General refraction problems with phase discontinuities on nonflat metasurfaces

CRISTIAN E. GUTIÉRREZ,1,* LUCA PALLUCCHINI,1 AND ERIC STACHURA2

1Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122, USA
2Department of Mathematics, Haverford College, Haverford, Pennsylvania 19041, USA
*Corresponding author: gutierre@temple.edu

Received 4 April 2017; revised 10 May 2017; accepted 24 May 2017; posted 26 May 2017 (Doc. ID 292078); published 22 June 2017

This paper provides a mathematical approach to study metasurfaces in nonflat geometries. Analytical conditions between the curvature of the surface and the set of refracted directions are introduced to guarantee the existence of phase discontinuities. The approach contains both the near and far field cases. A starting point is the formulation of a vector Snell’s law in the presence of abrupt discontinuities on the interfaces. © 2017 Optical Society of America

OCIS codes: (160.3918) Metamaterials; (080.2720) Mathematical methods (general); (080.2740) Geometric optical design.

https://doi.org/10.1364/JOSAA.34.001160

1. INTRODUCTION

For classical lens design, a typical problem is to find two surfaces so that the region sandwiched between them and filled with a homogeneous material refracts light in a desired manner. For metasurfaces, the surface is given and the question is to find a function on the surface (a phase discontinuity) so that the pair, surface together with the phase discontinuity (the metalens), refracts light in a desired manner. The subject of metalenses is a flourishing area of research and one of the nine runners-up for Science’s Breakthrough of the Year 2016 [1]. Metalenses have been designed for flat geometries with the scalar generalizations of the laws of refraction and refraction with phase discontinuities; see [2–4] and the comprehensive review article [5]. These general laws have been experimentally observed by using arrays of optical antennas on silicon. The review in [6] describes the past 15 years of progress on metasurfaces, from experimental realization of the generalized laws of refraction to applications in wave front and beam shaping. Recently, it has been proved [7] that at certain frequencies, a thin layer of nanoparticles on a perfectly conducting sheet acts as a metasurface. For more recent work in the area and an extensive up to date bibliography, we refer the reader to [8]; see also [9–12].

The purpose of this paper is to provide a mathematically rigorous foundation to deal with general metasurfaces and to determine the relationships between the curvature of the surface and the phase discontinuity. In fact, we present a mathematical method to construct phase discontinuities on a given surface so that radiation is steered into a prescribed set of directions. More precisely, given a surface $S$ in three-dimensional space, we determine when it is possible to have a function $\psi$ defined on a very thin (comparable with the wave length of the radiation) neighborhood of $S$ so that radiation emanating from a point source is refracted by the pair $(S, \psi)$, surface and function, into a set of directions prescribed in advance. In other words, for a given set of directions where we want to steer the radiation, we discover what kind of surfaces $S$ allow the existence of a function $\psi$ so that the pair $(S, \psi)$ directs the radiation in a desired way. This leads to ultrathin (not flat) optical components that produce abrupt changes over the scale of the free-space wavelength in the phase. This is in contrast with classical lens design, where the question is to engineer the gradual accumulation of phase delay as the wave propagates in the device, reshaping the scattered wave front and beam profile at will. In particular, in lenses, light propagates over distances much larger than the wavelength to shape wave fronts. The existence of phase discontinuity functions is intimately related with the shape of the given surface and the given set of directions. If these two objects satisfy condition Eq. (20), and the determinant of the matrix Eq. (31) is not zero, then the existence of the desired phase discontinuity is guaranteed; see Section 5. In particular, given a surface, one can see from these conditions what kind of sets of steering directions are allowed. In addition, and conversely, we show in Section 6 that given a phase discontinuity and a desired transmission direction, we describe what admissible surfaces are possible.

Our approach and results are new, given that the metasurfaces used in the literature are typically flat, the incoming radiation is normal to the surface, and the phase discontinuities are determined ad hoc. Our results open up the possibility of designing general nonflat metasurfaces, which we believe is potentially useful in optical applications. Our theory yields (for a given geometry) the functions needed for practical implementation. The realization of the metasurface consists of designing the arrangement of nano materials on the surface (such as gold
antenna arrays as in [2]) dictated by the phase discontinuity calculated theoretically.

