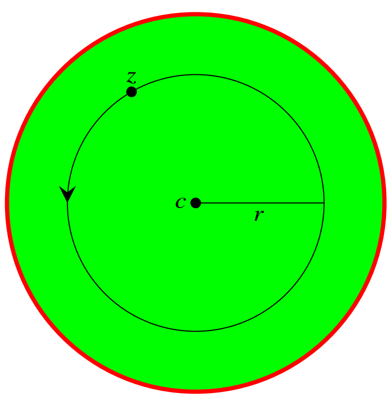

Fig. 4.2 The contour $C(c, r)$

For $C = C(c, r)$, $z'(\theta) = ire^{i\theta}$, $0 \le \theta \le 2\pi$, so

$$|C| = \int_0^{2\pi} |z'(\theta)|\, d\theta = \int_0^{2\pi} \left|ire^{i\theta}\right|\, d\theta = \int_0^{2\pi} r\, d\theta = 2\pi r.$$

In general, when t in $[t_1, t_2]$ is a continuously differentiable function $t(s)$ satisfying $t'(s) > 0$ for s in a closed interval $[s_1, s_2]$, and C_4 is given by $z_4(s) = z_1(t(s))$, $s_1 \le s \le s_2$, we say C_1 and C_4 are **equivalent**.

In this case, by the chain rule,

$$z_4'(s) = z_1'(t(s))t'(s), \qquad \text{hence} \qquad |z_4'(t)| = |z_1'(t(s))|\, t'(s).$$

By integral substitution (Theorem 3.6),

$$|C_1| = \int_{t_1}^{t_2} |z_1'(t)|\, dt = \int_{s_1}^{s_2} |z_1'(t(s))| t'(s)\, ds = \int_{s_1}^{s_2} |z_4'(s)|\, ds = |C_4|.$$

Thus *equivalent connected contours have the same length*.

Summarizing informally, we think of a connected contour as a road, and it does not matter how we drive along this road (as long as we do not go backward in time).

For example, the connected contour

$$z(t) = \frac{1-t^2}{1+t^2} + i\frac{2t}{1+t^2}, \qquad -1 \le t \le 1, \tag{4.2}$$

is equivalent to the connected contour $z(\theta) = e^{i\theta} = \cos t + i \sin t$, $-\pi/2 \le \theta \le \pi/2$, because $t = \tan(\theta/2)$ transforms $z(t)$ into $z(\theta)$. In fact, both yield the right-half unit circle, and both start at $-i$ and end at i. However, following $z(\theta)$ we are driving at a uniform speed, while, following $z(t)$, we are driving fastest at $z = 1$ and slowest at $z = \pm i$.

A **contour** C is a finite collection $C_1, C_2, \ldots, C_n$ of connected contours. In this case, it is natural to define the length $|C|$ to be

$$|C| = |C_1| + |C_2| + \cdots + |C_n|.$$

Because of this, we write

$$C = C_1 + C_2 + \cdots + C_n \tag{4.3}$$

and we call C the **sum** of the connected contours $C_1, C_2, \ldots, C_n$.

In particular, we write

$$n \cdot C = nC = C + C + \cdots + C, \qquad (n \text{ times}).$$

Then $|nC| = n|C|$.

If $C = C(c, r)$, the contour $2C$

$$z(\theta) = c + re^{i\theta}, \qquad 0 \le \theta \le 4\pi,$$

also gives the same circle, but now winding twice. This contour can also be written

$$z(\theta) = c + re^{2i\theta}, \qquad 0 \le \theta \le 2\pi.$$

The connected contours C and $2C$ are not equivalent, as they have distinct lengths,

$$|2C| = \int_0^{4\pi} |ire^{i\theta}|\, d\theta = 4\pi r = 2|C|.$$

We now broaden the definition of connected contour. A contour C is **connected** if C is given by a sum (4.3) of connected contours, as defined previously, with the additional requirement that C_2 starts where C_1 ends, C_3 starts where C_2 ends, and so on, up to and including C_n, which starts where C_{n-1} ends. Here there is no requirement that C_1 start where C_n ends.

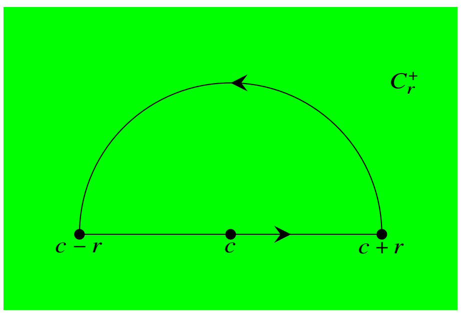

Fig. 4.3 $C_r^+ = C^+(c,r) + [c-r, c+r]$ is a connected contour

For example, let $C^+(c,r)$ be the upper half-circle $z(\theta) = c + re^{i\theta}$, $0 \le \theta \le \pi$, and let C_r^+ be

$$C_r^+ = C^+(c,r) + [c-r, c+r].$$

Then C_r^+ is connected contour (Figure 4.3).

In practice, all our contours will be circles, or arcs, or rectangles, or line segments, or sums of these.

We now define closed contours. A connected contour is **closed** if the last point equals the first point. For example $C(a,r)$ and C_r^+ are closed while $[a,b]$ is not, at least when $a \ne b$. A contour C is **closed** if it is the sum of closed connected contours.

If a is a complex number, the closed contour $z(t) = a$, $t_1 \le t \le t_2$, is the **constant contour**.

The closed contour

$$C = C(0,1) + C(i,1)$$

is made up of two counter-clockwise circles. The first circle has center 0 and radius 1, and the second circle has center i and radius 1.

The closed contour

$$C(0,5) - C(0,1)$$

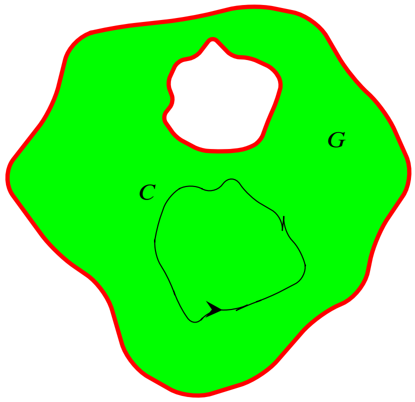

Fig. 4.4 A closed contour C

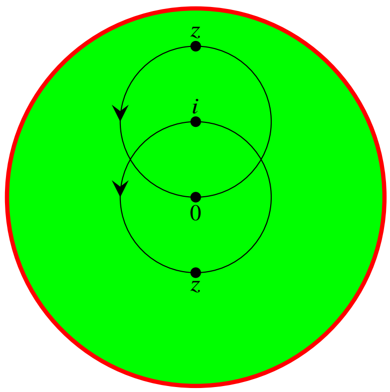

Fig. 4.5 The contour $C(0, 1) + C(i, 1)$

consists of two concentric circles, with the first circle taken counter-clockwise, and the second circle clockwise.

4.2 Open Sets and Regions

Let a be a complex number. The **open disk** with radius r and center a is the set

$$D(a, r) = \{z : |z - a| < r\}.$$

The **circle** with radius r and center a is the set

$$C(a, r) = \{z : |z - a| = r\}.$$

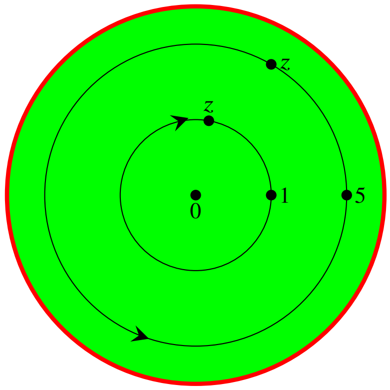

Fig. 4.6 The contour $C(0, 5) - C(0, 1)$

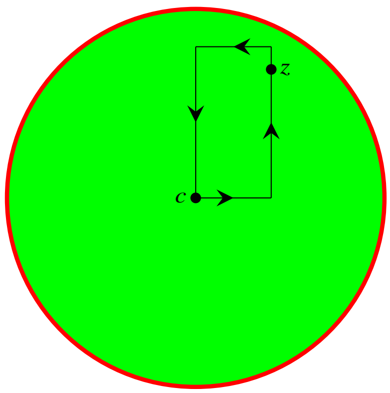

Fig. 4.7 A closed rectangle contour

Open Set

A set G in **C** is **open** if every point a in G can be surrounded by an open disk $D = D(a, r)$ centered at a and completely contained in G.

Clearly, the complex plane **C** is an open set. Examples of open sets are in Figures 4.9 and 4.10.

The **complement** of a set C in the complex plane is the set of points z that are not in C. Fo example, the complement of the contour $x \leq 0$ is G_1, which is an open set (Figure 4.10). More generally,

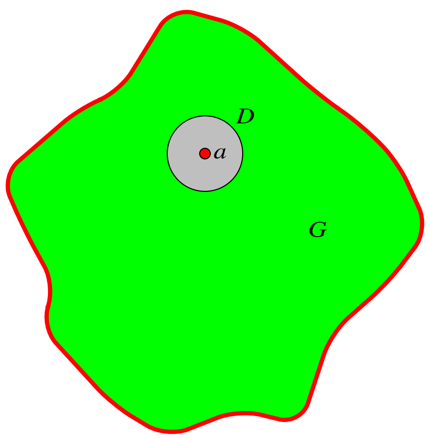

Fig. 4.8 An open set G

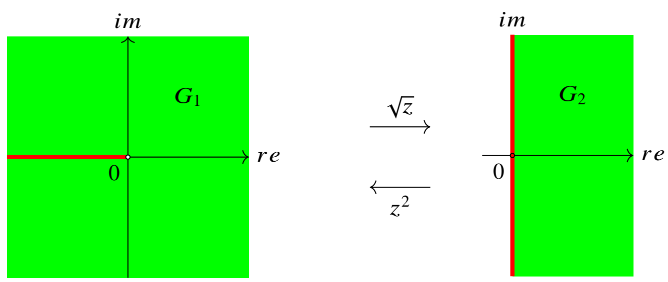

Fig. 4.9 The open set $G_1 = G_2^2$ and its image $G_2 = \sqrt{G_1}$

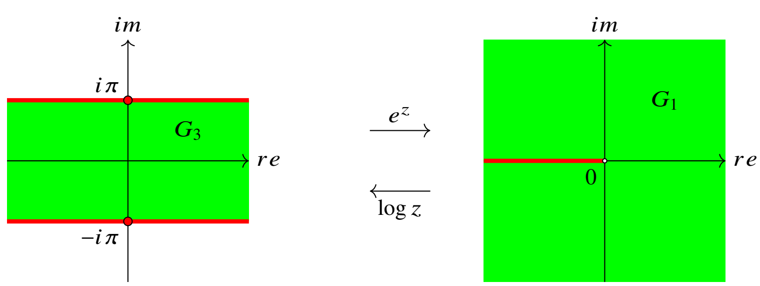

Fig. 4.10 The open set $G_3 = \log G_1$ and its image $G_1 = e^{G_3}$

4.1. The Complement of a Contour is Open

Let C be a contour, and let G be the set of points z not in C. Then G is open.

Proof If z is not in C, then there is a minimum distance between z and C. Let C be $z(t)$, $t_1 \le t \le t_2$, and let

$$r = \min_{t_1 \le t \le t_2} |z - z(t)|.$$

By continuity of $z(t)$, the minimum r is positive. Then the open disk $D(z, r/2)$ does not touch C, hence lies in G. □

Let G be a set in $\mathbf{C}$ and let a and b be points in G. We say a and b are **connected within** G if there is a connected contour starting at a and ending at b, and lying entirely in G.

If every two points in G are connected within G, then G is a **connected** set. A connected open set is a **region**.

An open disk D is a region, since any point z in D can be connected within D to the center c by the contour $[c, z]$.

Sets G and G' are **disjoint** if they have no point in common.

If we write $a \sim_G b$ when a and b are connected within G, then we have the following properties. The constant contour $z(t) \equiv a$ starts at a and ends at a. If C starts at a and ends at b, then $-C$ starts at b and ends at a. If C starts at a and ends at b, and C' starts at b and ends at c, then $C + C'$ starts at a and ends at c. Summarizing, $\sim_G$ is an equivalence relation,

1. $a \sim_G a$,
2. $a \sim_G b$ implies $b \sim_G a$, and
3. $a \sim_G b$ and $b \sim_G c$ imply $a \sim_G c$.

It follows that G is a disjoint union

$$G = G_1 \cup G_2 \cup G_3 \cup \dots$$

of connected sets $G_1, G_2, \dots$, the **connected components** of G.

Suppose G is open, and let a and b be in G. Then there is a disk D centered at b and contained in G. If $a \sim_G b$, then $a \sim_G z$ for any z in D. Hence the connected components $G_1, G_2, \dots$, are themselves open.

By the following theorem, the decomposition of an open set into a disjoint union of regions is unique, except possibly for the ordering of the regions.

4.2. Theorem

If G_1 and G_2 are disjoint open sets, then the open set $G = G_1 \cup G_2$ is not connected.

Proof Argue by contradiction: Suppose a_1 and a_2 are points in G_1 and G_2 respectively, and there is a connected contour $z(t)$, defined on an interval $I = [t_1, t_2]$, starting at a_1 and ending at a_2. Let I_1 be the set of times t such that the initial segment $z([t_1, t])$ lies in G_1, and let I_2 be the set of times t such that the terminal segment $z([t, t_2])$ lies in G_2. Then t_1 is in I_1 and t_2 is in I_2.

If s is a fixed time in (t_1, t_2), by continuity of $z(t)$, $z(t)$ is near $z(s)$ when t is near s. Since G_1 is open, some interval $(s - \delta, s + \delta)$ is in I_1 if s is in I_1. By the same token, since G_2 is open, some interval $(s - \delta, s + \delta)$ is in I_2 if s is in I_2. Thus I_1 is an interval of the form $[t_1, s_1)$, and I_2 is an interval of the form $(s_2, t_2]$.

But, since I_1 and I_2 are disjoint and their union is I, this is impossible. □

4.3 Differentiable Functions

Throughout, we consider complex functions $f(z)$ defined only on open sets, or functions whose **domains** are open sets.

Let $f(z)$ be a function defined on an open set G, and let z be a point in G. We say $f(z)$ is **differentiable at** z if the limit

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{4.4}$$

exists. Here h is a complex number, so $h \to 0$ means $|h| \to 0$. When the limit $f'(z)$ exists (and is unique), we call $f'(z)$ the **(complex) derivative at** z**.**

For $f'(z)$ to exist, by definition of limit, $f(z+h)$ must be defined for $z + h$ near z. This explains we restrict the domains of our functions $f(z)$ to open sets.

The simplest example of a differentiable function is $f(z) =$ constant. This $f(z)$ is differentiable at every point z with $f'(z) = 0$. The next simplest example is $f(z) = z$. This $f(z)$ is differentiable and $f'(z) = 1$.

4.3. Complex Derivative Rules

Let $f(z)$ and g be differentiable. Then so are $f \pm g$, fg, and f/g, at least where $g \neq 0$, and we have

$$(f+g)' = f' + g', \quad (fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Moreover if h is differentiable at $g(z)$, then the composition $h(g(z))$ is differentiable with

$$\frac{d}{dz}h(g(z)) = h'(g(z))g'(z).$$

The proof of this result is exactly like the proof of the corresponding result in real calculus. In the following, we use it without comment.

Using Theorem 4.3, we see every polynomial function and every rational function is differentiable at all points except where we are dividing by zero. For example,

$$f(z) = \frac{z^2+1}{z^3-2z+1}$$

is differentiable on the open set $G = \mathbf{C} - \{a, b, c\}$, where a, b, c are the roots of $z^3 - 2z + 1 = 0$.

Also

$$f(z) = e^{1/z}$$

is differentiable on $G = \mathbf{C} - 0$.

Let C be a connected contour $z(t)$, $t_1 \le t \le t_2$, and suppose $f(z)$ is differentiable at each point $z(t)$ of C. Let h be a real number. If $h \to 0$, then $z(t+h)) \to z(t)$. By differentiability,

$$f'(z(t)) = \lim_{h\to 0} \frac{f(z(t+h)) - f(z(t))}{z(t+h) - z(t)},$$

so

$$\begin{aligned}
\frac{d}{dt} f(z(t)) &= \lim_{h\to 0} \frac{f(z(t+h)) - f(z(t))}{h} \\
&= \lim_{h\to 0} \frac{f(z(t+h)) - f(z(t))}{z(t+h) - z(t)} \cdot \frac{z(t+h) - z(t)}{h} \\
&= \lim_{h\to 0} \frac{f(z(t+h)) - f(z(t))}{z(t+h) - z(t)} \cdot \lim_{h\to 0} \frac{z(t+h) - z(t)}{h} \\
&= f'(z(t))z'(t).
\end{aligned}$$

This is the

4.4. Contour Chain Rule

Let C be a connected contour $z(t)$, $t_1 \le t \le t_2$. If $f(z)$ is differentiable at each point of C, then

$$\frac{d}{dt} f(z(t)) = f'(z(t))z'(t) \tag{4.5}$$

for $t_1 \le t \le t_2$.

Let $f(z)$ be a complex function. Recall (§3.2) $f(z) = f(x+iy)$ may be viewed as a function $f(x, y)$ of two real variables (x, y). When h is real, $f(z+h) = f((x+h)+iy)$ corresponds to $f(x+h, y)$, and $f(z+ih) = f(x+i(y+h))$ corresponds to $f(x, y+h)$. Hence

$$\lim_{\substack{h\to 0\\ h \text{ real}}} \frac{f(z+h)-f(z)}{h} = \lim_{\substack{h\to 0\\ h \text{ real}}} \frac{f(x+h,y)-f(x,y)}{h} = \frac{\partial f}{\partial x}(z), \tag{4.6}$$

and

$$\begin{aligned}\lim_{\substack{h\to 0\\ h \text{ imag}}} \frac{f(z+h)-f(z)}{h} &= \lim_{\substack{h\to 0\\ h \text{ real}}} \frac{f(z+ih)-f(z)}{ih}\\ &= \frac{1}{i}\lim_{\substack{h\to 0\\ h \text{ real}}} \frac{f(x,y+h)-f(x,y)}{h} \\ &= \frac{1}{i}\frac{\partial f}{\partial y}(z).\end{aligned} \tag{4.7}$$

We will use the following result often to establish existence of $f'(z)$.