There has been recent interest in nonflat metasurfaces (some authors call them conformal metasurfaces); see [13–15]. Most surfaces considered in these papers are cylindrical, which is a particular case in our theory, and the results are numerical. Concerning losses in metasurfaces, see [16,17].

Of great importance for us in answering these questions in general geometries is the formulation of a generalized Snell’s law in vector form, Eq. (6), which is deduced using wave fronts in Section 3. In terms of wave vectors, a vector law is formulated in Eq. (2) [4]. However, Eq. (6) is effective and flexible for the actual calculation of phase discontinuities in general and to obtain our results. We illustrate these with explicit constructions for planar and spherical interfaces (Sections 4.A and 4.B, Sections 7.A and 7.B; see also Remark 5).

The outline of the paper is as follows. In Section 2, we briefly recall the classical Snell’s law for surfaces without phase discontinuities. Then in Section 3, we derive a generalized Snell’s law in the presence of a phase discontinuity using wave fronts, Eq. (6), and analyze the possible critical angles. The far field problem is studied in Section 4 for the plane and the sphere. In Section 5, we allow for variable directions m in the far field. In Section 6, conditions are derived so that, given a phase discontinuity, a surface exists. Finally, in Section 7, the near field problem is addressed.

2. BACKGROUND

We recall the classical Snell’s law in vector form here. Suppose \( \Gamma \) is a surface in \( \mathbb{R}^3 \) that separates two media, \( I \) and \( II \), that are homogeneous and isotropic, with refractive indices \( n_1 \) and \( n_2 \), respectively. If a ray of light having direction \( x \in S^2 \), the unit sphere in \( \mathbb{R}^3 \), and traveling through medium I strikes \( \Gamma \) at the point \( P \), then this ray is refracted in the direction \( m \in S^2 \) through medium II, according to the Snell’s law in vector form:

\[
n_1 (x \times \nu) = n_2 (m \times \nu),
\]

(1)

where \( \nu \) is the unit normal to the surface to \( \Gamma \) at \( P \) pointing toward medium II (see Subsection 4.1 [18]). Since the refraction angle depends on the frequency of the radiation, we assume that light rays are monochromatic. It is assumed here that \( x \cdot \nu \geq 0 \).

This has several consequences:

(a) the vectors \( x, m, \nu \) are all on the same plane (called the plane of incidence), and

(b) the well-known Snell’s law in scalar form holds:

\[
n_1 \sin \theta_1 = n_2 \sin \theta_2,
\]

where \( \theta_1 \) is the angle between \( x \) and \( \nu \) (the angle of incidence), and \( \theta_2 \) is the angle between \( m \) and \( \nu \) (the angle of refraction).

Equation (1) is equivalent to \( (n_1 x - n_2 m) \times \nu = 0 \), which means that the vector \( n_1 x - n_2 m \) is parallel to the normal vector \( \nu \). If we set \( \kappa = n_2/n_1 \), then

\[
x - \kappa m = \lambda \nu
\]

(2)

for some \( \lambda \in \mathbb{R} \). Notice that Eq. (2) univocally determines \( \lambda \). Taking dot products with \( x \) and \( m \) in Eq. (2), we get

\[
\lambda = \cos \theta_1 - \kappa \cos \theta_2, \quad \cos \theta_1 = x \cdot \nu > 0, \quad \text{and} \quad \cos \theta_2 = m \cdot \nu = \sqrt{1 - \kappa^2(1 - (x \cdot \nu)^2)}. \]

In fact, there holds

\[
\lambda = x \cdot \nu - \kappa \sqrt{1 - \kappa^2(1 - (x \cdot \nu)^2)}. \tag{3}
\]

The formulation (2) is useful to solve refraction problems for lens design [19–22]; see also [23] for a numerical implementation.