4.5. Theorem

Let $f(z)$ and $g(z)$ be continuous functions on an open set G. Then

1. $f'(z)$ exists at z and equals $g(z)$ for all z in G iff for all h complex and z such that the contour $[z, z+h]$ lies in G,

$$f(z+h) - f(z) = h \cdot \int_0^1 g(z+th)\,dt, \tag{4.8}$$

2. $\partial f/\partial x$ exists at z and equals $g(z)$ for all z in G iff for all h real and z such that the contour $[z, z+h]$ lies in G, (4.8) holds, and
3. $(1/i)\partial f/\partial y$ exists at z and equals $g(z)$ for all z in G iff for all h imaginary and z such that the contour $[z, z+h]$ lies in G, (4.8) holds.

Proof Suppose (4.8) holds. Then

$$\frac{f(z+h)-f(z)}{h} - g(z) = \int_0^1 (g(z+th) - g(z))\,dt.$$

Let $\epsilon(h)$ be the variation in $g(z)$ over the contour $[z, z+h]$,

$$\epsilon(h) = \max_{0\le t\le 1} |g(z+th) - g(z)|\,.$$

Since $g(z)$ is continuous, $\epsilon(h) \to 0$ as $h \to 0$. By the triangle inequality for integrals,

$$\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| \le \int_0^1 |g(z+th) - g(z)|\,dt \le \epsilon(h).$$

Passing to the limit $h \to 0$, by (4.4), $f'(z)$ exists and equals $g(z)$. If (4.8) holds only when h is real, passing to the limit $h \to 0$, by (4.6), $\partial f/\partial x$ exists at z and equals $g(z)$. If (4.8) holds only when h is imaginary, by (4.7), passing to the limit $h \to 0$, $(1/i)\partial f/\partial y$ exists at z and equals $g(z)$.

For the converse, if $f'(z) = g(z)$, by the contour chain rule applied to $z(t) = z+th$, and the fundamental theorem of calculus,

$$f(z+h) - f(z) = \int_0^1 f'(z(t))z'(t)\,dt = h \cdot \int_0^1 g(z(t))\,dt.$$

The other two cases are similar. □

As a consequence,

4.6. Theorem

If $f(z)$ is differentiable on an open set G and $f'(z) = 0$ everywhere on G, then $f(z)$ is a constant on each connected component of G.

Proof If $[a, b]$ is in G, by (4.8) with $g(z) = f'(z) = 0$, we have $f(a) = f(b)$. Fix a point a in G, and let G_1 be the set of points b in G connected to a by a multi-segment contour (Figure 4.11)

$$[a_0, a_1] + [a_1, a_2] + \cdots + [a_{n-1}, a_n], \qquad a_0 = a, a_n = b.$$

Then $f(a) = f(b)$, thus $f(z)$ is constant on G_1. If b is in G_1 and z is near b, then z is in G_1. If b is not in G_1 and z is near b, then z is not in G_1. Thus G_1 and $G - G_1$ are open sets, hence G_1 is the connected component of G containing a. □

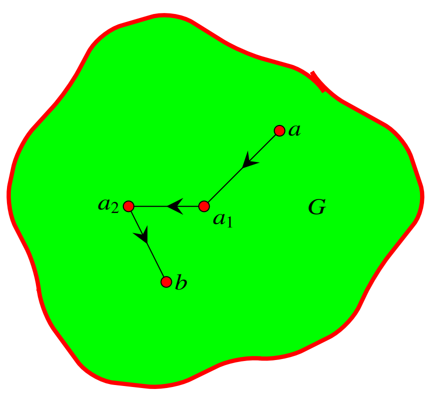

Fig. 4.11 A multi-segment contour

4.4 Complex Taylor Series

A **power series centered at a point** c is a complex series of the form

$$\sum_{n=0}^{\infty} a_n(z-c)^n = a_0 + a_1(z-c) + a_2(z-c)^2 + \dots \tag{4.9}$$

When $c = 0$, the series

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots,$$

is similar to the Taylor series in §2.5.

Recall (3.5) the **geometric series**

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \dots \tag{4.10}$$

converges in the open unit disk $|z| < 1$. Replacing z by $z - c$,

$$\frac{1}{1-(z-c)} = \sum_{n=0}^{\infty} (z-c)^n = 1 + (z-c) + (z-c)^2 + \dots$$

converges in the open disk $D(c, 1)$.

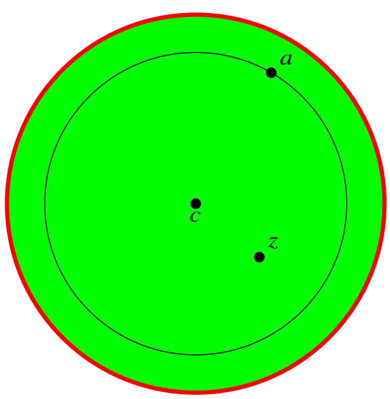

Fig. 4.12 If the power series converges at a, it converges in the disk enclosed by a

4.7. Power Series Converge in a Disk

Let c be a complex number. Then the power series (4.9) (1) converges only at c, or (2) converges everywhere, or (3) , for some $R > 0$, converges absolutely if $|z - c| < R$ and diverges if $|z - c| > R$.

The disk $D(c, R)$ is the **disk of convergence**. For example, for the geometric series, $R = 1$, and the disk of convergence is $D(0, 1)$.

Proof Suppose the series converges at some $a \neq c$. By the n-th term test, the sequence of terms is bounded, hence there is a constant M with

$$|a_n|\,|a - c|^n \leq M, \qquad n \geq 0.$$

Suppose $r = |z - c|/|a - c| < 1$ (Figure 4.12). Then

$$\sum_{n=0}^{\infty} |a_n|\,|z - c|^n = \sum_{n=0}^{\infty} |a_n||a - c|^n \left(\frac{|z - c|}{|a - c|}\right)^n \leq M \sum_{n=0}^{\infty} r^n = \frac{M}{1 - r}.$$

Thus (4.9) converges absolutely at all points in the open disk centered at c with radius $|a - c|$.

Now let R be the least upper bound of the distances $|a - c|$ to c of the points a at which the power series converges. Then there are three possibilities. If $R = 0$, the series converges only at c. If $R = \infty$, the series converges everywhere. If $0 < R < \infty$, then the series converges absolutely when $|z-c| < R$ and diverges when $|z-c| > R$. □

4.8. Power Series are Differentiable

Suppose the power series (4.9) converges for z in a disk D centered at c, and call the sum $f(z)$. Then all derivatives $f^{(n)}(z)$, $n \geq 0$, exist in D, and are given by differentiating the series (4.9) term-by-term, with

$$f^{(n)}(c) = n!a_n, \qquad n \geq 0.$$

As a consequence of this theorem, a power series $f(z)$ may be written as a **complex Taylor series**

$$f(z) = \sum_{n=0}^{\infty} f^{(n)}(c)\frac{(z - c)^n}{n!} = f(c) + f'(c)(z - c) + f''(c)\frac{(z - c)^2}{2!} + \dots,$$

at any point z in the disk of convergence D.

The geometric series (4.10) may be differentiated term-by-term, yielding

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} nz^{n-1} = 1 + 2z + 3z^2 + \dots, \qquad |z| < 1. \tag{4.11}$$

In fact, this particular result was already derived as (3.10). We can now prove Theorem 4.8.

Proof We first show the differentiated power series

$$\sum_{n=1}^{\infty} na_n (z-c)^{n-1} \tag{4.12}$$

converges for z in D.

Let z be in D. Then (Figure 4.12) there is an a in D and an $0 < r < 1$ satisfying $|z-c| \le r|a-c|$. Since the series (4.9) converges at a, by the n-th term test, the sequence of terms is bounded. Hence there is a constant M with

$$|a_n|\,|a-c|^n \le M, \qquad n \ge 0. \tag{4.13}$$

Then by (4.11),

$$\begin{aligned}\sum_{n=1}^{\infty} n|a_n|\,|z-c|^{n-1} &= \frac{1}{|a-c|}\sum_{n=1}^{\infty} n|a_n||a-c|^n \cdot \left(\frac{|z-c|}{|a-c|}\right)^{n-1} \\ &\le \frac{M}{|a-c|}\sum_{n=1}^{\infty} nr^{n-1} = \frac{M}{|a-c|}\cdot\frac{1}{(1-r)^2}.\end{aligned}$$

Thus (4.12) converges absolutely at z.

By the binomial theorem,

$$\begin{aligned}(z+h-c)^n &= \sum_{j=0}^{n}\binom{n}{j}(z-c)^{n-j}h^j \\ &= (z-c)^n + n(z-c)^{n-1}h + \sum_{j=2}^{n}\binom{n}{j}(z-c)^{n-j}h^j,\end{aligned}$$

thus for $h \ne 0$,

$$\frac{(z+h-c)^n - (z-c)^n}{h} - n(z-c)^{n-1} = \sum_{j=2}^{n}\binom{n}{j}(z-c)^{n-j}h^{j-1}.$$

Now choose any $\delta > 0$. Then for h satisfying $0 < |h| < \delta$,

$$\begin{aligned}\left|\frac{(z+h-c)^n-(z-c)^n}{h}-n(z-c)^{n-1}\right| &= \left|\sum_{j=2}^{n}\binom{n}{j}(z-c)^{n-j}h^{j-1}\right|\\ &\le |h|\sum_{j=2}^{n}\binom{n}{j}|z-c|^{n-j}|h|^{j-2}\\ &\le |h|\sum_{j=2}^{n}\binom{n}{j}|z-c|^{n-j}\delta^{j-2}\\ &= \frac{|h|}{\delta^2}\sum_{j=2}^{n}\binom{n}{j}|z-c|^{n-j}\delta^{j}\\ &\le \frac{|h|}{\delta^2}(|z-c|+\delta)^n,\end{aligned}$$

where we have used the binomial theorem again. To summarize,

$$\left|\frac{(z+h-c)^n-(z-c)^n}{h}-n(z-c)^{n-1}\right| \le \frac{|h|}{\delta^2}(|z-c|+\delta)^n \tag{4.14}$$

for $0<|h|<\delta$.

Now we show that $f'(z)$ exists and equals (4.12). Choose δ such that $|z-c|+\delta<|a-c|$. Since (4.9) converges absolutely at a, the series $g(w)=\sum|a_n|w^n$ converges for $|w|<|a-c|$. By the triangle inequality and (4.14), for $0<|h|<\delta$,

$$\begin{aligned}&\left|\frac{f(z+h)-f(z)}{h}-\sum_{n=1}^{\infty}na_n(z-c)^{n-1}\right|\\ &\qquad\le \sum_{n=1}^{\infty}|a_n|\left|\frac{(z+h-c)^n-(z-c)^n}{h}-n(z-c)^{n-1}\right|\\ &\qquad\le \sum_{n=1}^{\infty}|a_n|\frac{|h|}{\delta^2}(|z-c|+\delta)^n\\ &\qquad= \frac{|h|}{\delta^2}g(|z-c|+\delta).\end{aligned}$$

Letting $h\to 0$ in the last inequality establishes $f'(z)$ exists and equals (4.12). In particular, $f'(c)=a_1$.

Repeating the argument with (4.12) playing the role of (4.9) establishes

$$f''(z)=\left(\sum_{n=1}^{\infty}na_n(z-c)^{n-1}\right)'=\sum_{n=2}^{\infty}n(n-1)a_n(z-c)^{n-2},$$

hence $f''(c)=2!a_2$. Continuing in this manner, we obtain the result for all derivatives of $f(z)$. □

As a consequence, the complex functions e^z, $\sin z$, $\cos z$ given by (3.12), (3.13), (3.14) are differentiable everywhere on the complex plane. Differentiating their series definitions term-by-term,

$$(e^z)' = e^z, \qquad (\sin z)' = \cos z, \qquad (\cos z)' = -\sin z.$$

An alternate proof of Theorem 4.8 is to use Theorem 4.5. But this requires the differentiated series (4.12) to be continuous, which in turn requires discussions of uniform convergence. The above proof avoids these issues. Of course, as a consequence of Theorem 4.8, (4.12) is in fact continuous.

4.5 Cauchy-Riemann Equation

Let $f(z)$ be a complex function. Recall (§3.2) $f(z) = f(x+iy)$ may be viewed as a function $f(x, y)$ of two real variables (x, y). With this viewpoint, we may compute $\partial f/\partial x$ and $\partial f/\partial y$.

Cauchy-Riemann (CR) equation

Let $f(z) = f(x + yi)$ be a complex function. The Cauchy-Riemann equation is

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

The CR equation is valid for some functions $f(z)$, and not valid for other functions $f(z)$. For example, if

$$f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + 2xyi,$$

then

$$\frac{\partial f}{\partial x} = 2x + 2yi = \frac{1}{i}(2xi - 2y) = \frac{1}{i}\frac{\partial f}{\partial y},$$

so the CR equation is valid for this $f(z)$ at every point z.

On the other hand, if

$$f(z) = \bar{z}^2 = (x-iy)^2 = (x^2 - y^2) - 2xyi,$$

then

$$\frac{\partial f}{\partial x} = 2x - 2yi = -\frac{1}{i}(-2xi - 2y) = -\frac{1}{i}\frac{\partial f}{\partial y},$$

so the CR equation is not valid for this $f(z)$, for any point $z \neq 0$.

Let z be a point in G and suppose the derivative $f'(z)$ exists at z. In the definition (4.4), the limit $f'(z)$ is taken as h approaches 0 *from any direction*. In particular, if h approaches 0 along the horizontal direction, then h is real, and if h approaches along the vertical direction, then h is imaginary. From (4.6) and (4.7), we conclude

4.9. Theorem

If $f'(z)$ exists at every point in an open set G, then $f(z)$ satisfies the CR equation and $f'(z) = \partial f/\partial x$ at every point in G.

Now suppose $f(z)$ is defined and the partial derivatives $\partial f/\partial x$, $\partial f/\partial y$ are continuous on an open set G. Let C be a connected contour $z(t) = x(t) + iy(t)$ in G. Then $z(t) = x(t)+iy(t)$ corresponds to the pair of functions $(x(t), y(t))$, and $f(z(t))$ corresponds to $f(x(t), y(t))$. By the chain rule[1] for functions of two real variables,

$$\begin{aligned}\frac{d}{dt}f(z(t)) &= \frac{d}{dt}f(x(t), y(t))\\ &= \frac{\partial f}{\partial x}(x(t), y(t))x'(t) + \frac{\partial f}{\partial y}(x(t), y(t))y'(t)\\ &= \frac{\partial f}{\partial x}(z(t))x'(t) + \frac{\partial f}{\partial y}(z(t))y'(t).\end{aligned}$$

If moreover $f(z)$ satisfies the CR equation on G, this reduces to

$$\frac{d}{dt}f(z(t)) = \frac{\partial f}{\partial x}(z(t))(x'(t) + iy'(t)) = \frac{\partial f}{\partial x}(z(t))z'(t). \tag{4.15}$$

In this case, we may obtain the converse of Theorem 4.9.

4.10. Theorem

If $f(z)$ satisfies the CR equation at every point in an open set G, and $\partial f/\partial x$ is continuous at every point in G, then $f'(z)$ exists and equals $\partial f/\partial x$ at every point in G.

Proof Let z be a point in G and suppose the contour $[z, z + h]$ lies in G. Since $z'(t) = h$ on this contour, by the fundamental theorem of calculus (Theorem 3.5) and (4.15),

$$f(z + h) - f(z) = h \cdot \int_0^1 \frac{\partial f}{\partial x}(z + th)\, dt.$$

By Theorem 4.5, $f'(z)$ exists and equals $\partial f/\partial x$ at z. □

[1] Continuity of $\partial f/\partial x$, $\partial f/\partial y$ is needed for this chain rule.

For example, let

$$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} e^x(\cos y + i \sin y) = e^z,$$

and

$$\frac{1}{i}\frac{\partial f}{\partial y} = \frac{1}{i}\frac{\partial}{\partial y} e^x(\cos y + i \sin y) = \frac{1}{i} e^x(-\sin y + i \cos y) = e^z.$$

Hence the CR equation is valid for e^z everywhere, so e^z is differentiable at every point in **C**, with

$$(e^z)' = e^z.$$

Of course we already know this, but this time we are illustrating the CR equation.

The function $f(z) = f(x + iy) = y^2$ satisfies

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 2y$$

everywhere. Therefore this function is differentiable at all points on the real line, and nowhere else.

When $f(z)$ is differentiable at every point in an open disk about a point a, we say $f(z)$ **is holomorphic** at a. So a function is differentiable at every point of an open set G iff it is holomorphic at every point of G.

Polynomials (in z) and e^z, $\sin z$, $\cos z$ are holomorphic on $G = \mathbf{C}$, and rational functions are holomorphic everywhere except at the roots of the denominator.

The function $f(z) = y^2$ is differentiable on the real line but nowhere holomorphic, as the real line is not an open set.

A function that is holomorphic at every point in **C** is **entire**. So every polynomial (in z) is entire and e^z, $\sin z$, $\cos z$ are entire.

4.6 The Principal Logarithm

From real calculus, the logarithm is defined to be the inverse of the exponential. We want to do the same for e^z. We want to define a holomorphic inverse $f(z)$ to e^z,

$$e^{f(z)} = z \qquad \text{and} \qquad f(e^z) = z.$$

Because $e^z = e^{z+2\pi i}$, the second equation implies

$$z = f(e^z) = f(e^{z+2\pi i}) = z + 2\pi i,$$

which is impossible. Hence no such function $f(z)$ can satisfy the second equation.

As for the first equation, inserting $z = 0$ yields $e^{f(0)} = 0$, which can't happen since e^z is never zero. So we will not be able to define the logarithm of 0.

Even if we stay away from 0, we still have problems. Below, using integration, we see that we will not be able to define a holomorphic logarithm on all of $G = \mathbf{C} - 0$. But we will be able to define a holomorphic logarithm on all of G_1 as in Figure 4.10.