3. DERIVATION OF A VECTOR SNEILL’S LAW WITH PHASE DISCONTINUITY USING WAVE FRONTS

Let \( n_1, n_2 \) be the refractive indices of two homogeneous media I and II, respectively. Suppose a surface \( \Gamma \) separates the two media, and an incoming light ray in medium I with wave vector \( k_1 \), strikes \( \Gamma \). Assume that there is a real-valued function \( \psi \), the phase discontinuity, defined in a neighborhood of the surface \( \Gamma \). Notice that \( \psi \) must be defined in a neighborhood of \( \Gamma \) because the gradient of \( \psi \) will be considered.

If \( \nu \) denotes the unit normal vector to \( \Gamma \), then the refracted wave vector \( k_2 \) satisfies Eq. (2) [4]:

\[
\nu \times (k_2 - k_1) = \nu \times \nabla \psi. \tag{4}
\]

We give an alternate formulation and derivation of this result by using wave fronts; our starting point is Section 2.2 in [24]. For each \( \tau \), \( \Psi(x,y,z,t) = 0 \) denotes a surface in the variables \( x, y, z \) that separates the part of the space that is at rest from the part of the space that is disturbed by the electric and magnetic fields. This surface is called a wave front, and the light rays are the orthogonal trajectories to the wave fronts at each time \( t \). We assume that \( \Psi_t \neq 0 \), and so we can solve \( \Psi(x,y,z,t) = 0 \) in \( t \), obtaining that \( \Phi(x,y,z) = ct \); so, letting \( t \) run, the wave fronts are then the level sets of \( \Phi(x,y,z) \).

Let \( n_1, n_2, \Gamma \) be as above. An incoming wave front \( \Psi_1 \) on medium I strikes the surface \( \Gamma \), and it is then transmitted into a wave front \( \Psi_2 \) in medium II (of course, there is also a wave front reflected back). Assuming, as before, that \( \Psi_1 \neq 0 \), \( j = 1, 2 \), and solving in \( t \), we get that the wave fronts are given by \( \Phi_j(x,y,z) = ct \) for \( j = 1, 2 \), respectively. Suppose the surface \( \Gamma \) is parameterized by \( x = f(\xi, \eta), y = g(\xi, \eta), z = h(\xi, \eta) \). If there were no phase discontinuity on the surface \( \Gamma \), then we would have \( \Phi_1 = \Phi_2 \) along \( \Gamma \). But since there is now a phase discontinuity \( \psi \) on \( \Gamma \), we have the following jump condition along \( \Gamma \):

\[
\Phi_1(f(\xi, \eta), g(\xi, \eta), h(\xi, \eta)) - \Phi_2(f(\xi, \eta), g(\xi, \eta), h(\xi, \eta)) = \psi(f(\xi, \eta), g(\xi, \eta), h(\xi, \eta)).
\]

Taking derivatives in \( \xi \) and \( \eta \) yields

\[
(\nabla \Phi_1 - \nabla \Phi_2 - \nabla \psi) \cdot (f_x g_\xi h_\eta, f_y g_\xi h_\eta, f_z g_\xi h_\eta) = 0
\]

and

\[
(\nabla \Phi_1 - \nabla \Phi_2 - \nabla \psi) \cdot (f_y g_\eta h_\xi, f_z g_\eta h_\xi, f_z g_\eta h_\xi) = 0.
\]

That is, the vector \( \nabla \Phi_1 - \nabla \Phi_2 - \nabla \psi \) must be normal to \( \Gamma \); as such, there exists a real number \( \lambda \) such that

\[
\nabla \Phi_1 - \nabla \Phi_2 - \nabla \psi = \lambda \nu,
\]

(5)

where \( \nu \) is the unit normal to \( \Gamma \).