Let G be an open set. A **branch of** $\log z$ on G is a holomorphic function $f(z)$ satisfying

$$e^{f(z)} = z, \qquad z \text{ in } G.$$

When there is a branch of $\log z$ on G, we say $\log z$ *is holomorphic on* G.

Notice if there is a branch $f(z)$ of $\log z$ on G, then we can't have 0 in G, since e^z is never 0.

If $f(z)$ is a branch of $\log z$ on G, then by the chain rule

$$1 = z' = \left(e^{f(z)}\right)' = e^{f(z)} f'(z) = z f'(z),$$

so $f'(z) = 1/z$.

Conversely, suppose we have a holomorphic function $f(z)$ on G whose derivative is $1/z$ on G. Then by the quotient rule,

$$\left(\frac{e^{f(z)}}{z}\right)' = \frac{e^{f(z)} f'(z) z - e^{f(z)}}{z^2} = 0,$$

so $e^{f(z)}/z$ is a constant c on G, or

$$e^{f(z)} = cz, \qquad z \text{ in } G.$$

If we choose any complex number d satisfying $e^d = c$, then we see $f(z) - d$ is a branch of $\log z$ on G. Thus

4.11. Derivative of $\log z$

Let G be an open set in $\mathbf{C} - 0$ and let $f(z)$ be a holomorphic function on G. Then $f(z)$ plus a constant is a branch of $\log z$ on G if and only if

$$f'(z) = \frac{1}{z}, \qquad z \text{ in } G.$$

Every nonzero complex number $z = x + iy$ can be written as $re^{i\theta}$ for some r and θ, so it makes sense to define

$$\log z = \log r + i\theta.$$

This is not well-defined because θ is only determined up to multiples of 2π. However, if we restrict z to be in the open set G_1 in Figure 4.9, then we can restrict θ to be in $(-\pi, \pi)$, giving us a well-defined function $\log z$. We will show

4.12. Principal Logarithm

Let G_1 be the open set in Figure 4.9. Then

$$\log z = \log r + i\theta, \qquad r > 0, -\pi < \theta < \pi,$$

is a branch of $\log z$ on G_1, with

$$(\log z)' = \frac{1}{z}. \tag{4.16}$$

Let G_1 and G_2 be as in Figure 4.9. By definition, the principal logarithm is a function satisfying

$$e^{\log z} = z, \qquad z \text{ in } G_1.$$

We show $\log z$ is holomorphic on G_2 using the CR equation. On G_2, we have

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\left(\frac{y}{x}\right), \qquad r > 0, -\pi/2 < \theta < \pi/2.$$

Hence

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2},$$

which leads to

$$\frac{\partial \log z}{\partial x} = \frac{\partial}{\partial x}(\log r + i\theta) = \frac{x}{r^2} - i\frac{y}{r^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}.$$

Similarly,

$$\frac{1}{i}\frac{\partial \log z}{\partial y} = \frac{1}{i}\frac{\partial}{\partial y}(\log r + i\theta) = \frac{1}{i}\left(\frac{y}{r^2} + i\frac{x}{r^2}\right) = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}.$$

This shows $\log z$ is holomorphic on G_2 and establishes (4.16). To extend this result to G_1, we bring in $\sqrt{z}$.

Let G be an open set. A **branch of** $\sqrt{z}$ on G is a holomorphic function $f(z)$ satisfying

$$(f(z))^2 = z, \qquad z \text{ in } G.$$

When there is a branch of $\sqrt{z}$ on G, we say $\sqrt{z}$ *is holomorphic on* G.

We will also show

4.13. Principal Square Root

Let G_1 be the open set in Figure 4.9. Then

$$\sqrt{z} = \sqrt{r}e^{i\theta/2}, \qquad r > 0, -\pi < \theta < \pi,$$

is a branch of $\sqrt{z}$ on G_1, with

$$(\sqrt{z})' = \frac{1}{2\sqrt{z}}. \tag{4.17}$$

By definition, the principal square root is a function satisfying $(\sqrt{z})^2 = z$. To show $\sqrt{z}$ is holomorphic on G_1, we first establish

$$\sqrt{r}e^{i\theta/2} = \frac{z+r}{\sqrt{2x+2r}}, \qquad r > 0, -\pi < \theta < \pi. \tag{4.18}$$

Since both sides square to z, they are either equal or negatives of each other. But both sides lie in G_2, so cannot be negatives of each other. This establishes (4.18).

Using the CR equation, we derive (4.17). If $D = \sqrt{2x+2r}$, then we can write

$$w = \sqrt{z} = \frac{z+r}{D}.$$

Then

$$\frac{\partial D}{\partial x} = \frac{1}{D}\frac{\partial}{\partial x}(x+r) = \frac{1+x/r}{D} = \frac{(x+r)}{Dr}$$

and

$$\frac{\partial D}{\partial y} = \frac{y}{Dr}.$$

Moreover we have

$$w\bar{w} = |w|^2 = |w^2| = |z| = r.$$

We introduce notation that streamlines the computation. The CR equation may be rewritten

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0.$$

It is useful to give the left side a name.

The d-bar Operator

The CR equation is

$$\bar{\partial} f = 0,$$

where

$$\bar{\partial} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

is the **d-bar operator**.

For example,

$$\bar{\partial} D = \frac{z+r}{Dr}.$$

We compute

$$\bar{\partial} z = \frac{\partial z}{\partial x} + i\frac{\partial z}{\partial y} = 1 + i \cdot i = 1 - 1 = 0$$

and

$$\bar{\partial}(z+r) = \bar{\partial} r = \frac{x}{r} + i\frac{y}{r} = \frac{z}{r}.$$

Then

$$\bar{\partial} w = \frac{Dz/r - (z+r)(z+r)/Dr}{D^2} = \frac{zD^2 - (z+r)^2}{rD^3} = \frac{2z(x+r) - (z+r)^2}{rD^3}.$$

Because $r^2 = x^2 + y^2$,

$$2z(x+r) - (z+r)^2 = 2zx - z^2 - r^2 = -(z-x)^2 - y^2 = 0,$$

hence $\bar{\partial} w = 0$. This shows $w = \sqrt{z}$ satisfies the CR equation, hence $\sqrt{z}$ is holomorphic on G_1.

Since

$$\frac{\partial}{\partial x}(z+r) = 1 + \frac{x}{r},$$

we have

$$\begin{aligned} w' = \frac{\partial w}{\partial x} &= \frac{(1+x/r)\cdot D - (z+r)\cdot(x+r)/Dr}{D^2} \\ &= \frac{2(x+r)^2 - (z+r)(x+r)}{2rD(x+r)} \\ &= \frac{2x + 2r - z - r}{2rD} \\ &= \frac{\bar{z} + r}{2rD} = \frac{\bar{w}}{2r} = \frac{\bar{w}}{2w\bar{w}} = \frac{1}{2\sqrt{z}}. \end{aligned}$$

This establishes (4.17).

It remains to be shown that $\log z$ is holomorphic on G_1. For this, we use the chain rule. By definition of $\sqrt{z}$ and $\log z$ on G_1, $2\log\sqrt{z}$ equals

$$2\log\left(\sqrt{r}e^{i\theta/2}\right) = 2\left(\log\sqrt{r} + i\frac{\theta}{2}\right) = 2\left(\frac{1}{2}\log r + i\frac{\theta}{2}\right) = \log r + i\theta = \log z,$$

so we have

$$\log z = 2\log\sqrt{z}$$

on G_1. Since $\sqrt{z}$ lies in G_2, this expresses $\log z$ as a composition of a holomorphic function $\sqrt{z}$ on G_1 with a holomorphic function $\log z$ on G_2. By the chain rule (Theorem 4.3), $\log z$ is holomorphic on G_1, with

$$(\log z)' = (2\log\sqrt{z})' = 2\frac{1}{\sqrt{z}} \cdot \frac{1}{2\sqrt{z}} = \frac{1}{z}.$$

Recall arctan takes values in $(-\pi/2, \pi/2)$, and

$$\arctan\left(\frac{y}{x}\right) + \arctan\left(\frac{x}{y}\right) = \frac{\pi}{2}, \qquad x > 0, y > 0.$$

Another way to show $\log z$ is holomorphic on G_1 is to check

$$\log z = \log r + i\theta = \begin{cases} \log\left(\sqrt{x^2+y^2}\right) + i\arctan\left(\frac{y}{x}\right), & x > 0, \\ \log\left(\sqrt{x^2+y^2}\right) + i\left(\frac{\pi}{2} - \arctan\left(\frac{x}{y}\right)\right), & y > 0, \\ \log\left(\sqrt{x^2+y^2}\right) - i\left(\frac{\pi}{2} - \arctan\left(\frac{x}{-y}\right)\right), & y < 0. \end{cases}$$

is consistently defined on the overlaps, and to check the CR equation directly.

Exercises

Problem 4.1 With $f(z) = u(x, y) + iv(x, y)$, show the CR equation becomes the pair of equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Problem 4.2 Let $F(u, v)$ be a differentiable non-constant function of two real variables u and v. If $u = u(x, y)$ and $v = v(x, y)$ are the real and imaginary parts of a holomorphic $f(z)$ on an open set G, then $F(u, v) = 0$ on G implies $f'(z) = 0$ on G.

Chapter 5
Contour Integration

5.1 Contour Integrals

If C is a connected contour $z(t)$, $t_1 \le t \le t_2$, as in (4.1) and $f(z)$ is a continuous function on C, the **contour integral** is

$$\int_C f(z)\,dz = \int_{t_1}^{t_2} f(z(t))\,z'(t)\,dt.$$

Note the integral on the right is a complex integral as in §3.5. Thus a contour integral is a particular kind of complex integral.

Since z inside the integral is a dummy variable, we may use any other variable,

$$\int_C f(z)\,dz = \int_C f(w)\,dw.$$

The simplest contour is $[a, a+h]$. Over this contour, $z(t) = a + th$, $0 \le t \le 1$, so $z'(t) = h$, leading to

$$\int_{[a,a+h]} f(z)\,dz = h \cdot \int_0^1 f(a+th)\,dt. \tag{5.1}$$

If $C = C(c, r)$,we have $z(\theta) = c + re^{i\theta}$, $0 \le \theta \le 2\pi$, and $z'(\theta) = ire^{i\theta}$, so

$$\int_C \frac{dz}{z-c} = \int_C \frac{1}{z-c}\,dz = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}\,d\theta = 2\pi i. \tag{5.2}$$

More generally, for any sum of connected contours (§4.1)

$$C = C_1 + C_2 + \cdots + C_n,$$

we set

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz.$$

This we call **contour additivity**. In particular, we have

$$\int_{nC} f(z)\,dz = n \int_C f(z)\,dz,$$

for any contour C.

When t in $[t_1, t_2]$ is a continuously differentiable function $t(s)$ of s in $[s_1, s_2]$, and C_1 is the contour given by $z_1(s) = z(t(s))$, $s_1 \le s \le s_2$, by Theorem 3.6,

$$\int_{t_1}^{t_2} f(z(t))\, z'(t)\,dt = \int_{s_1}^{s_2} f(z(t(s)))\, z'(t(s))t'(s)\,ds = \int_{s_1}^{s_2} f(z_1(s))\, z_1'(s)\,ds,$$

so

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz.$$

Thus *the contour integrals over equivalent (§4.1) connected contours agree*.

For example, the contour integral over e^{it}, $0 \le t \le 2\pi$, is the same as the contour integral over e^{it}, $-\pi \le t \le \pi$. However,

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz,$$

because

$$\int_{-t_2}^{-t_1} f(z(-t))\, z'(-t)\,dt = -\int_{t_1}^{t_2} f(z(t)\, z'(t(s))dt.$$

Thus

$$C + (-C) = C - C = 0,$$

in the sense

$$\int_C f(z)\,dz + \int_{-C} f(z)\,dz = 0,$$

for any $f(z)$ continuous on C.

We begin with a simple but fundamental observation. Let $f(z)$ be a continuous function defined on an open set G. We say the contour integrals of $f(z)$ are **path-independent** in G if for any two points a and b in G and for any two connected contours C, C' in G starting at a and ending at b,

$$\int_C f(z)\,dz = \int_{C'} f(z)\,dz.$$

5.1. Path-Independence of Contour Integrals

Let $f(z)$ be a continuous function defined on an open set G. Then the contour integrals of $f(z)$ are path-independent in G if and only if

$$\int_C f(z)\,dz = 0$$

for every closed contour C in G.

Proof Let a, b be in G and suppose C, C' are connected contours starting at a and ending at b. Then $C - C'$ is a closed connected contour. If the integrals over closed contours vanish, then

$$\int_C f(z)\,dz - \int_{C'} f(z)\,dz = \int_{C-C'} f(z)\,dz = 0,$$

so the contour integrals of $f(z)$ are path-independent in G. Conversely, suppose the contour integrals of $f(z)$ are path-independent in G, let C be a closed connected contour in G, and let a be a point on C. Then C starts and ends at a. By path-independence, the integral over C equals the integral over the constant contour $z(t) \equiv a$, which is zero. Since a closed contour is the sum of connected closed contours, the result follows. □

Let $f(z)$ be a function on an open set G. If $F(z)$ is holomorphic and satisfies $F'(z) = f(z)$ on G, we say $F(z)$ is an **anti-derivative** of $f(z)$ on G. When this happens, we have path-independence.

5.2. Contour Fundamental Theorem of Calculus

Let C be a connected contour starting at a, and ending at b. Let $f(z)$ be a continuous function defined on an open set G containing C, and let $F(z)$ be an anti-derivative of $f(z)$ on G. Then

$$\int_C f(z)\,dz = F(b) - F(a).$$

Proof This follows from the contour chain rule (4.4) and the complex fundamental theorem of calculus, since

$$\begin{aligned}\int_C f(z)\,dz &= \int_{t_1}^{t_2} F'(z(t))z'(t)\,dt = \int_{t_1}^{t_2} \frac{d}{dt}F(z(t))\,dt \\ &= F(z(t_2)) - F(z(t_1)) = F(b) - F(a).\end{aligned}$$

Let G be an open set containing the unit circle C and suppose we have a branch $f(z)$ of $\log z$ on G. Then

$$\int_C \frac{dz}{z} = \int_C f'(z)\,dz = f(1) - f(1) = 0,$$

which contradicts the answer $2\pi i$ in (5.2). Therefore an open set containing the unit circle cannot support a branch of $\log z$. This is another illustration why $G = G_1$ (Figure 4.9) is cut along the negative reals.

Here is a result that will be used frequently.

5.3. Path-Independence iff the Anti-Derivative Exists

Let G be an open set, and let $f(z)$ be continuous on G. Then there is an antiderivative $F(z)$ on G if and only if

$$\int_C f(z)\,dz = 0, \tag{5.3}$$

for every closed contour C in G.

Proof If $f(z) = F'(z)$ on G, by Theorem 5.2, (5.3) holds for every closed connected contour C, hence for every closed contour C. Conversely, suppose (5.3) holds for every closed contour C. We define an anti-derivative $F(z)$ separately on each connected component G_1 of G.

Fix a point c in G_1. For z in G_1, define

$$F(z) = \int_C f(w)\,dw, \tag{5.4}$$

following any connected contour C starting at c and ending at z. By assumption, integrals over closed contours vanish. By Theorem 5.1, we have path-independence, hence $F(z)$ is well-defined. We show $F'(z) = f(z)$.

Assume h is such that the contour $[z, z+h]$ lies in G_1. By definition, $F(z)$ is the integral over any connected contour C starting at c and ending at z. Pick one such connected contour C. Since $F(z+h)$ is the integral over any connected contour starting at c and ending at $z+h$, we may assume $F(z+h)$ is the integral over the connected contour $C + [z, z+h]$, i.e., over C followed by the segment $[z, z+h]$. With this choice of connected contour, $F(z+h) - F(z)$ is the integral over $[z, z+h]$, hence by (5.1),

$$F(z+h) - F(z) = h \cdot \int_0^1 f(z+th)\,dt.$$

By Theorem 4.5, $F'(z)$ exists and equals $f(z)$. □

To be consistent with contour integral notation, we write

$$|dz| = |dx + i dy| = \sqrt{dx^2 + dy^2} = \sqrt{x'(t)^2 + y'(t)^2}\, dt = |z'(t)|\, dt.$$

Then the length of a contour C is

$$|C| = \int_C |dz|.$$

5.4. Triangle Inequality for Contour Integrals

Let M be the maximum value of $|f(z)|$ over a contour C. Then

$$\left|\int_C f(z)\, dz\right| \le \int_C |f(z)|\, |dz| \le M\, |C|.$$

Proof By contour additivity and the triangle inequality, it is enough to derive the inequality when C is connected as in (4.1). Let C be given by $z(t)$, $t_1 \le t \le t_2$. By Theorem 3.4,

$$\begin{aligned}\left|\int_C f(z)\, dz\right| &= \left|\int_{t_1}^{t_2} f(z(t))z'(t)\, dt\right| \\ &\le \int_{t_1}^{t_2} |f(z(t))z'(t)|\, dt = \int_{t_1}^{t_2} |f(z(t))|\, |z'(t)|\, dt = \int_C |f(z)|\, |dz|.\end{aligned}$$

Since $|f(z)| \le M$ on C,

$$\int_C |f(z)|\, |dz| \le \int_C M\, |dz| = M\, |C|.$$

Using contour integrals, we can construct many examples of holomorphic functions.

5.5. Differentiation Under the Contour Integral

Let $f(z)$ be a continuous function on a contour C. Then

$$F(z) = \int_C \frac{f(w)\,dw}{w-z} \tag{5.5}$$

has derivatives $F^{(n)}(z)$ of all orders for all z not on C, with

$$F^{(n)}(z) = n! \int_C \frac{f(w)\,dw}{(w-z)^{n+1}} \tag{5.6}$$

for $n \geq 1$.

Proof The complement G of C is open (Theorem 4.1). For $\delta > 0$, let G_δ be the set of points z with distance from C greater than δ,

$$|z-w| > \delta, \qquad w \text{ in } C.$$

Then G_δ is open, and z not in C implies z is in G_δ for some $\delta > 0$. This is the same as saying G is the union of G_δ over all $\delta > 0$.

Let $F_n(z)$ denote the right side of (5.6), and let M be the maximum of $f(w)$ on C. By the triangle inequality,

$$|F_n(z)| \leq \frac{|C|\,n!M}{\delta^{n+1}}$$

for z in G_δ, so $F_n(z)$ is bounded on G_δ, for every n.

By (5.5), $F_0(z) = F(z)$. We establish (5.6) by showing $F_n'(z) = F_{n+1}(z)$. Then

$$F_n(z) = (F_{n-1}(z))' = ((F_{n-2}(z))')' = (F_{n-2}(z))'' = \cdots = F_0^{(n)}(z),$$

which establishes (5.6).

Suppose z and h are such that the contour $[z, z+h]$ lies in G_δ. Differentiating,

$$\frac{d}{dt}\frac{n!}{(w-z-th)^{n+1}} = h \cdot \frac{(n+1)!}{(w-z-th)^{n+2}}.$$

By the fundamental theorem of calculus,

$$\frac{n!}{(w-z-h)^{n+1}} - \frac{n!}{(w-z)^{n+1}} = h \cdot \int_0^1 \frac{(n+1)!}{(w-z-th)^{n+2}}\,dt.$$

Multiplying by $f(w)$ and integrating over C, then switching the order of integration (Theorem 3.7),

$$F_n(z+h) - F_n(z) = h \cdot \int_0^1 F_{n+1}(z+th)\,dt. \tag{5.7}$$

Since $F_n(z)$ is bounded on G_δ, for every n, (5.7) implies $F_n(z)$ is continuous on G_δ, for every n. By Theorem 4.5 (applied on $G = G_\delta$), $F_n'(z)$ exists and equals $F_{n+1}(z)$, on G_δ, for every n. □

For example, if $f(x)$ is a continuous function of $0 \le x \le 1$, the integral

$$F(a) = \int_0^1 \frac{f(x)\,dx}{x+a}, \qquad a \text{ in } G_1,$$

is holomorphic. When $f(x) \equiv 1$, we already know this, since

$$\int_0^1 \frac{dx}{x+a} = \log(1+a) - \log a, \qquad a \text{ in } G_1.$$

5.2 Winding Numbers

Let C be any contour and let c be a number not on C. We define

$$N(C, c) = \frac{1}{2\pi i} \int_C \frac{dz}{z-c}.$$

Then, by contour additivity,

$$N(C_1 + C_2 + \cdots + C_n, c) = N(C_1, c) + N(C_2, c) + \cdots + N(C_n, c),$$

whenever c does not lie on any of $C_1, C_2, \ldots, C_n$, and

$$N(-C, c) = -N(C, c),$$

whenever c does not lie on $\pm C$. When C is closed, we call $N(C, c)$ the **winding number** of C around c.

For example, if $C = C(c, r)$ is the circle with radius r and center c (Figure 4.2), (5.2) shows $N(C, c) = 1$. So *a circle winds once in the counter-clockwise direction about its center.*

Another example is a rectangle centered at the origin 0. Let R be a rectangle centered at the origin 0, and let C be the perimeter of R taken in the counter-clockwise direction. Following Figure 5.1, the substitution $z \to -z$ shows

$$\int_{C_+} \frac{dz}{z} = \int_{C_-} \frac{dz}{z},$$

so $N(C_+, 0) = N(C_-, 0)$. With $\log z$ the principal logarithm on G_1, we have $(\log z)' = 1/z$. By Theorem 5.2,

$$\int_{C_+} \frac{dz}{z} = \log i - \log(-i) = \left(\log 1 + i\frac{\pi}{2}\right) - \left(\log 1 - i\frac{\pi}{2}\right) = i\pi.$$

Since $C = C_+ + C_-$, this implies $N(C, 0) = 1$. Since the substitution $z \to z + c$ translates a rectangle centered at c to a rectangle centered at 0, the same conclusion holds in general, or *the perimeter of a rectangle winds once in the counter-clockwise direction about its center.*

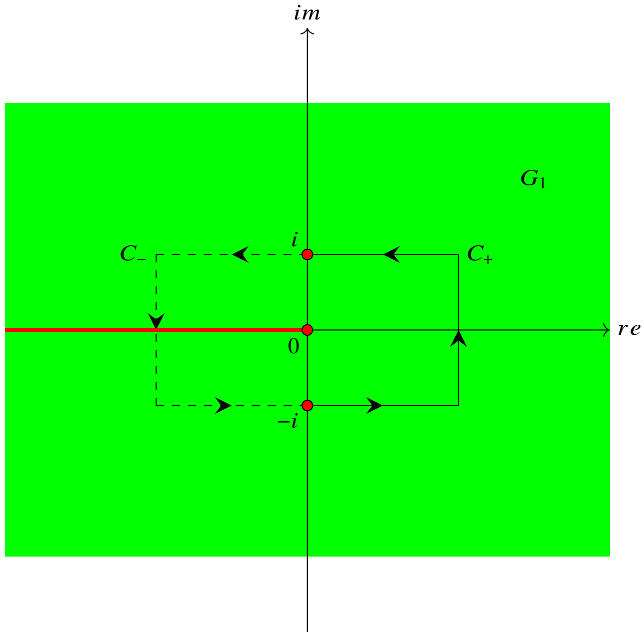

Fig. 5.1 A rectangle with perimeter $C = C_+ + C_-$

For a general closed contour, we have

5.6. Winding Number is an Integer

If C is a closed contour, and c is not on C, then the winding number $N(C, c)$ is an integer.

Proof By contour additivity, we may assume C is a closed connected contour. Let C be given by $z(t)$, $t_1 \leq t \leq t_2$, and let

$$h(t) = \int_{t_1}^{t} \frac{z'(s)}{z(s) - c}\, ds, \qquad t_1 \leq t \leq t_2.$$

Then $h(t_2) = 2\pi i N(C, c)$, $h(t_1) = 0$, $z(t_1) = z(t_2)$, and

$$h'(t) = \frac{z'(t)}{z(t) - c}, \qquad t_1 \le t \le t_2.$$

Now

$$\frac{d}{dt} e^{-h(t)}(z(t) - c) = e^{-h(t)}\left(-h'(t)(z(t) - c) + z'(t)\right) = 0,$$

so

$$e^{-h(t_2)}(z(t_2) - c) = e^{-h(t_1)}(z(t_1) - c).$$

Since $z(t_1) = z(t_2)$, this implies

$$e^{-h(t_2)} = 1,$$

so $N(C, c) = h(t_2)/2\pi i$ is an integer. □

If C is a closed contour, then the complement $G = \mathbf{C} - C$ is an open set. Therefore (§4.2) $\mathbf{C} - C$ is a disjoint union of connected components

$$\mathbf{C} - C = G_1 \cup G_2 \cup G_3 \cup \dots$$

When C is a closed contour, we show $N(C, a)$ is a constant function of a on each connected component of the complement of C.

5.7. Winding Number is Constant

Let C be a closed contour. Then $N(C, z)$ is constant on each connected component of the complement of C. Moreover, $N(C, z) = 0$ on the unbounded component of the complement of C, which we write as $N(C, \infty) = 0$. In particular, when C is the perimeter of a disk, or C is the perimeter of a rectangle, both taken counter-clockwise,

$$N(C, z) = \begin{cases} 1 & z \text{ inside} \\ 0 & z \text{ outside.} \end{cases} \tag{5.8}$$

Proof Let $f(w) = 1/2\pi i$ and let $F(z)$ be as in (5.5). Then $F(z) = N(C, z)$. By Theorem 5.5, $N(C, z)$ is holomorphic on the complement of C. Since C is closed, by Theorem 5.3,

$$2\pi i \frac{d}{dz} N(C, z) = \int_C \frac{1}{(w - z)^2}\, dw = \int_C \frac{d}{dw}\left(\frac{-1}{w - z}\right) dw = 0.$$

By Theorem 4.6, $N(C, z)$ is constant on connected components of $\mathbf{C} - C$.

Now, for any contour C, we have

$$\lim_{z\to\infty} N(C,z) = \lim_{a\to\infty} \int_C \frac{dz}{z-a} = 0.$$

Since $N(C,z)$ is constant on the unbounded component of the complement of C, this shows $N(C,z) = 0$ for z in the unbounded component of the complement of C.

Let C be the perimeter of a disk or a rectangle, both centered at c. The complement of C consists of two connected components, the inside, and the outside. Since we already showed $N(C,c) = 1$ and $N(C,z)$ is constant on the inside, $N(C,z) = 1$ for z inside. Since $N(C,z) = 0$ is constant on the outside and $N(C,\infty) = 0$, $N(C,z) = 0$ for z outside. □

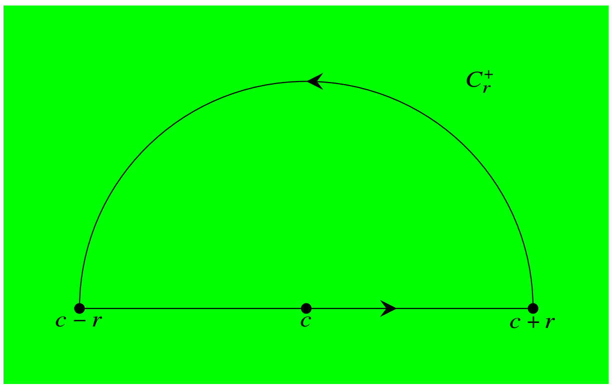

Fig. 5.2 The closed contour C_r^+

Let C be a closed contour and let z be a point not on C. Then the winding number $N(C,z)$ is an integer $0, \pm1, \pm2, \dots$. We say C **winds around** a if $N(C,z) \neq 0$, and C **does not wind around** a if $N(C,z) = 0$. With this terminology, we may say *no closed contour winds about* ∞.

Let $C^+(c,r)$ be the upper half-circle $z(\theta) = c + re^{i\theta}$, $0 \le \theta \le \pi$, and let C_r^+ be the closed connected contour in Figure 5.2,

$$C_r^+ = C^+(c,r) + [c-r, c+r].$$

We show $N(C_r^+, z) = 1$ *when* z *is inside* C_r^+*, and* $N(C_r^+, z) = 0$ *when* z *is outside* C_r^+.

Let $C^-(c,r)$ be the lower half-circle $z(\theta) = c + re^{i\theta}$, $-\pi \le \theta \le 0$, and let C_r^- be the closed connected contour

$$C_r^- = C^-(c,r) + [c+r, c-r] = C^-(c,r) - [c-r, c+r].$$

Then

$$C(c,r) = C^+(c,r) + C^-(c,r) = C_r^+ + C_r^-.$$

When z is outside $C_r^\pm$, we have $N(C_R^\pm, z) = 0$, because z can be connected to ∞ and $N(C_R^\pm, \infty) = 0$. Let z be inside C_r^+. Then $N(C_r^-, z) = 0$ because a is outside C_r^-, and

$$N(C_r^+, z) = N(C_r^+, z) + N(C_r^-, z) = N(C(c, r), z) = 1. \tag{5.9}$$

In short, when we combine C_r^+ and its reflection C_r^-, the contours along the diameters cancel, and we obtain $C(c, r)$.

5.3 Branches

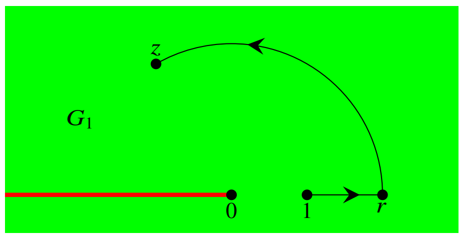

Fig. 5.3 The contour C in G_1

Given two complex numbers a and b, let $[a, b]$ denote the line segment contour $(1-t)a + tb$, $0 \le t \le 1$ (Figure 4.1). Using winding numbers, we will see which open sets G carry branches of the logarithm. When this happens, we say $\log z$ *is holomorphic on* G. For example, the principal logarithm (Theorem 4.12) is a branch of $\log z$ on G_1.

5.8. Theorem

Let G be an open set not containing 0. Then $\log z$ is holomorphic on G if and only if $N(C, 0) = 0$ for every closed contour C in G.

Proof By Theorem 4.11, $f(z)$ is a branch of the logarithm if and only if $f(z)$ is an anti-derivative of $1/z$ on G. By Theorem 5.3, an anti-derivative exists if and only if

$$2\pi i N(C, 0) = \int_C \frac{dz}{z} = 0.$$

For example, since no closed contour winds around the origin in G_1 (Figure 4.10), Theorem 5.8 implies there is a branch of $\log z$ on G_1. Also, since the unit circle winds about the origin, Theorem 5.8 implies there is no branch of $\log z$ on $\mathbf{C} - 0$. We now show how we can recover the principal logarithm on G_1 from Theorem 5.8.

Let $F(z)$ be *any* branch of $\log z$ on G_1, and fix a point $z = re^{i\theta}$, $-\pi < \theta < \pi$, $r > 0$, in G_1. Then $F(z) - F(1)$ equals the integral

$$\int_C \frac{dw}{w}$$

over any contour C starting at 1 and ending at z. If we use the contour C in Figure 5.3, we obtain

$$F(z) - F(1) = \int_C \frac{dw}{w} = \int_1^r \frac{dx}{x} + \int_0^\theta \frac{ire^{it}\,dt}{re^{it}} = \log r + \int_0^\theta i\,dt = \log r + i\theta,$$

so we recover the principal logarithm.

We now extend Theorem 5.8 to more general logs like $\log(z^2 - 1)$ or $\log(\sin z)$.

We say $f(z)$ is **continuously holomorphic** if $f(z)$ is holomorphic and $f'(z)$ is continuous. Theorem 6.4 shows a holomorphic function is continuously holomorphic, so this definition is superfluous. In other words, this section may be placed after Theorem 6.4 without affecting the logical development, in which case this definition would not be needed or mentioned. We present the material in this section at this point as a source of examples for the theory.

5.9. Theorem

Suppose $f(z)$ is continuously holomorphic and never zero on G and let C be a closed contour in G. Then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$$

equals the winding number $N(f(C), 0)$ of the image $f(C)$ about the origin.

Because $f(z)$ is continuously holomorphic and nonzero, $f'(z)/f(z)$ is continuous, so the integral is well-defined. Note we are not saying 0 is not in G. We are saying 0 is not in the image $f(G)$.

Proof Let C be $z(t)$, $t_1 \le t \le t_2$. Then $f(C)$ is $w(t) = f(z(t))$, $t_1 \le t \le t_2$. Since $w'(t) = f'(z(t))z'(t)$,

$$\int_C \frac{f'(z)}{f(z)}dz = \int_{t_1}^{t_2} \frac{f'(z(t))z'(t)}{f(z(t))}\,dt = \int_{t_1}^{t_2} \frac{w'(t)}{w(t)}\,dt = \int_{f(C)} \frac{dw}{w}.$$

The result follows. □

Let $f(z)$ be a holomorphic function on an open set G, and suppose $f(z)$ is never zero on G. A **branch of** $\log(f(z))$ is by definition a holomorphic function $g(z)$ satisfying

$$e^{g(z)} = f(z), \qquad z \text{ in } G.$$

If there is such a branch, we say $\log(f(z))$ *is holomorphic on* G.

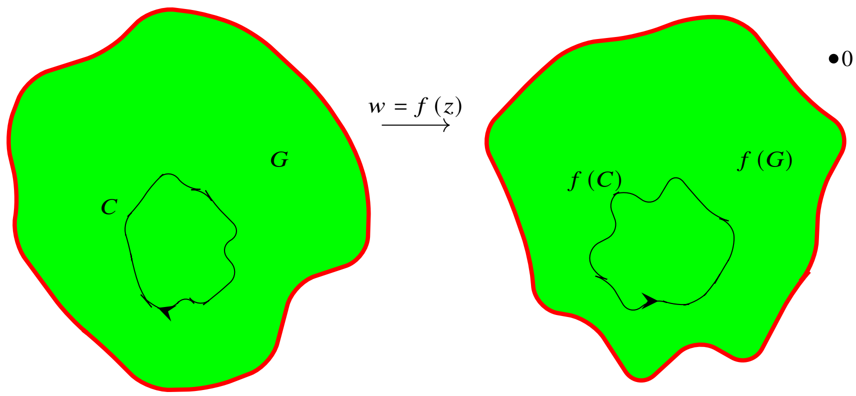

Fig. 5.4 C and its image $f(C)$ with $f(z) \neq 0$

5.10. Theorem

Let $f(z)$ be a continuously holomorphic function on an open set G, and suppose $f(z)$ is never zero on G. Then $\log(f(z))$ is holomorphic on G if and only if $N(f(C), 0) = 0$ for every closed contour C in G.

Proof If $g(z)$ plus a constant is a branch of $\log(f(z))$, then $e^{g(z)}/f(z)$ is a constant on G. Taking the derivative, this happens iff

$$0 = \frac{e^{g(z)}g'(z)f(z) - e^{g(z)}f'(z)}{f(z)^2},$$

which happens iff

$$g'(z) = \frac{f'(z)}{f(z)}$$

on G. Thus $\log(f(z))$ is holomorphic on G iff $f'(z)/f(z)$ has a holomorphic antiderivative on G. By Theorems 5.3 and 5.9, this happens iff $N(f(C), 0) = 0$ for every closed contour C in G. □

We now show $\log(z^2 - 1)$ is holomorphic on G_4 (Figure 5.5). To see this, note $f(z) = z^2 - 1$ is never zero on G_4, and

$$\frac{f'(z)}{f(z)} = \frac{2z}{z^2 - 1} = \frac{1}{z-1} + \frac{1}{z+1}.$$

If C is a closed contour in G_4, then integrating this last equation over C and recalling Theorem 5.9 yields

$$N(f(C), 0) = N(C, 1) + N(C, -1).$$

But C is in G_4 so C doesn't touch $(-\infty, -1]$. By continuity, $N(C, -1) = N(C, \infty) = 0$. Similarly, $N(C, 1) = 0$. Hence $N(f(C), 0) = 0$. By Theorem 5.10, $\log(z^2 - 1)$ is holomorphic on G_4.

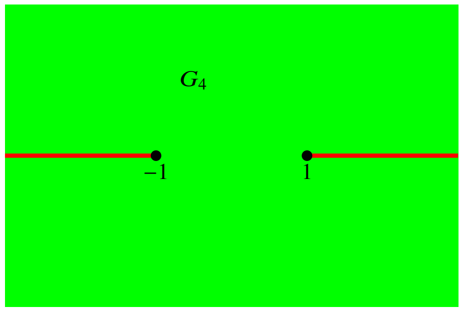

Fig. 5.5 $\log(z^2 - 1)$ is holomorphic on G_4

We now turn to square roots. To deal with them, we need to modify Theorem 5.3.