Let \( \gamma_j(t) \) denote the light rays in medium \( j \) having speed \( v_j \), for \( j = 1, 2 \), i.e., the orthogonal trajectories to \( \Phi_j \). In particular, we have that \( \Phi_j(\gamma_j(t)) = ct \), and by the chain rule
\( \nabla \phi_j(y_j(t)) \cdot y_j'(t) = c, \quad j = 1, 2. \)

If we parameterize the rays so that \( |y_j'(t)| = v_j \), then we obtain
\[
|\nabla \phi_j(y_j(t))| = \frac{c}{v_j} = n_j, \quad j = 1, 2,
\]

since \( \nabla \phi_j \) is parallel to \( y_j' \). Letting
\[
x = \frac{\nabla \phi_1(y_1(t))}{|\nabla \phi_1(y_1(t))|}, \quad m = \frac{\nabla \phi_2(y_2(t))}{|\nabla \phi_2(y_2(t))|},
\]

we obtain from Eq. (5) the following formula:
\[
n_1x - n_2m = \lambda \nu + \nabla \psi. \tag{6}
\]

Taking cross-products with the unit normal \( \nu \) in Eq. (6), we obtain the equivalent formula
\[
\nu \times (n_1x - n_2m) = \nu \times \nabla \psi. \tag{7}
\]

Recall that \( x \) is the unit direction of the incident ray, \( m \) is the unit direction of the refracted ray, \( \nu \) is the unit outer normal at the incident point on \( \Gamma \), and \( \nabla \psi \) is calculated at the incident point. Note that in the case that \( \psi \) is constant, we recover the classical Snell’s law in vector form, Eq. (1). In fact, if \( \psi = \) constant, then \( n_1\nu \times x = n_2\nu \times m \). Taking dot product with \( m \) yields
\[
n_1m \cdot (\nu \times x) = 0.
\]

This means that \( m \) is on the plane through the origin having normal \( \nu \times x \), which is the plane generated by \( \nu \) and \( x \). Therefore \( \nu, x, m \) are all on the same plane, i.e., the plane of incidence. On the other hand, if \( \psi \) is not necessarily constant, then from Eq. (7), \( n_1\nu \times x = n_2\nu \times m + \nu \times \nabla \psi \). Again, taking dot product with \( m \) yields
\[
n_1m \cdot (\nu \times x) = m \cdot (\nu \times \nabla \psi), \quad \text{that is,}
\]
\[
m \cdot (\nu \times (n_1x - \nabla \psi)), \quad \text{where} \ \nabla \psi \text{is calculated at the point on the surface} \ \Gamma \text{where the ray with direction} \ x \text{strikes it. This shows that in the general case, the refracted vector} \ m \text{is not on the plane generated by} \ \nu \text{and} \ x.
\]

Starting from Eq. (6), we now calculate \( \lambda \). Taking dot products in Eq. (6) and solving for \( x \cdot m \) yields
\[
x \cdot m = \frac{n_1 - \lambda \nu \cdot x \cdot \nabla \psi}{n_2}.
\]

Next, taking dot products in Eq. (6) with itself, expanding, and substituting \( x \cdot m \) from the previous expression yields that \( \lambda \) satisfies the quadratic equation:
\[
\lambda^2 - [2(n_1x - \nabla \psi) \cdot \nu] + \left| n_1x - \nabla \psi \right|^2 - n_2^2 = 0. \tag{8}
\]

Solving for \( \lambda \) yields
\[
\lambda = \frac{(n_1x - \nabla \psi) \cdot \nu}{n_2} \pm \sqrt{n_2^2 - \left| n_1x - \nabla \psi \right|^2 - \left| (n_1x - \nabla \psi) \cdot \nu \right|^2}. \tag{9}
\]

Since \( \lambda \) must be a real number, the quantity under the square root must be nonnegative, i.e.,
\[
n_2^2 \geq \left| n_1x - \nabla \psi \right|^2 - \left| (n_1x - \nabla \psi) \cdot \nu \right|^2. \tag{10}
\]