5.11. Theorem

Let G be an open set and let $f(z)$ be continuous on G. Then there is a nonzero holomorphic $g(z)$ on G satisfying

$$g'(z) = g(z)f(z)$$

if and only if

$$\exp\left(\int_C f(z)\,dz\right) \tag{5.10}$$

is path-independent in G.

If $f(z)$ has an anti-derivative $F(z)$ in G, then $g(z) = \exp(F(z))$ satisfies $g'(z) = g(z)f(z)$. This theorem covers cases where this is not so.

Proof Suppose a nonzero holomorphic $g(z)$ satisfying $g'(z) = g(z)f(z)$ exists on G. Let C and C' be connected contours in G having the same endpoints. Then

$$\begin{aligned}\exp\left(\int_C f(z)\,dz - \int_{C'} f(z)\,dz\right) &= \exp\left(\int_{C-C'} \frac{g'(z)}{g(z)}\,dz\right)\\ &= \exp\left(2\pi i N(g(C - C'), 0)\right).\end{aligned}$$

Since $C-C'$ is a closed contour, $g(C-C')$ is a closed contour, hence $N(g(C-C'), 0)$ is an integer. We conclude (5.10) is path-independent in G.

Conversely, suppose (5.10) is path-independent in G. We define $g(z)$ separately on each connected component G_1 of G. Fix a point c in G_1 and define

$$g(z) = \exp\left(\int_C f(w)\,dw\right), \tag{5.11}$$

over any connected contour C starting at c and ending at z in G_1. By path-independence, $g(z)$ is well-defined. Choose a contour C starting at c and ending at z. Let h be such that the contour $[z, z+h]$ lies in G_1. Then we may define $g(z+h)$ by integrating over the contour $C_h = C + [z, z+h]$. Since

$$g(z+th) = \exp\left(\int_{C_{th}} f(w)\,dw\right) = \exp\left(\int_C f(w)\,dw + h\int_0^t f(z+sh)\,ds\right),$$

we have[1]

$$\frac{g(z+th)}{g(z)} = \exp\left(h\int_0^t f(z+sh)\,ds\right).$$

Differentiating,

$$\frac{d}{dt}g(z+th) = h\,g(z+th)\,f(z+th).$$

Thus

$$g(z+h) - g(z) = h\int_0^1 g(z+th)\,f(z+th)\,dt.$$

By Theorem 4.5, $g'(z)$ exists and equals $g(z)f(z)$. □

Let $f(z)$ be a holomorphic function on an open set G, and suppose $f(z)$ is never zero on G. A **branch of** $\sqrt{f(z)}$ is by definition a holomorphic function $g(z)$ satisfying

$$g(z)^2 = f(z), \qquad z \text{ in } G.$$

When this happens, we say $\sqrt{f(z)}$ *is holomorphic on* G. For example, the principal square root (Theorem 4.13) is a branch of $\sqrt{z}$ on G_1.

5.12. Theorem

Let $f(z)$ be a continuously holomorphic function on an open set G, and suppose $f(z)$ is never zero on G. Then $\sqrt{f(z)}$ is holomorphic on G if and only if $N(f(C), 0)$ is even for every closed contour C in G.

Proof Suppose C and C' are connected contours in G having the same endpoints. Then $C - C'$ is a closed contour, hence

[1] Since $f(z)$ is bounded in a disk centered at z, this shows $g(z)$ is continuous at z, hence continuous on G.

$$\exp\left(\frac{1}{2}\int_C \frac{f'(z)}{f(z)}\,dz - \frac{1}{2}\int_{C'} \frac{f'(z)}{f(z)}\,dz\right) = \exp\left(\frac{1}{2}\int_{C-C'} \frac{f'(z)}{f(z)}\,dz\right)$$
$$= \exp\left(\pi i N(f(C-C'),0)\right).$$

By assumption, $N(f(C-C'),0)$ is even, so $\exp(\pi i N(f(C-C'),0)) = 1$, so

$$\exp\left(\frac{1}{2}\int_C \frac{f'(z)}{f(z)}\,dz\right)$$

is path-independent in G. By Theorem 5.11 with f replaced by $f'/2f$, there is a nonzero holomorphic function $g(z)$ on G satisfying

$$g'(z) = \frac{1}{2}g(z)\frac{f'(z)}{f(z)}. \tag{5.12}$$

Since

$$\left(\frac{g(z)^2}{f(z)}\right)' = \frac{2g(z)g'(z)f(z) - g(z)^2 f'(z)}{f(z)^2} = 0,$$

$g(z)$ is a branch of $\sqrt{cf(z)}$, for some constant c. It follows that $g(z)/\sqrt{c}$ is a branch of $\sqrt{f(z)}$.

Conversely, suppose $g(z)$ is holomorphic and satisfies $g(z)^2 = f(z)$. Then (5.12) holds, hence

$$N(f(C),0) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_C \frac{2g'(z)}{g(z)}\,dz = 2N(g(C),0).$$

Since $N(g(C),0)$ is an integer, $N(f(C),0)$ is even. □

The simplest case is $f(z) = z$. Then Theorem 5.12 says $\sqrt{z}$ is holomorphic on G iff $N(C,0)$ is even for every closed contour C in G. For example, since no closed contour winds around the origin in G_1 (Figure 4.10), Theorem 5.12 implies there is a branch of $\sqrt{z}$ on G_1. Also, since the unit circle winds about the origin, Theorem 5.12 implies there is no branch of $\sqrt{z}$ on $\mathbf{C} - 0$. We now show how we can recover the principal square root on G_1 from Theorem 5.12.

Let $g(z)$ be *any* branch of $\sqrt{z}$ on G_1, and fix a point z in G_1. If $z = re^{i\theta}$, $-\pi < \theta < \pi$, $r > 0$, is a point in G_1, then by (5.11),

$$\frac{g(z)}{g(1)} = \exp\left(\frac{1}{2}\int_C \frac{dw}{w}\right),$$

over any contour C starting at 1 and ending at w in G_1. If we use the contour C in Figure 5.3, we obtain

$$\frac{g(w)}{g(1)} = \exp\left(\frac{1}{2}(\log r + i\theta)\right) = \sqrt{r}e^{i\theta/2},$$

so we recover the principal square root.

The next simplest case is a quadratic with roots a and b,

$$\sqrt{f(z)} = \sqrt{(z-a)(z-b)}, \qquad a \neq b.$$

We show $\sqrt{(z-a)(z-b)}$ is holomorphic on $G_5 = \mathbf{C} - [a, b]$ (Figure 5.6).

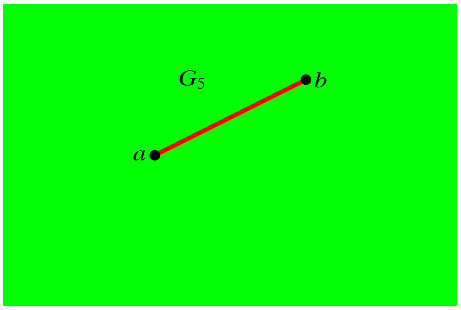

Fig. 5.6 $\sqrt{(z-a)(z-b)}$ is holomorphic on G_5

To see this, note $f(z)$ is never zero on G_5, and

$$\frac{f'(z)}{f(z)} = \frac{2z-(a+b)}{(z-a)(z-b)} = \frac{1}{z-a} + \frac{1}{z-b}.$$

If C is a closed contour in G_5, then integrating this last equation over C and recalling Theorem 5.9 yields

$$N(f(C), 0) = N(C, a) + N(C, b).$$

But C is in G_5 so C doesn't touch $[a, b]$. By continuity, $N(C, a) = N(C, b)$. Thus $N(f(C), 0)$ is even. By Theorem 5.12, $\sqrt{(z-a)(z-b)}$ is holomorphic on G_5.

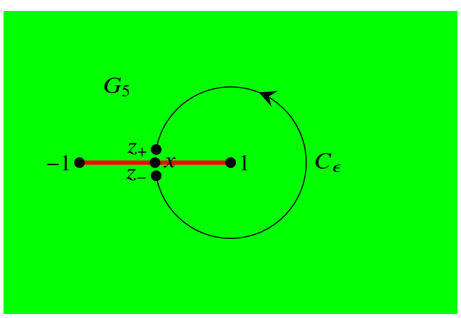

Fig. 5.7 Computing $\sqrt{z^2-1}$ on opposite sides of $[-1, 1]$

For example, consider $\sqrt{z^2-1}$ on G_5. Let x be in $(-1, 1)$, and look at circle $C(1, 1-x)$ with center 1 and radius $1-x$ as in Figure 5.7. Let ϵ be a small positive number, and let $z_\pm$ be on $C(1, 1-x)$ with $z_+ - z_- = 2i\epsilon$. By the same logic as for $\sqrt{z}$,

$$\frac{\sqrt{z_+^2-1}}{\sqrt{z_-^2-1}} = \exp\left(\frac{1}{2}\int_C \frac{2z\,dz}{z^2-1}\right) = \exp\left(\frac{1}{2}\int_C \frac{dz}{z-1} + \frac{1}{2}\int_C \frac{dz}{z+1}\right),$$

over any contour C starting at z_- and ending at z_+ in G_5. Let C_ϵ be the contour in Figure 5.7; if we use C_ϵ in the exponent, then $C_0 = C(1, 1-x)$, and we obtain

$$\lim_{\epsilon\to 0}\frac{\sqrt{z_+^2-1}}{\sqrt{z_-^2-1}} = \exp(\pi i N(C_0, 1) + \pi i N(C_0, -1)) = \exp(\pi i(1+0)) = -1.$$

Thus the values of $\sqrt{z^2-1}$ are opposites across the cut $[-1, 1]$.

The same logic shows $\sqrt{\sin \pi z}$ is holomorphic on G_6 (Figure 5.8). Let $f(z) = \sin \pi z$. Then $f(z) = 0$ if and only if z is an integer. Let C be any closed contour in $\mathbf{C}$ not passing through any integer. Since C is contained in $D(0, R)$ for R large enough, C winds only around finitely many integers.

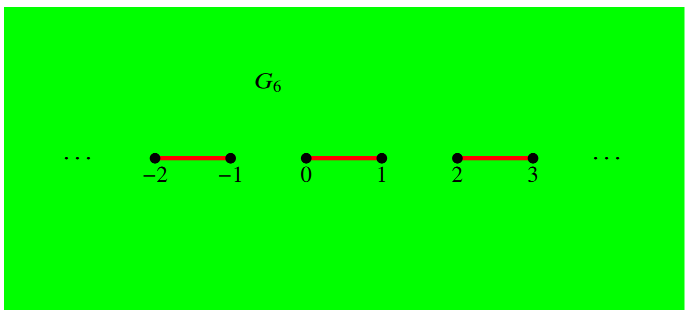

Fig. 5.8 $\sqrt{\sin(\pi z)}$ is holomorphic on G_6

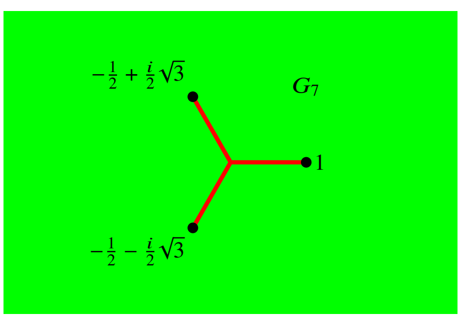

Fig. 5.9 $\sqrt[3]{z^3-1}$ is holomorphic on G_7

In §7.1, we show (see (7.4))

$$N(f(C),0) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_C \frac{\pi\cos\pi z}{\sin\pi z}\,dz = \sum_{n=-\infty}^{\infty} N(C,n). \qquad (5.13)$$

Now, if C is in G_6, then C does not touch any of the line segments $[2n, 2n+1]$, so by continuity, $N(C,2n) = N(C,2n+1)$, for every integer n. This shows the sum in (5.13) is even. By Theorem 5.12, $\sqrt{\sin(\pi z)}$ is holomorphic on G_6.

The same logic shows $\sqrt[3]{z^3-1}$ is holomorphic on G_7 (Figure 5.9).

Exercises

Chapter 6
Cauchy's Theorems

6.1 The Perimeter of a Rectangle

We start with the simplest version of Cauchy's theorem.

6.1. Cauchy's Theorem for the Perimeter of a Rectangle

Suppose $f(z)$ is holomorphic in an open set containing a rectangle R, and let C be the perimeter of R. Then

$$\int_C f(z)\,dz = 0.$$

Proof Let C be the perimeter of R and let

$$I = \int_C f(z)\,dz.$$

Our goal is to show $I = 0$.

Divide R into four sub-rectangles R', R'', R''', R'''', let C', C'', C''', C'''' be the perimeters of R', R'', R''', R'''', and let I', I'', I''', I'''' be the integrals over C', C'', C''', C'''' respectively. Since the inside paths are traversed twice, their contributions to the integrals cancel, and (Figure 6.1)

$$I = I' + I'' + I''' + I''''.$$

By the triangle inequality,

$$|I| \le |I'| + |I''| + |I'''| + |I''''|,$$

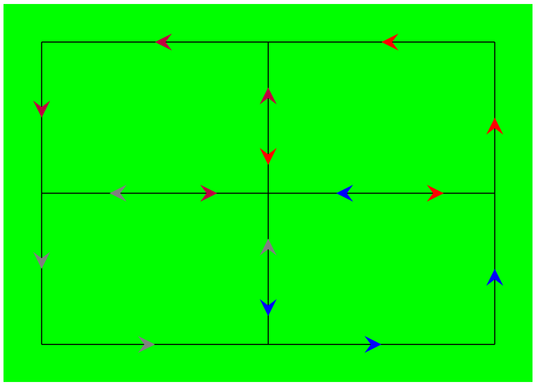

Fig. 6.1 A rectangle R divided into four rectangles

so at least one of the numbers $|I'|$, $|I''|$, $|I'''|$, $|I''''|$ is not less than $|I|/4$. In other words, for at least one of the four sub-rectangles, call it R_1 with perimeter C_1 and corresponding integral I_1, we have

$$|I_1| \geq \frac{1}{4}\,|I|\,.$$

Now divide the sub-rectangle R_1 into four sub-sub-rectangles, and repeat the same logic: For at least one sub-sub-rectangle, call it R_2 with perimeter C_2 and corresponding integral I_2, we have

$$|I_2| \geq \frac{1}{4}\,|I_1|\,, \qquad \text{so} \qquad |I_2| \geq \frac{1}{4^2}\,|I|\,.$$

Repeating this process indefinitely, we have rectangles $R \supset R_1 \supset R_2 \supset R_3 \supset \dots$, with perimeters $C_1, C_2, C_3, \dots$, and corresponding integrals

$$I_n = \int_{C_n} f(z)\,dz,$$

satisfying

$$|I_n| \geq \frac{1}{4^n}\,|I|\,, \qquad n \geq 1. \tag{6.1}$$

Let L and d be the lengths of the perimeter C and diagonal of R, and let L_n and d_n be the lengths of the perimeter C_n and diagonal of R_n. Then

$$L_n = 2^{-n}L \qquad \text{and} \qquad d_n = 2^{-n}d, \qquad n \geq 1. \tag{6.2}$$

Since the coordinates of the lower left corners of R_n are increasing with n, and the coordinates of the upper right corners of R_n are decreasing with n, it follows the rectangles R_n and their perimeters C_n converge to a specific point a in R, as $n \to \infty$.

Since 1 and z have holomorphic anti-derivatives z and $z^2/2$, the contour integrals of 1 and z over the closed contours C_n are zero, hence

$$I_n = \int_{C_n} f(z)\,dz = \int_{C_n} [f(z) - f(a) - f'(a)(z-a)]\,dz, \qquad n \geq 1.$$

Let ϵ_n be the maximum value of the error

$$\left|\frac{f(z)-f(a)}{z-a} - f'(a)\right| \tag{6.3}$$

as z varies over the perimeter C_n. Since C_n converges to a, by definition of the derivative $f'(a)$, the error converges to zero,

$$\epsilon_n \to 0 \qquad \text{as} \qquad n \to \infty.$$

Since (6.3) implies

$$|f(z) - f(a) - f'(a)(z-a)| \leq \epsilon_n |z-a|$$

for z on C_n, we have by the triangle inequality for integrals (Theorem 5.4),

$$|I_n| \leq \int_{C_n} |f(z) - f(a) - f'(a)(z-a)|\,|dz| \leq \epsilon_n \int_{C_n} |z-a|\,|dz|.$$

But $|z-a| \leq d_n$ on C_n, and the length of C_n is L_n, so from (6.2) and Theorem 5.4,

$$|I_n| \leq \epsilon_n L_n d_n = 4^{-n}\epsilon_n L d.$$

From (6.1), we get

$$4^{-n}|I| \leq |I_n| \leq 4^{-n}\epsilon_n L d$$

which implies

$$|I| \leq \epsilon_n L d.$$

Since this is true for all $n \geq 1$ and $\epsilon_n \to 0$, we conclude $I = 0$. □

If R is a rectangle and a is a point inside R, then R with a removed is a **punctured rectangle**. More generally, a punctured rectangle is a rectangle with several puncture points removed.