Assuming this for now, it remains to check which sign \( (\pm) \) to take in Eq. (9). Dotting Eq. (6) with \( \nu \) and using Eq. (9) yields
\[
n_1x \cdot \nu - n_2m \cdot \nu = (n_1x - \nabla \psi) \cdot \nu \pm \sqrt{n_2^2 - \left| n_1x - \nabla \psi \right|^2 - \left| (n_1x - \nabla \psi) \cdot \nu \right|^2} + \nabla \psi \cdot \nu,
\]

so
\[
n_1x \cdot \nu - n_2m \cdot \nu = \pm \sqrt{n_2^2 - \left| n_1x - \nabla \psi \right|^2 - \left| (n_1x - \nabla \psi) \cdot \nu \right|^2} \cdot \nabla \psi \cdot \nu.
\]

Since \( n_2 > 0 \) and \( m \cdot \nu \geq 0 \), we obtain that
\[
\lambda = (n_1x - \nabla \psi) \cdot \nu \pm \sqrt{n_2^2 - \left| n_1x - \nabla \psi \right|^2 - \left| (n_1x - \nabla \psi) \cdot \nu \right|^2}. \tag{11}
\]

We next analyze Eq. (10), which will yield the critical angles. Equation (10) is equivalent to
\[
\left| \left( x - \frac{\nabla \psi}{n_1} \right) \cdot \nu \right|^2 \geq \left| \frac{\nabla \psi}{n_1} \right|^2 - \kappa^2.
\]

Thus, if \( x \) is such that
\[
\left| x - \frac{\nabla \psi}{n_1} \right| \leq \kappa,
\]

then Eq. (10) holds. On the other hand, if
\[
\left| x - \frac{\nabla \psi}{n_1} \right| > \kappa,
\]

then Eq. (10) holds when either
\[
x \cdot \nu \geq \frac{\nabla \psi}{n_1} \cdot \nu + \sqrt{\left| x - \frac{\nabla \psi}{n_1} \right|^2 - \kappa^2} \quad \text{or}
\]
\[
x \cdot \nu \leq \frac{\nabla \psi}{n_1} \cdot \nu - \sqrt{\left| x - \frac{\nabla \psi}{n_1} \right|^2 - \kappa^2}.
\]

Therefore, the critical angles between \( x \) and \( \nu \) are \( \theta_c \) with
\[
x \cdot \nu = \cos \theta_c = \frac{\nabla \psi}{n_1} \cdot \nu + \sqrt{\left| x - \frac{\nabla \psi}{n_1} \right|^2 - \kappa^2}
\]

or
\[
x \cdot \nu = \cos \theta_c = \frac{\nabla \psi}{n_1} \cdot \nu - \sqrt{\left| x - \frac{\nabla \psi}{n_1} \right|^2 - \kappa^2}.
\]

**Remark 1.** In two dimensions, the critical angles are considered in [2]. It is assumed there that the interface \( \Gamma \) is the \( x \) axis, the region \( y > 0 \) is filled with a material with refractive index \( n_1 \), and the region \( y < 0 \) is filled with a material with refractive index \( n_2 \). Also, the phase discontinuity satisfies that \( \nabla \psi \) is constant and is tangential to the interface, i.e., \( \nabla \psi = (a, 0) \) with, for example, \( a > 0 \). Therefore, the above calculations applied to this case yield
\[
\cos \theta_c = \frac{x \cdot \nu}{\frac{\nabla \psi}{n_1}} = \sqrt{\left| x - \frac{\nabla \psi}{n_1} \right|^2 - \kappa^2}
\]

where \( \kappa = \frac{n_2}{n_1} \). Squaring both sides, we obtain
\[
\cos^2 \theta_e = 1 - \frac{2|\nabla \psi|}{n_1} \sin \theta_e + \frac{|\nabla \psi|^2}{n_1^2} - \kappa^2,
\]
and the critical angles \( \theta_e \) are therefore the solutions to the equation
\[
\sin^2 \theta_e - \frac{2|\nabla \psi|}{n_1} \sin \theta_e + \frac{|\nabla \psi|^2}{n_1^2} - \kappa^2 = 0,
\]
i.e.,
\[
\theta_e = \arcsin \left( \frac{|\nabla \psi|}{n_1} \pm \kappa \right),
\]
which is in agreement with formula (3) [2].

In three dimensions, the critical angles are considered in [4]. The interface \( \Gamma \) is the \( x-y \) plane, the region \( z > 0 \) is filled with a material with refractive index \( n_1 \), and the region \( z < 0 \) is filled with a material with refractive index \( n_2 \). Also, the phase discontinuity is tangential to the interface, i.e., \( \nabla \psi = (\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, 0) \), and without loss of generality, we may assume \( x = (0, y, z) \). Once again, the above calculations applied to this case yield
\[
\cos \theta_e = x \cdot \nu = \sqrt{x - \frac{\nabla \psi^2}{n_1} - \kappa^2}
\]
\[
= \sqrt{1 - \frac{2|\nabla \psi|}{n_1} \cos(\pi/2 - \theta_e) + \frac{|\nabla \psi|^2}{n_1^2} - \kappa^2}.
\]
Proceeding as before, we find
\[
\theta_e = \arcsin \left( \frac{1}{n_1} \frac{\partial \psi}{\partial y} \pm \sqrt{\kappa^2 - \frac{1}{n_1^2} \left( \frac{\partial \psi}{\partial x} \right)^2} \right),
\]
recovering formula (8) [4].

**Remark 2.** The reflection case is when \( n_1 = n_2 \), so Eq. (6) and Eq. (11) become
\[
x - m = \frac{1}{n_1} \lambda \nu + \frac{\nabla \psi}{n_1},
\]
\[
\lambda = (n_1 x - \nabla \psi) \cdot \nu + \sqrt{n_1^2 - (|n_1 x - \nabla \psi|^2 - [(n_1 x - \nabla \psi) \cdot \nu]^2)},
\]
with \( x \) the unit incident direction, \( m \) the unit reflected vector, \( \nu \) the unit normal to the interface at the striking point, and \( \nabla \psi \) at the striking point. Notice that the choice of the plus sign in front of the square root is because for reflection \( m \cdot \nu \leq 0 \).

### 4. FAR FIELD UNIFORMLY REFRACTING PLANAR AND SPHERICAL METALENSES

Let \( \Gamma \) be a surface in three-dimensional space and \( V \) be a vector valued function defined on \( \Gamma \); \( V: \Gamma \to \mathbb{R}^3 \). If \( x \) is an incident unit direction striking \( \Gamma \) at a point \( P \), and \( m \) is the unit refracted direction, then we obtain, dividing by \( n_1 \) in the generalized Snell's law Eq. (6), that
\[
x - \kappa m = \lambda \nu(P) + V(P),
\]
where \( \nu(P) \) is the unit outer normal to \( \Gamma \) at \( P \) for some \( \lambda \in \mathbb{R}; \kappa = n_2/n_1 \). Suppose rays emanate from the origin and we are given a fixed unit vector \( m \). Our goal is to answer the following two questions. First, given a surface \( \Gamma \) separating media \( n_1 \) and \( n_2 \), find a field \( V \) defined on \( \Gamma \) so that all rays from the origin are refracted into the direction \( m \); see Fig. 1. The second question is, given a field \( V \) defined in a region of \( \mathbb{R}^3 \), find a separation surface \( \Gamma \) between \( n_1 \) and \( n_2 \) within that region so that all rays emanating from the origin are refracted into the direction \( m \).

We begin in this section answering the first question, when \( \Gamma \) is either a plane or a sphere, surfaces of traditional interest in optics, showing explicit phase discontinuities. For general surfaces, the first question is considered in Section 5, even for the more general case of variable \( m \). The second question is answered in Section 6.

#### A. Case of the Plane

Let \( \Gamma \) be the plane \( x_1 = a \) in \( \mathbb{R}^3 \) with \( a > 0 \). We want to determine a field \( V = (V_1, V_2, V_3) \) defined on \( \Gamma \) so that all rays emanating from the origin are refracted into the unit direction \( m = (m_1, m_2, m_3) \), with \( m_1 > 0 \); see Fig. 2(a). Using spherical coordinates \( x(u, v) = (\cos u \sin v, \sin u \sin v, \cos v) \), \( 0 \leq u \leq 2\pi \), \( 0 \leq v \leq \pi \), \( \Gamma \) is described parametrically by
\[
r(u, v) = \frac{a}{\cos u \sin v} x(u, v) = a \left( 1, \tan u, \frac{1}{\cos u \tan v} \right).
\]

**Fig. 1.** Metalens refracting into a fixed direction (rays are monochromatic; colors are used only for visual purposes).