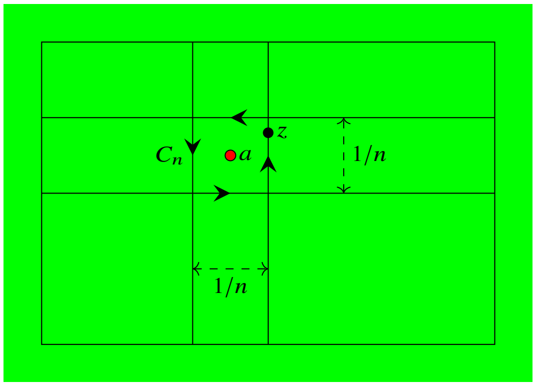

Fig. 6.2 A punctured rectangle R' divided into nine rectangles

6.2. Cauchy's Theorem for the Perimeter of a Punctured Rectangle

Let R' be a rectangle punctured by finitely many points a. Suppose $f(z)$ is holomorphic in R', and suppose

$$\lim_{z \to a} (z - a) f(z) = 0 \tag{6.4}$$

at each puncture point a. If C is the perimeter of R', then

$$\int_C f(z)\, dz = 0.$$

Proof First, by dividing R' into sub-rectangles, we may assume there is only one puncture a. Next, divide R' into nine sub-rectangles, with a at the center of one of them, as in Figure 6.2, in such a way that the sub-rectangle R_n centered at a is a square with side length $1/n$. If the perimeter of R_n is C_n, then we can apply Theorem 6.1 to every sub-rectangle but R_n, obtaining

$$I = \int_C f(z)\, dz = \int_{C_n} f(z)\, dz.$$

If we let ϵ_n be the maximum value of $|z-a|\,|f(z)|$ on C_n, then by assumption $\epsilon_n \to 0$ as $n \to \infty$. Then by the triangle inequality for integrals (Theorem 5.4),

$$\left|\int_{C_n} f(z)\, dz\right| = \left|\int_{C_n} (z-a) f(z) \frac{dz}{z-a}\right| \le \epsilon_n \int_{C_n} \frac{|dz|}{|z-a|}.$$

If z is in C_n, then $|z - a| \ge 1/2n$. Since the length of C_n is $4/n$,

$$\int_{C_n} \frac{|dz|}{|z-a|} \le 8,$$

to conclude

$$|I| = \left| \int_{C_n} f(z)\, dz \right| \le 8\epsilon_n.$$

Since $\epsilon_n \longrightarrow 0$, this shows $I = 0$. □

Now we derive

6.3. Cauchy's Integral Formula for the Perimeter of a Rectangle

Suppose $f(z)$ is holomorphic in an open set containing a rectangle R, and let C be the perimeter of R. If z is inside R, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw. \tag{6.5}$$

Notice what Cauchy's integral formula says: If you know $f(z)$ along the perimeter, then you know $f(a)$ at any point a inside!

Proof Fix z inside R and define

$$g(w) = \begin{cases} \dfrac{f(w)-f(z)}{w-z}, & w \neq z, \\ f'(z), & w = z. \end{cases}$$

Then $g(w)$ is holomorphic at all points of R, except possibly at z, where it satisfies

$$\lim_{w \to z} (w-z) g(w) = 0.$$

Applying Theorem 6.2 to $g(w)$, we obtain

$$\int_C \frac{f(w)-f(z)}{w-z}\, dw = 0.$$

But a rectangle winds once around its center, so $N(C, z) = 1$. Rewriting this, using $N(C, z) = 1$, yields (6.5). □

As a consequence,

6.4. Holomorphic Functions are Infinitely Differentiable

Suppose $f(z)$ is holomorphic in an open set G. Then all derivatives $f^{(n)}(z)$, $n \geq 2$, exist at all points of G.

Proof Fix a point c in G and let R be a small rectangle centered at c completely contained in G. Then by (6.5) and Theorem 5.5, we may differentiate under the integral arbitrarily many times. Thus $f^{(n)}(z)$ exist at all points z inside R, for all $n \geq 2$. Since c was any point in G, the result follows. □

6.2 The Disk and the Rectangle

Now we show what was true for the perimeter of a rectangle is true for any closed contour in a rectangle, and in a disk.

6.5. Cauchy's Theorem in a Disk and in a Rectangle

Suppose $f(z)$ is holomorphic in an open disk G or in an open rectangle G. Then

$$\int_C f(z)\,dz = 0,$$

for every closed contour C in G.

Proof The proof is similar to that of Theorem 5.3, except here we can only use contour segments $[a, z]$ that are vertical or horizontal. Because of this, here we must appeal to the CR equation.

Suppose c is the center of G. Given z in G, let

$$F(z) = \int_C f(w)\,dw,$$

where C is the contour $[c, b] + [b, z]$ (Figure 6.3). Then, for h real, $F(z + h)$ is the contour integral along $[c, b] + [b, z + h]$. Thus $F(z + h) - F(z)$ is the contour integral of $f(w)$ over $[z, z + h]$,

$$F(z + h) - F(z) = h \cdot \int_0^1 f(z + th)\,dt. \tag{6.6}$$

Since h is real, by Theorem 4.5, $\partial f/\partial x$ exists at z and equals $f(z)$.

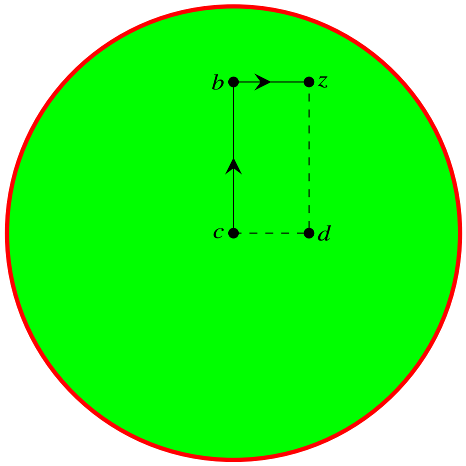

Fig. 6.3 Proof of Cauchy's theorem in a disk

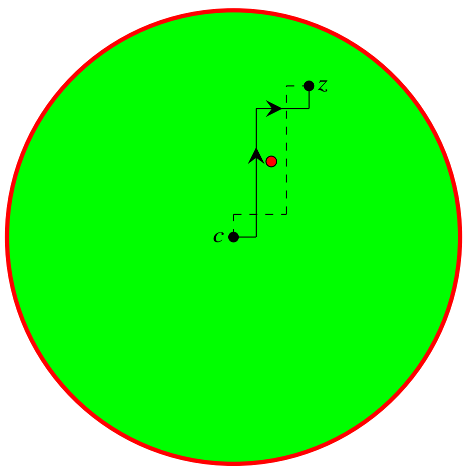

Fig. 6.4 Proof of Cauchy's theorem in a punctured disk

By Theorem 6.1, we may replace the contour in the definition of $F(z)$ by $[c,d]+[d,a]$ (Figure 6.3). Then, for h imaginary, $F(z+h)$ is the contour integral along $[c,d]+[d,z+h]$. Thus $F(z+h)-F(z)$ is the integral over $[z,z+h]$, and (6.6) holds. Since h is imaginary, by Theorem 4.5, $(1/i)\partial f/\partial y$ exists at z and equals $f(z)$. By Theorem 4.10, $F'(z)$ exists and equals $f(z)$. By Theorem 5.2, the integral of $f(z)$ over any closed contour is zero. □

If D is a disk and a is a point inside D, then D with the point a removed is a **punctured disk**. More generally, a punctured disk is a disk with several puncture points removed.

6.6. Cauchy's Theorem in a Punctured Disk and in a Punctured Rectangle

Let G' be an open disk or open rectangle punctured by finitely many points a. Suppose $f(z)$ is holomorphic in G', and suppose

$$\lim_{z\to a}(z-a)f(z)=0, \tag{6.7}$$

at each puncture point a. Then

$$\int_C f(z)\,dz=0,$$

for every closed contour C in G'.

Proof Suppose first the center c is not a puncture point. This time let

$$F(z)=\int_C f(w)\,dw,$$

where C is either of the four-segment contours shown in Figure 6.4. By Theorem 6.2, $F(z)$ does not depend on the choice of the middle segments, as long as the contours do not pass through any of the puncture points. By Theorem 6.2, we may use either of the two four-segment contours to define $F(z)$.

If the center c is a puncture point, then start instead at a non-puncture point c' near c, and draw the same contours. The rest of the proof is as in Theorem 6.5. □

Now we derive

6.7. Cauchy's Integral Formula in a Disk and in a Rectangle

Suppose $f(z)$ is holomorphic in an open disk G or open rectangle G, and let z be a point in G. If C is a closed contour in G not passing through z, and $n=N(C,z)$,

$$n\cdot f(z)=\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw. \tag{6.8}$$

Proof Define

$$g(w)=\begin{cases}\dfrac{f(w)-f(z)}{w-z}, & w\neq z,\\ f'(z), & w=z.\end{cases}$$

Then $g(w)$ is holomorphic at all points of G, except possibly at z, where it satisfies

$$\lim_{w \to z} (w - z) g(w) = 0.$$

Applying Theorem 6.6 to $g(w)$, we obtain

$$\int_C \frac{f(w) - f(z)}{w - z} \, dw = 0.$$

Rewriting this, using $n = N(C, z)$, yields (6.8). □

6.8. Cauchy's Integral Formula in a Punctured Disk and in a Punctured Rectangle

Let G' be an open disk or open rectangle punctured by finitely many points a. Suppose $f(z)$ is holomorphic in G', and suppose

$$\lim_{z \to a} (z - a) f(z) = 0,$$

at each puncture point a. Let z be a point in G', let C be a closed contour in G' not passing through z, and let $n = N(C, z)$. Then

$$n \cdot f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} \, dw. \tag{6.9}$$

The proof is identical to the previous version, except we use Cauchy's theorem in a punctured disk and in a punctured rectangle. We now apply Cauchy's integral formula to the case when C is a circle.

6.9. Mean Value Property

Let $f(z)$ be holomorphic on the open disk $|z - c| < R$. Then $f(c)$ is the average of $f(z)$ over $C(c, r)$ for any $r < R$,

$$f(c) = \frac{1}{2\pi} \int_0^{2\pi} f(c + re^{i\theta}) \, d\theta.$$

Proof Apply Cauchy's integral formula with $C = C(c, r)$. Then $n = N(C, c) = 1$. On C, $z = c + re^{i\theta}$, $0 \le \theta \le 2\pi$. Inserting this into (6.8) yields the result. □

Let $f(z)$ be holomorphic on a disk D centered at c, and let $C = C(c, r)$ lie in D. If $|z - c| < r$, then z is inside C, so $N(C, z) = 1$. By (6.8),

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} \, dw, \qquad |z - c| < r.$$

By Theorem 5.5,

6.10. Cauchy's Integral Formula for Derivatives

Let $f(z)$ be holomorphic on a disk D. If $r > 0$ is such that $C(c, r)$ lies in D, the derivatives are given by

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C(c,r)} \frac{f(w)}{(w-z)^{n+1}}\, dw, \tag{6.10}$$

for $|z - c| < r$ and $n \geq 0$.

This implies

6.11. Cauchy's Estimates

Let $f(z)$ be holomorphic on the disk $D(c, R)$ and let $r < R$. If $|f(z)| \leq M$ on $C(c, r)$, then

$$|f^{(n)}(c)| \leq \frac{n!M}{r^n}, \qquad n \geq 0.$$

Proof Using Theorem 6.10 with $a = c$, and the triangle inequality for integrals (Theorem 5.4),

$$\begin{aligned} |f^{(n)}(c)| &= \left| \frac{n!}{2\pi i} \int_{C(c,r)} \frac{f(z)}{(z-c)^{n+1}}\, dz \right| \leq \frac{n!}{2\pi} \int_{C(c,r)} \frac{|f(z)|}{|z-c|^{n+1}}\, |dz| \\ &= \frac{n!}{2\pi} \int_{C(c,r)} \frac{|f(z)|}{r^{n+1}}\, |dz| \leq \frac{n!}{2\pi r^{n+1}} \cdot 2\pi r \cdot M = \frac{n!M}{r^n}. \end{aligned}$$

As a corollary, we have

6.12. Liouville's theorem

A bounded entire function is constant.

Proof Assume $|f(z)| \leq M$ on all of $\mathbf{C}$. By Cauchy's estimates,

$$|f'(c)| \leq \frac{M}{r}$$

for all $r > 0$. Letting $r \to \infty$ yields $f'(c) = 0$. Since c is any point, we conclude $f'(z) = 0$ on all of $\mathbf{C}$, so $f(z)$ is constant. □

Liouville's theorem leads to a proof of the

6.13. Fundamental Theorem of Algebra

If $f(z)$ is a nonconstant polynomial, then $f(a) = 0$ for some complex number a. Consequently, a degree n polynomial has n complex roots.

Proof Suppose not. Then $1/f(z)$ is analytic on all of **C**, hence entire. Since

$$f(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n = z^n \left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n}{z^n}\right),$$

we have $f(z) \to \infty$ as $z \to \infty$. Hence $1/f(z)$ is a bounded entire function, hence constant, which is a contradiction. □

From §4.4, a power series is a holomorphic function, on any disk of convergence. We now prove the converse.

Suppose $f(z)$ is holomorphic on an open disk D centered at c. By Theorem 6.4, we know all derivatives $f^{(n)}(z)$ exist in D. Hence it is meaningful to write the Taylor series

$$f(c) + f'(c)(z-c) + \frac{1}{2!} f''(c)(z-c)^2 + \frac{1}{3!} f'''(c)(z-c)^3 + \ldots$$

at c. It is natural to ask where is this series convergent, and where does it sum to $f(z)$? We show that this is so at every z in D.

Choose $r > 0$ such that the circle $C = C(c, r)$ lies in D, and let

$$a_n = \frac{1}{n!} f^{(n)}(c) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-c)^{n+1}}\, dw.$$

Given $|z - c| < r$, let δ be the minimum distance of z to C. Then

$$|w - z| \geq \delta, \qquad w \text{ in } C.$$

Let M be the maximum value of $|f(z)|$ on C.

Now write

$$\frac{1}{w-z} = \frac{1}{(w-c)-(z-c)} = \frac{1}{w-c} \cdot \frac{1}{1-(z-c)/(w-c)}.$$

If $t_n(z)$ is the n-th tail of the geometric series, then replacing z by $(z-c)/(w-c)$ in (3.7), we obtain

$$\frac{1}{w-z} = \frac{1}{w-c} \left(\sum_{k=0}^{n} \left(\frac{z-c}{w-c}\right)^k + \frac{w-c}{w-z} \left(\frac{z-c}{w-c}\right)^{n+1} \right),$$

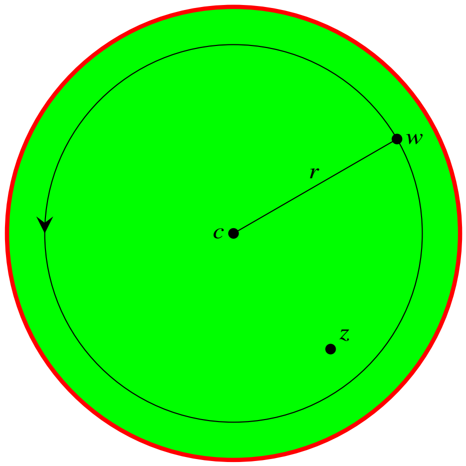

Fig. 6.5 A holomorphic function may be expanded into powers of $(z-c)$ at c

so

$$\frac{1}{w-z} = \sum_{k=0}^{n} \frac{(z-c)^k}{(w-c)^{k+1}} + \frac{1}{w-z}\left(\frac{z-c}{w-c}\right)^{n+1}. \tag{6.11}$$

Multiply both sides of (6.11) by $f(w)$ and integrate over w in C. By (6.10), we obtain

$$f(z) = \sum_{n=0}^{n} a_k(z-c)^k + \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\left(\frac{z-c}{w-c}\right)^{n+1} dw.$$

Since $|w-c| = r$, by the triangle inequality for integrals,

$$\left| f(z) - \sum_{n=0}^{n} a_k(z-c)^k \right| \le \frac{Mr}{\delta}\left(\frac{|z-c|}{r}\right)^{n+1}.$$

Since $|z-c|/r < 1$, this goes to zero as $n \to \infty$. Thus

$$f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$$

for z in $D(c,r)$. Since $r > 0$ is any radius for which $C(c,r)$ lies in D, we established

6.14. Holomorphic Functions are Power Series

If $f(z)$ is holomorphic on a disk D centered at c, then $f(z)$ equals its Taylor series,

$$f(z) = f(c) + f'(c)(z-c) + \frac{1}{2!}f''(c)(z-c)^2 + \frac{1}{3!}f'''(c)(z-c)^3 + \dots$$

for z in D.

From this, we see *the disk of convergence of the Taylor series of a holomorphic function extends to the nearest singularity.* For example, the function

$$\tau(z) = \frac{z}{e^z - 1}$$

is holomorphic at the origin, since (Theorem 6.17)

$$\lim_{z\to 0} \frac{z}{e^z - 1} = 1.$$

Since $e^{2\pi i} = 1$, $2\pi i$ is the singularity of $\tau(z)$ nearest to the origin. Hence $\tau(z)$ may be expanded into a complex power series in $D(0, 2\pi)$,

$$\tau(z) = 1 + B_1 z + B_2 \frac{z^2}{2!} + B_3 \frac{z^3}{3!} + \dots, \qquad |z| < 2\pi.$$

This is the **Bernoulli series** and the coefficients B_n are the **Bernoulli numbers**.

Also, from the above derivation, we obtain an explicit formula for N-th tail of the complex Taylor series,

$$t_n(z) = \left(\frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)(w-c)^{n+1}}\,dw\right)(z-c)^{n+1}.$$

Now we use Theorem 6.14 to derive a converse of Cauchy's theorem.

6.15. Morera's Theorem

Suppose $f(z)$ is a continuous function on an open set G, and suppose

$$\int_C f(z)\,dz = 0$$

for *every* closed contour C in G. Then $f(z)$ is holomorphic in G.

Proof By Theorem 5.3, $f(z)$ has a holomorphic anti-derivative $F(z)$. By Theorem 6.14, $F'(z) = f(z)$ is holomorphic. □

6.3 Zeros and Poles

We start with zeros of a holomorphic function, then study poles of a holomorphic function.

A **zero** of a holomorphic function is a point c with $f(c) = 0$. A zero c is **isolated** if there is a disk D centered at c with $f(z) \neq 0$ on $D - c$.

6.16. Theorem

Let $f(z)$ be holomorphic on an open set G and let c be a zero of $f(z)$. Then there are two possibilities. Either (1) c is an isolated zero, and there is a unique $n \geq 1$ such that

$$\frac{f(z)}{(z-c)^n}$$

is nonzero and holomorphic at a, or (2) $f(z) = 0$ on the connected component G_1 of G containing c.