**Fig. 2.** Planar and spherical metalenses.
Since the normal to the plane $\Gamma$ is $\nu = (1, 0, 0)$, then Eq. (12) implies that $u \sin v \nu - \kappa m_2 = V_2(r(u, v))$ and $\cos v - \kappa m_1 = V_1(r(u, v))$. Hence, $V_2$ and $V_3$ are univocally determined. Also, from Eq. (12), we get

$$V_1(r(u, v)) = \cos u \sin v - \kappa m_1 - \lambda(u, v).$$  \hspace{1cm} (14)

Notice also that from Eq. (11),

$$\lambda = \nu \cdot (x - V) - \sqrt{\nu \cdot (x - V)^2 - |x - V|^2 + k^2},$$

which in the present case yields

$$\lambda = \cos u \sin v - V_1 - \sqrt{k^2 - (\sin u \sin v - V_2)^2 - (\cos v - V_3)^2} = \cos u \sin v - V_1 - \sqrt{k^2 - (\kappa m_2)^2 - (\kappa m_3)^2} = \cos u \sin v - V_1 - \kappa m_1$$

since $m_1 > 0$

$$= \frac{a}{\sqrt{a^2 + x_2^2 + x_3^2}} \cdot V_1(a, x_2, x_3) - \kappa m_1.$$

This means that in Eq. (14), each $V_1$ determines $\lambda$ and vice versa.

We now write the field $V$ in rectangular coordinates $x_1, x_2, x_3$. Since $\sqrt{a^2 + x_2^2 + x_3^2} = \frac{a}{\cos \frac{a}{\sin x}}$, we can write

$$V_2(a, x_2, x_3) = \frac{a}{\sqrt{a^2 + x_2^2 + x_3^2}} - \kappa m_2,$$

$$= \frac{\partial}{\partial x_2} \sqrt{x_2^2 + x_2^2 + x_3^2} \bigg|_{x_1 = a} - \kappa m_2,$$

$$V_3(a, x_2, x_3) = \frac{a}{\sqrt{a^2 + x_2^2 + x_3^2}} - \kappa m_3,$$

$$= \frac{\partial}{\partial x_3} \sqrt{x_1^2 + x_2^2 + x_3^2} \bigg|_{x_1 = a} - \kappa m_3,$$

$$V_1(a, x_2, x_3) = \frac{a}{\sqrt{a^2 + x_2^2 + x_3^2}} - \kappa m_1 - \lambda,$$

$$= \frac{\partial}{\partial x_1} \sqrt{x_1^2 + x_2^2 + x_3^2} \bigg|_{x_1 = a} - \kappa m_1 - \lambda$$

for $-\infty < x_2, x_3 < \infty$. From Eq. (13), $u = \arctan(x_2/x_3)$ and $v = \arctan(\sqrt{x_1^2 + x_2^2})$, so $\lambda(u, v) = b(x_2, x_3)$. Let $y(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2} - \kappa m_1 x_1 - \kappa m_2 x_2 - \kappa m_3 x_3 + C$, with $C$ a constant of integration. Therefore, if on the plane $x = a$, we give the field

$$V(x_1, x_2, x_3) := \nabla y(x_1, x_2, x_3) - b(x_2, x_3)i, \hspace{1cm} (15)$$

then the resulting metasurface does the desired refraction job. If we want $V$ to be the gradient of a function, then $b(x_2, x_3)i$ must be a gradient, which is only possible when $b(x_2, x_3) = C_0$ is a constant; that is, $V = \nabla y(x_1, x_2, x_3) = C_0$. As a particular case, when $m_1 = 1, m_2 = m_3 = 0$, and $C_0 = 0$, we obtain the equivalent formula (2) [5] (where a different orientation of the coordinates is used) with $x_1 = a = f$. Notice also that if we want $V$ in Eq. (15) to be tangential to the plane $x_1 = a$, that is, $(\nabla y(x_1, x_2, x_3) - b(x_2, x_3)i) \cdot (1, 0, 0) = 0$, then $b = \frac{a}{\sqrt{a^2 + x_2^2 + x_3^2}} - \kappa m_1$.