Proof This follows from Theorem 6.14. We have two possibilities. Either all derivatives $f^{(n)}(c) = 0$, $n \geq 0$, or there is at least one derivative that is not zero at c, $f^{(n)}(c) \neq 0$.

Let G_1' be the set of points a in G_1 such that all derivatives are zero, $f^{(n)}(a) = 0$, $n \geq 0$. By Theorem 6.14, if a is in G_1', then $f^{(n)}(z) = 0$, $n \geq 0$, for all z in a disk about a. Hence G_1' is open. On the other hand, if a is not in G_1', then, for some n, $f^{(n)}(a) \neq 0$, which implies $f^{(n)}(z) \neq 0$ for z near a, so $G_1'' = G_1 - G_1'$ is also open. But G_1 is connected, so cannot be written as a disjoint union of two open sets $G_1' \cup G_1''$ (Theorem 4.2). Hence one of G_1' or G_1'' is empty.

If G_1' is empty, for some n, $f^{(n)}(c) \neq 0$. If we let N be the least such n, then

$$\begin{aligned} f(z) &= \frac{1}{N!} f^{(N)}(c)(z-c)^N + \frac{1}{(N+1)!} f^{(N+1)}(c)(z-c)^{N+1} + \dots \\ &= (z-c)^N \left(\frac{1}{N!} f^{(N)}(c) + \frac{1}{(N+1)!} f^{(N+1)}(c)(z-c) + \dots \right), \end{aligned}$$

which implies $f(z)/(z-c)^N$ is nonzero and holomorphic for z near c, so c is an isolated zero. If G_1'' is empty, then $f(z) = 0$ on G_1. □

This theorem shows that, near an isolated zero c, a holomorphic function looks like $(z-c)^n$ for a unique $n \geq 1$. This integer $n \geq 1$ is the **order** of the zero c. When $n = 1$, the zero c is **simple**.

Now we look at points where $f(z)$ is not defined. An **isolated singularity** is[1] a point c such that $f(z)$ is defined and holomorphic in a punctured disk $D - c$ centered about c. For example, 0 is an isolated singularity for

$$f(z) = \frac{1}{z^3}.$$

At an isolated singularity, there are three possibilities:

- The point c is a **removable singularity** if

$$\lim_{z \to c} (z - c) f(z) = 0.$$

- The point c is a **pole** if $\lim_{z \to c} |f(z)| = \infty$.
- The point c is an **essential singularity** if otherwise.

Note the condition for a removable singularity is exactly what we had for puncture points in Cauchy's theorem and Cauchy's integral formula.

For example, 0 is a pole for $f(z) = 1/z^3$, and 0 is an essential singularity for $f(z) = e^{1/z}$. The following theorem shows removable singularities are no cause for concern.

6.17. Theorem

Let c be a removable singularity for $f(z)$ in the disk $D = D(c, R)$. Then $f(z)$ is well-defined and holomorphic at c.

Proof By assumption, c is a puncture point for $f(z)$, so Cauchy's integral formula in a punctured disk (Theorem 6.8) applies. Hence

$$f(a) = \frac{1}{2\pi i} \int_{C(c,r)} \frac{f(z)}{z - a} \, dz, \qquad a \neq c.$$

But the right side is a holomorphic function also at $a = c$ (Theorem 5.5), hence so is $f(z)$. □

Now we examine non-removable singularities. Note if $f(z)$ is bounded near an isolated singularity c, then c is a removable singularity. Hence, near a non-removable singularity, a holomorphic function is unbounded. Then there are two possibilities: either c is a pole, or c is an essential singularity.

[1] According to this definition, $f(z)$ could be holomorphic at c.

6.18. Theorem

Let c be a pole of a holomorphic function $f(z)$ in a disk $D = D(c, R)$. Then there is a unique $n \geq 1$ such that

$$(z - c)^n f(z)$$

is a nonzero and holomorphic at c.

Proof If c is a pole of $f(z)$, then c is a zero of $1/f(z)$, so this follows from Theorem 6.16.

The integer n is the **order** of the pole c. When $n = 1$, the pole c is **simple**.
If we write the Taylor series of $(z - c)^n f(z)$ at c,

$$(z - c)^n f(z) = a_0 + a_1(z - c) + a_2(z - c)^2 + \ldots,$$

we obtain the **Laurent series** of $f(z)$ at a,

$$f(z) = \frac{1}{(z - c)^n}\left(a_0 + a_1(z - c) + a_2(z - c)^2 + \ldots\right).$$

For example, for the function

$$f(z) = \frac{e^z}{z^n}, \qquad n = 0 \pm 1, \pm 2, \ldots$$

the origin $c = 0$ is

- a pole of order n when $n \geq 1$,
- a zero of order $-n$ when $n \leq -1$, and
- neither when $n = 0$.

In this case the Laurent series at 0 is

$$\frac{e^z}{z^n} = \frac{1}{z^n} + \frac{1}{z^{n-1}} + \frac{1}{2!z^{n-2}} + \frac{1}{3!z^{n-3}} + \ldots$$

6.4 The General Theorems

Now we go from a disk to a general open set. Let G be an open set, let C be a closed contour in G, and let a be a point not in G.

In previous sections, we measured how C wound around the point a by integrating $1/(z - a)$ over C, obtaining the winding number $N(C, a)$.

We also studied showed how the vanishing of the winding number over all closed contours in G related to the existence of a branch of $\log z$ on G.

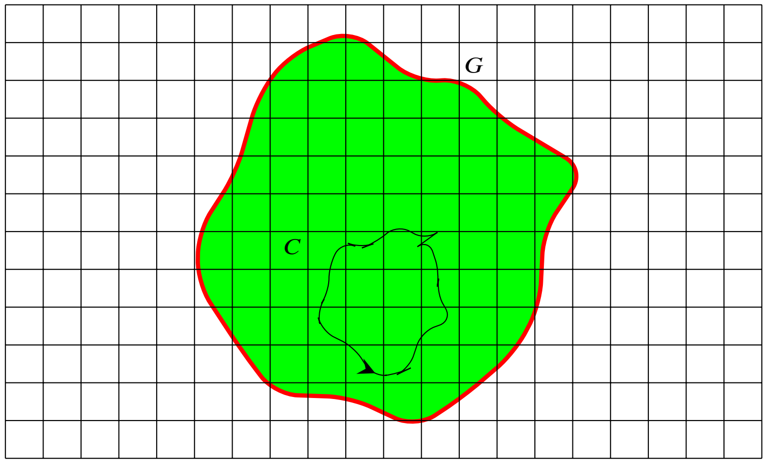

Fig. 6.6 An open set G containing a closed contour C

Now we consider a more general question. Fix a closed contour C in G. What conditions should we require on C so that

$$\int_C f(x)\,dz = 0$$

for every holomorphic $f(z)$ on G?

Since $1/(z-a)$ is holomorphic on G whenever a is not in G, a necessary condition is the vanishing of the winding number

$$N(C,a) = \frac{1}{2\pi i}\int_C \frac{dz}{z-a} = 0$$

for all a not in G. The following asserts this condition is sufficient.

6.19. Cauchy's Theorem

Let C be a closed contour lying in an open set G. If C does not wind around any point not in G, then

$$\int_C f(z)\,dz = 0,$$

for every $f(z)$ holomorphic on G.

To rephrase, Cauchy's theorem is saying: If

$$\int_C \frac{dz}{z-a} = 0$$

for every point a not in G, then

$$\int_C f(z)\,dz = 0$$

for every holomorphic function $f(z)$.

Proof Draw an infinite grid over the whole complex plane of *closed* squares Q of edge-lengths $\delta > 0$. This divides **C** into closed squares. For each square Q, let ∂Q denote its perimeter, taken as a closed contour in the counter-clockwise direction. Now let C_δ be the sum of the contours ∂Q, over all squares Q contained in G,

$$C_\delta = \sum_{Q\subset G} \partial Q.$$

If δ is small enough, there is at least one Q completely contained in G. Let G_δ be the interior of the union of these squares Q,

$$G_\delta = \text{interior}\left(\bigcup_{Q\subset G} Q\right).$$

Then G_δ is open and has no point in common with C_δ; in fact, C_δ is the boundary of G_δ. Choose δ small enough so C is contained in G_δ.

Now pick a point a in the inside of one of the squares Q_0 completely contained in G. Then

$$\frac{1}{2\pi i}\int_{\partial Q} \frac{f(z)}{z-a}\,dz = \begin{cases} f(a) & Q = Q_0 \\ 0 & Q \neq Q_0. \end{cases}$$

It follows

$$f(a) = \frac{1}{2\pi i}\int_{C_\delta} \frac{f(z)}{z-a}\,dz$$

for all a in the insides of the squares Q completely contained in G. Since both sides are continuous functions of a on G_δ, this holds for all a in G_δ. *Needs to be completed after figure is drawn* □

We say an open set G is **simply connected** if *every* closed contour in G does not wind about *any* point not in G. *For a simply connected G, Cauchy's theorem holds for any closed contour C in G.*

If $G = D(c,R) - D(c,r)$, $r < R$, is the region between two concentric circles, then G is not simply connected, because any circle $C(c,t)$, $r < t < R$, winds about the center c, which is not in G.

Every holomorphic $f(z)$ on a simply connected G necessarily has a holomorphic anti-derivative. In particular, if $f(z) \neq 0$ on such a G, then $f'(z)/f(z)$ has an anti-derivative $F(z)$. By Theorem 5.10, we have the following.

6.20. Theorem

Let G be simply connected and let $f(z) \neq 0$ be holomorphic on G. Then $\log(f(z))$ is holomorphic on G, and $\sqrt{f(z)}$ is holomorphic on G.

We say a set G is **star-shaped** if there is a point a in G such for any z in G, the line segment $[a, z]$ lies in G.

6.21. Theorem

A star-shaped open set G is simply connected.

Proof Let a in G be a point such that $[a, z]$ lies in G for every z in G. Let b be a point not in G, and let $[b, \infty)$ be the segment of the ray emanating from a and passing through b. Then $[b, \infty)$ lies entirely outside G. Let C be a closed contour in G. Since $N(C, \infty) = 0$, we have $N(C, z) = 0$ for every z on the segment $[b, \infty)$, hence $N(C, b) = 0$. □

We say a set G is **convex** if the line segment $[a, b]$ lies in G for any two points a and b in G. Since a convex set is star-shaped,

6.22. Theorem

A convex open set is simply connected.

The open set G_2 (Figure 4.9) is convex, hence simply connected. The holomorphic function $w = z^2$ maps G_2 to G_1, and has a holomorphic inverse $z = \sqrt{w}$. We show G_1 is also simply connected.

Let C be a closed contour in G_1, and let $f(w)$ be a holomorphic function on G_1. Then by Theorem 3.6,

$$\int_C f(w)\,dw = \int_{\sqrt{C}} f(z^2)\,2z\,dz = 0,$$

since $\sqrt{C}$ is a closed contour in G_2. Inserting $f(w) = 1/(w - a)$ with a not in G_1, we conclude C does not wind about any point not in G_1, or G_1 is simply connected. This argument is valid more broadly.

A **biholomorphism** is an injective holomorphic function $f(z)$ whose inverse is holomorphic. Open sets G_1 and G_2 are **biholomorphic** if there is a biholomorphism $f(z)$ with domain G_1 and image $G_2 = f(G_1)$. Then we have

Simple Connectedness is a Biholomophic Invariant

A biholomorphic image of a simply connected open set is simply connected.

Now we state the general

6.23. Cauchy's Integral Formula

Let C be a closed contour lying in an open set G and let a be a point in G not on C. If C does not wind around any point not in G and $n = N(C, a)$, then

$$n \cdot f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

for every $f(z)$ holomorphic on G.

The simplest case is $f(z) \equiv 1$, in which case the integral formula becomes the definition of winding number $N(C, a)$.

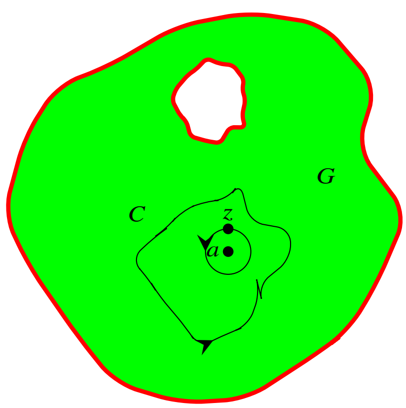

Fig. 6.7 Proof of Cauchy's integral formula

Proof The idea (Figure 6.7) behind the proof is to replace C by a small circle $C(a, r)$ about a, and then use Cauchy's integral formula on $D(a, r)$. To start, since G is open, there is a small disk $D(a, r)$ contained in G. With $n = N(C, a)$, let

$$C' = C - n \cdot C(a, r),$$

where $n \cdot C(a, r)$ is $C(a, r)$ traversed n times.

We want to apply Cauchy's theorem to the contour C' and the punctured open set $G' = G - a$, so we need to check that C' does not wind around any point not in G'.

If a point is not in G', it is either a or it is a point b not in G. In the first case, we know from (5.8) that

$$N(C', a) = N(C - n \cdot C(a, r), a) = N(C, a) - N(n \cdot C(a, r), a) = n - n = 0,$$

while in the second case, b is not in $D(a, r)$ and not in G, so from (5.8) again

$$N(C', b) = N(C, b) - n \cdot N(C(a, r), b) = 0 - 0 = 0.$$

By Cauchy's theorem applied to the holomorphic function $f(z)/(z - a)$ on G',

$$\int_{C'} \frac{f(z)}{z - a}\, dz = 0.$$

This implies

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z - a}\, dz = \frac{1}{2\pi i} \int_{n \cdot C(a,r)} \frac{f(z)}{z - a}\, dz = n \cdot \frac{1}{2\pi i} \int_{C(a,r)} \frac{f(z)}{z - a}\, dz,$$

which, by Cauchy's integral formula for a disk, equals $n \cdot f(a)$. □

When G is simply connected, Cauchy's theorem holds for any closed contour in G, so *when G is simply connected, Cauchy's integral formula holds for any closed contour in G*.

Exercises

Chapter 7
The Residue Theorem

7.1 Residue Theorem

In this section we rewrite Cauchy's integral formula in a manner useful for explicitly computing integrals.

A function that is holomorphic on an open set G except for poles is **meromorphic** on G. The simplest meromorphic function is a rational function $p(z)/q(z)$, the ratio of polynomial functions $p(z)$ and $q(z)$.

Since $\sin(\pi z) = 0$ at the integers $z = n$,

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$$

has poles at the integers. Since $(\sin(\pi z))' = \pi \cos(\pi z)$ is nonzero at the integers, these poles are simple. In particular, $\cot(\pi z)$ is meromorphic on $\mathbf{C}$.

More generally, if $p(z)$ and $q(z)$ are holomorphic functions on an open set G, then the ratio

$$f(z) = \frac{p(z)}{q(z)}$$

is not defined at the zeros of $q(z)$. By Theorems 6.16 and 6.18, these points must be poles of $f(z)$. Thus the ratio of holomorphic functions is meromorphic, provided the denominator is not identically zero.

Care must be exercised in computing the order of poles. For example, although the origin 0 is a simple zero for both $\sin z$ and z, 0 is not a pole for the meromorphic function $\sin z/z$, since

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots,$$

hence

$$\lim_{z\to 0} \frac{\sin z}{z} = 1.$$

Suppose $p(z)$ and $q(z)$ are holomorphic with $p(c) \neq 0$ and $q(c) = 0$. Then c is a pole of $f(z) = p(z)/q(z)$. Since

$$\lim_{z\to c}(z-c)f(z) = \lim_{z\to c}\frac{z-c}{q(z)-q(c)}\cdot p(z) = \frac{p(c)}{q'(c)},$$

we see c is a simple pole iff in addition $q'(c) \neq 0$.

Let $f(z)$ be a meromorphic function in an open set G. Let c be a pole in G of order $n \geq 1$. The **residue** at c is

$$\operatorname{Res}(f,c) = \frac{1}{(n-1)!}\,\frac{d^{n-1}}{dz^{n-1}}(z-c)^n f(z)\bigg|_{z=c}.$$

For example, if $p(z)$ and $q(z)$ are holomorphic with $p(c) \neq 0$, $q(c) = 0$, $q'(c) \neq 0$, then

$$\operatorname{Res}\left(\frac{p(z)}{q(z)}, c\right) = \frac{p(c)}{q'(c)}. \tag{7.1}$$

If instead c is a double pole (pole of order 2), so $p(c) \neq 0$, $q(c) = q'(c) = 0$, but $q''(c) \neq 0$, then

$$q(z) = (z-a)^2\left(\frac{1}{2}q''(a) + \frac{1}{6}q'''(a)(z-a) + \dots\right),$$

so

$$\operatorname{Res}\left(\frac{p(z)}{q(z)}, c\right) = \frac{d}{dz}\,\frac{(z-c)^2 p(z)}{q(z)}\bigg|_{z=c} = \frac{2p'(c)}{q''(c)} - \frac{2p(c)q'''(c)}{3q''(c)^2}. \tag{7.2}$$

In particular, if c is a simple zero for $q(z)$, $q(c) = 0$, $q'(c) \neq 0$, and $p(c) \neq 0$, then c is a double pole for $p(z)/q(z)^2$. Replacing q by q^2 in (7.2), we get

$$\operatorname{Res}\left(\frac{p(z)}{q(z)^2}, c\right) = \frac{p'(c)}{q'(c)^2} - \frac{p(c)q''(c)}{q'(c)^3}. \tag{7.3}$$

7.1. Cauchy's Residue Theorem

Let $f(z)$ be meromorphic in an open set G, and let C be a closed contour in G not winding about any point not in G, and not passing through the poles of $f(z)$. Then

$$\int_C f(z)\,dz = 2\pi i \sum_a n_a \cdot \operatorname{Res}(f,a),$$

where n_a is the winding number of C about a, and the sum is over all poles a.

In applications, the contour is very explicit and the winding numbers are usually 1, or the open set G is simply connected, so the winding condition is automatic. The proof is similar to that of Cauchy's integral formula.

Proof For each pole a, let D_a be a small disk contained in G and centered at a, such that there is no other pole in D_a. If C_a is the perimeter of D_a, taken counterclockwise, then as before we check that the contour

$$C' = C - \sum_a n_a \cdot C_a$$

does not wind about any pole a, nor does it wind about any point not in G, so, by Cauchy's theorem,

$$\int_C f(z)\,dz = \sum_a n_a \int_{C_a} f(z)\,dz.$$

Now, if a is a pole of order n, then $g(z) = (z-a)^n f(z)$ is holomorphic on D_a, so we may apply Cauchy's integral formula on D_a for derivatives [with $g(z)$ instead of $f(z)$],

$$\frac{1}{2\pi i}\int_{C_a} f(z)\,dz = \frac{1}{2\pi i}\int_{C_a} \frac{g(z)}{(z-a)^n}\,dz = \frac{1}{(n-1)!}g^{(n-1)}(a) = \operatorname{Res}(f,a),$$

obtaining the result. □

For example, the meromorphic function

$$\pi\cot\pi z = \frac{\pi\cos\pi z}{\sin\pi z}$$

has simple poles at the simple zeros of $\sin\pi z$, which are the integers. Hence

$$\operatorname{Res}(\pi\cot\pi z, n) = \left.\frac{\pi\cos\pi z}{(\sin\pi z)'}\right|_{z=n} = 1, \qquad n = 0, \pm1, \pm2, \ldots$$

Let C be any closed contour in $\mathbf{C}$ not passing through the integers. Then C winds around at most finitely many integers, and, by Theorem 7.1,

$$\frac{1}{2\pi i}\int_C \pi\cot\pi z\,dz = \sum_{n=-\infty}^{\infty} N(C,n)\cdot\operatorname{Res}(\pi\cot\pi z, n) = \sum_{n=\infty}^{\infty} N(C,n). \tag{7.4}$$

7.2 Evaluation of Real Integrals

Let

$$q(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$$

be a polynomial. Assume the degree n is at least 2. We evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{q(x)}.$$

We use the residue theorem applied to the green rectangle G and the closed contour

$$C_r^+ = C^+(0,r) + [-r,r]$$

in Figure 7.1. Since G is simply connected, the residue theorem applies to any closed contour in G.

Suppose $q(x)$ has a real zero r. Factoring, we may write

$$q(x) = (x-r)^k p(x)$$

for some polynomial $p(x)$ with $p(r) \neq 0$, and some $k \geq 1$, the order of the zero r. Since

$$\int \frac{dx}{x-r} = \log(x-r) \qquad \text{and} \qquad \int \frac{k-1}{(x-r)^k}\, dx = \frac{-1}{(x-r)^{k-1}}, k > 1,$$

the integral I will diverge at $x = r$.

By the fundamental theorem of algebra (Theorem 6.13), $q(z)$ has n complex zeros. If the degree n is odd, then $q(x) \to \pm\infty$ as $x \to \pm\infty$. By the intermediate value theorem, there is at least one real zero r of $q(x)$. Hence, to guarantee convergence of I, we *assume the degree* $n \geq 2$ *is even, and* $q(x)$ *has no real zeros*. In this case, the polynomial $q(x)$ is positive on the real line.

To evaluate I, we evaluate a contour integral over C_r^+, for r sufficiently large. Since

$$I = \lim_{r\to\infty} \int_{-r}^{r} \frac{dx}{q(x)}$$

and

$$\int_{C_r^+} \frac{dz}{q(z)} = \int_{-r}^{r} \frac{dx}{q(x)} + \int_{C^+(0,r)} \frac{dz}{q(z)},$$

we need to first show

$$\lim_{r\to\infty} \int_{C^+(0,r)} \frac{dz}{q(z)} = 0. \tag{7.5}$$

This is shown as in the proof of the fundamental theorem of algebra.

To estimate the integral in (7.5), by the triangle inequality, we need to estimate the maximum of the integrand absolute value over the contour. Since $q(z)$ is in the denominator, we need to estimate the minimum of $|q(z)|$ over $C^+(0,r)$.

Let $\epsilon(r)$ be the minimum value of $|q(z)|/r^n$ over $C^+(0,r)$. Then $|q(z)| \geq \epsilon(r) r^n$ for z on $C^+(0,r)$. Since

$$\lim_{z\to\infty} \frac{q(z)}{z^n} = \lim_{z\to\infty} \left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n}{z^n}\right) = 1,$$

we have $\epsilon(r) \to 1$ as $r \to \infty$. By the triangle inequality,

$$\left|\int_{C^+(0,r)} \frac{dz}{q(z)}\right| \le \int_{C^+(0,r)} \frac{|dz|}{|q(z)|} \le \frac{\pi r}{r^n \epsilon(r)}.$$

Since $n \ge 2$, (7.5) follows. This establishes

$$I = \lim_{r\to\infty} \int_{C_r^+} \frac{dz}{q(z)}.$$

By the residue theorem (the winding numbers here are all 1 — see (5.9)),

$$\int_{C_r^+} \frac{dz}{q(z)} = \sum_{\substack{q(a)=0\\ \mathrm{Im}(a)>0}} \mathrm{Res}\left(\frac{1}{q(z)}, a\right).$$

Here the sum is over the zeros of $q(z)$ ($q(a) = 0$) lying in the upper-half plane ($\mathrm{Im}(a) > 0$). Since there are finitely many zeros, for r sufficiently large, these zeros all lie inside C_r^+.

When the zeros a of $q(z)$ are simple, $p(a) = 0$ and $q(a) = 0$ and $q'(a) \ne 0$, by (7.1), the last equation reduces to

$$\int_{C_r^+} \frac{dz}{q(z)} = \sum_{\substack{q(a)=0\\ \mathrm{Im}(a)>0}} \frac{1}{q'(a)},$$

where again the sum is over the zeros of $q(z)$ lying in the upper-half plane.

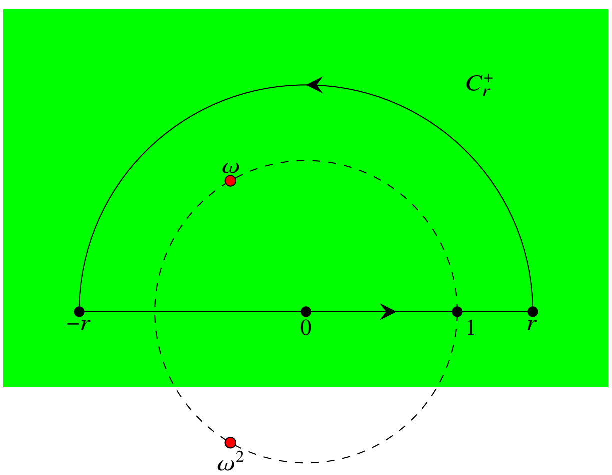

Fig. 7.1 Computing I_3 with $\omega = e^{2\pi i/3}$

We conclude

7.2. Evaluation of a Real Integral I

Let $q(x)$ be a polynomial with even degree $n \geq 2$ and with no real zeros. Then

$$\int_{-\infty}^{\infty} \frac{dx}{q(x)} = 2\pi i \sum_{\substack{q(a)=0 \\ \mathrm{Im}(a)>0}} \mathrm{Res}\left(\frac{1}{q(z)}, a\right), \tag{7.6}$$

where the sum is over the zeros of $q(z)$ lying in the upper-half plane. If these zeros are simple, this reduces to

$$\int_{-\infty}^{\infty} \frac{dx}{q(x)} = 2\pi i \sum_{\substack{q(a)=0 \\ \mathrm{Im}(a)>0}} \frac{1}{q'(a)}. \tag{7.7}$$

The integral of any rational function is handled in the same manner. Let $p(z)$, $q(z)$ be poynomials with no common zeros, and suppose none of the zeros of $q(z)$ are real.

7.3. Evaluation of a Real Integral II

Let $p(x)$, $q(x)$ be polynomials with no common factors and with $\deg(q) \geq \deg(p) + 2$, and with no real zeros for $q(x)$. Then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx = 2\pi i \sum_{\substack{q(a)=0 \\ \mathrm{Im}(a)>0}} \mathrm{Res}\left(\frac{p(z)}{q(z)}, a\right), \tag{7.8}$$

where the sum is over the zeros of $q(z)$ lying in the upper-half plane. If these zeros are simple, this reduces to

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx = 2\pi i \sum_{\substack{q(a)=0 \\ \mathrm{Im}(a)>0}} \frac{p(a)}{q'(a)}. \tag{7.9}$$

We use (7.7) to evaluate

$$I_3 = \int_{-\infty}^{\infty} \frac{dx}{1 + x + x^2} \tag{7.10}$$

and

$$I_5 = \int_{-\infty}^{\infty} \frac{dx}{1 + x + x^2 + x^3 + x^4}. \tag{7.11}$$

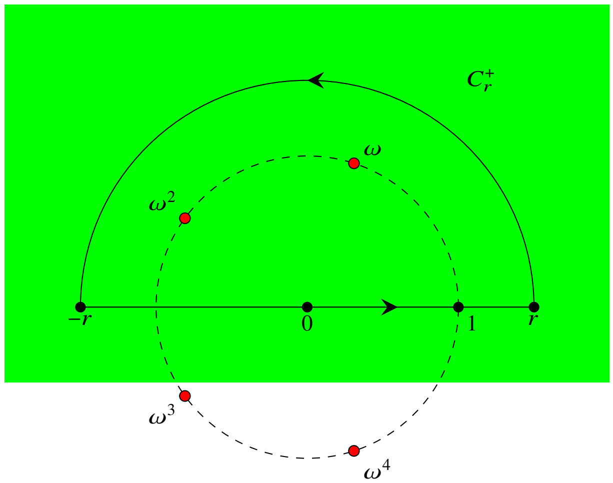

Fig. 7.2 Computing I_5 with $\omega = e^{2\pi i/5}$

More generally, let

$$p_n(z) = 1 + z + z^2 + \cdots + z^{n-1}.$$

We evaluate

$$I_n = \int_{-\infty}^{\infty} \frac{dx}{1 + x + x^2 + x^3 + \cdots + x^{n-1}} = \int_{-\infty}^{\infty} \frac{dx}{p_n(x)}. \tag{7.12}$$

Since

$$z^n - 1 = (z-1)(1 + z + z^2 + \cdots + z^{n-1}) = (z-1)p_n(z),$$

we have

$$p_n(z) = \frac{z^n - 1}{z - 1}. \tag{7.13}$$

Thus *the zeros of $p_n(z)$ are exactly the n-th roots of unity that are not* 1. In particular, the zeros of $p_n(z)$ are simple.

It follows that the only possible real zeros of $p_n(z)$ are ± 1. Clearly 1 is never a zero of $p_n(z)$, while -1 is a zero iff n is even. Thus, to guarantee convergence of I_n, *we assume $n \geq 3$ and n is odd.*

Now $z^n - 1$ has n zeros (3.15),

$$1, \omega, \omega^2, \ldots, \omega^{n-1}, \qquad \omega = e^{2\pi i/n}.$$

It follows that $p_n(z)$ has the $n - 1$ zeros

$$\omega, \omega^2, \omega^3, \ldots, \omega^{n-1}.$$

Since we assume n is odd, these zeros are all non-real (Figures 7.1 and 7.2 and 7.3).

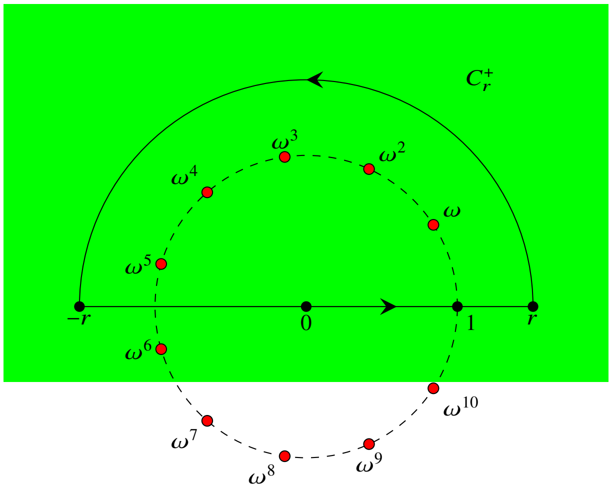

Fig. 7.3 Computing I_{11} with $\omega = e^{2\pi i/11}$

Thus the zeros of $p_n(z)$ lying in the upper-half plane are

$$\omega, \omega^2, \dots, \omega^{(n-1)/2}.$$

When $r > 1$, these zeros are inside C_r^+.

Using (7.13), check that

$$p_n'(a) = \frac{na^{n-1}}{a-1},$$

for any zero a of $p_n(z)$. Then, by (7.7),

$$\begin{aligned}
I_n = 2\pi i \sum_{\substack{p_n(a)=0\\ \mathrm{Im}(a)>0}} \frac{1}{p_n'(a)}. &= 2\pi i \sum_{\substack{a^n=1\\ a\neq 1\\ \mathrm{Im}(a)>0}} \frac{a-1}{na^{n-1}}\\
&= \frac{2\pi i}{n} \sum_{\substack{a^n=1\\ a\neq 1\\ \mathrm{Im}(a)>0}} \frac{a(a-1)}{a^n}\\
&= \frac{2\pi i}{n} \sum_{\substack{a^n=1\\ a\neq 1\\ \mathrm{Im}(a)>0}} a(a-1)\\
&= \frac{2\pi i}{n} \sum_{k=1}^{(n-1)/2} \omega^k(\omega^k-1)\\
&= \frac{2\pi i}{n} \sum_{k=0}^{(n-1)/2} \omega^k(\omega^k-1).
\end{aligned}$$

By definition of $p_n(z)$, this equals

$$\frac{2\pi i}{n}\left(\sum_{k=0}^{(n+1)/2-1} \omega^{2k} - \sum_{k=0}^{(n+1)/2-1} \omega^k\right) = \frac{2\pi i}{n}\left(p_{(n+1)/2}(\omega^2) - p_{(n+1)/2}(\omega)\right). \quad (7.14)$$

To arrive at an explicitly real expression, we use the exponential form of sin and cos (§3.4). Set $\theta = \pi/n$; then $\omega = e^{2i\theta}$.

By (7.13),

$$\begin{aligned}
p_{(n+1)/2}(\omega^2) - p_{(n+1)/2}(\omega) &= \frac{(\omega^2)^{(n+1)/2}-1}{\omega^2-1} - \frac{\omega^{(n+1)/2}-1}{\omega-1}\\
&= \frac{2\omega}{\omega^2-1} - \frac{\omega^{(n+1)/2}}{\omega-1}.
\end{aligned}$$

But

$$\frac{2\omega}{\omega^2-1} = \frac{2}{\omega-\omega^{-1}} = \frac{1}{i}\frac{2i}{e^{2i\theta}-e^{-2i\theta}} = \frac{1}{i}\cdot\frac{1}{\sin(2\theta)},$$

and

$$-\frac{\omega^{(n+1)/2}}{\omega-1} = \frac{\sqrt{\omega}}{\omega-1} = \frac{1}{\sqrt{\omega}-1/\sqrt{\omega}} = \frac{1}{2i}\cdot\frac{1}{\sin\theta}.$$

Hence

$$I_n = \frac{2\pi i}{n}\left(\frac{1}{i}\cdot\frac{1}{\sin(2\theta)} + \frac{1}{2i}\cdot\frac{1}{\sin\theta}\right) = \frac{2\theta}{\sin(2\theta)} + \frac{\theta}{\sin\theta}.$$

Summarizing,

7.4. Evaluation of a Real Integral III

Suppose n is a positive integer with $n \geq 3$ and odd, and let $\theta = \pi/n$. Then

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x + x^2 + x^3 + \cdots + x^{n-1}}$$

coverges and equals

$$\frac{2\theta}{\sin(2\theta)} + \frac{\theta}{\sin\theta}.$$

When n is even, the integral diverges at $x = -1$.

In particular (Table 2.11)

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x + x^2} = \frac{2\pi}{\sqrt{3}},$$

and

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x + x^2 + x^3 + x^4} = \frac{2\pi}{5}\sqrt{\frac{8}{5+\sqrt{5}}} + \frac{\pi}{5}\sqrt{\frac{8}{5-\sqrt{5}}}.$$

This can be generalized.

7.5. Evaluation of a Real Integral IV

Suppose m and n are positive integers with $n \geq m + 2$. Let $\theta = \pi/n$. Then

$$\int_{-\infty}^{\infty} \frac{1 + x + x^2 + x^3 + \cdots + x^{m-1}}{1 + x + x^2 + x^3 + \cdots + x^{n-1}}\, dx$$

converges and equals

$$\frac{2\theta \sin(m\theta)}{\sin\theta \sin((m+1)\theta)}$$

when n and m are both even, and equals

$$\frac{\theta \sin(m\theta)}{\sin\theta \sin((m+1)\theta)} + \frac{\theta}{\sin\theta} - (-1)^m \frac{\theta}{\sin((m+1)\theta)},$$

when n is odd. When n is even and m is odd, the integral diverges at $x = -1$.

The integral (7.6) may be modified by an exponential factor e^{iz}. Since

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix}e^{-y}| \leq 1$$

on the upper-half plane $y > 0$, we also have

$$\int_{C^+(0,r)} e^{iz} \frac{p(z)}{q(z)}\, dz \to 0 \qquad \text{as } r \to \infty,$$

as long as $\deg(q) \geq (p) + 2$. Thus the same procedure as before works, and we obtain

7.6. Evaluation of a Real Integral V

Let $p(x)$, $q(x)$ be polynomials with no common factors and with $\deg(q) \geq \deg(p) + 2$, and with no real zeros for $q(x)$. Then

$$\int_{-\infty}^{\infty} e^{iz} \frac{p(x)}{q(x)}\, dx = 2\pi i \sum_{\substack{q(a)=0 \\ \operatorname{Im}(a)>0}} \operatorname{Res}\left(e^{iz} \frac{p(z)}{q(z)}, a\right), \tag{7.15}$$

where the sum is over the poles of $p(z)/q(z)$ lying in the upper-half plane. If these poles are simple, this reduces to

$$\int_{-\infty}^{\infty} e^{ix} \frac{p(x)}{q(x)}\, dx = 2\pi i \sum_{\substack{q(a)=0 \\ \operatorname{Im}(a)>0}} e^{ia} \frac{p(a)}{q'(a)}. \tag{7.16}$$

Exercises

References

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Symbols

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