**B. Case of the Sphere**

Now, the surface $\Gamma$ considered is a sphere of radius $R$ centered at the origin, that is, $r(x, v) = Rx(u, v)$, with $(u, v)$ spherical coordinates. We denote by $x = (u, v)$; see Fig. 2(b). Since $\Gamma$ is a sphere, the normal $\nu = x$, and from Eq. (12), we get $(x - Vm - V)x = 0, so$

$$(V + \kappa m) \times x = 0.$$  \hspace{1cm} (16)

That is,

$$[x_2 x_3 0] [x_1 x_3] = \kappa m_1 [x_2 x_2 0] [x_1 x_3] = 0.$$  \hspace{1cm} (17)

Notice that det $[x_2 x_3 0] [x_1 x_3] = 0$. Set $W_i = V_i + \kappa m_i$, so the system is equivalent to

$$[0 0 0] [x_1 x_3] = \kappa m_1 [x_2 x_2 0] [x_1 x_3] = 0.$$  \hspace{1cm} (18)

If $x_1 x_2 x_3 \neq 0$, the last matrix has rank two, so the space of solutions has dimension one, and the solutions are given by

$$W_1, W_2, W_3 = \begin{pmatrix} x_1 & x_2 & x_3 & 1 \end{pmatrix} W_3,$$

with $W_3$ being arbitrary. Therefore,

$$V_1(Rx(u, v)) = \frac{x_1}{x_3} (V_3(Rx(u, v)) + \kappa m_3) - \kappa m_1,$$

$$V_3(Rx(u, v)) = \frac{x_2}{x_3} (V_3(Rx(u, v)) + \kappa m_3) - \kappa m_2,$$

with $V_3$ being arbitrary. Notice that if in Eq. (16), we take cross-product with $x_2$, we get

$$0 = x \times [(V + \kappa m) \times x] = (V + \kappa m)(x \times x) - x[(V + \kappa m) \times x] = V + \kappa m - [\kappa (m \times x) + V \times x] x.$$

Hence, if we want to pick $V$ tangential to the sphere, we obtain

$$V(Rx) = -\kappa m + \kappa (m \times x) x \quad \text{with} \quad |x| = 1.$$  \hspace{1cm} (19)

$V$ is a field defined on the sphere of radius $R$. We shall determine a function $\psi$ defined in a neighborhood of the sphere of radius $R$ such that $V(Rx) = V\nabla \psi(Rx)|_{x=1}$, and satisfying $\psi_j(Rx) = -\kappa m_j + \kappa (m \times x) x_j$, for $|x| = 1, j \leq 3$. \hspace{1cm} (20)

In fact, we have $(x = x(u, v))$

$$\frac{\partial V(Rx(u, v))}{\partial u}$$

$$\begin{array}{c}
\quad = R \sum_{k=1}^{3} \frac{\partial y(Rx(u, v))}{\partial x_k} u = R(\frac{\partial y(Rx(u, v))}{\partial x_k} \cdot x_k) \\
\end{array}$$

$$\begin{array}{c}
\quad = R[-\kappa m \cdot x_u + \kappa (m \times x)(x \times x_u)] \\
\end{array}$$

$$\begin{array}{c}
\quad = -\kappa R(m \cdot x_u) \quad \text{and similarly,} \\
\end{array}$$
\[ \frac{\partial \psi(R(x, v))}{\partial v} = -\kappa R \frac{\partial}{\partial v} (m \cdot x). \]

Integrating the derivative in \( u \) yields
\[ \psi(R(x, v)) = -\kappa R(m \cdot x) + g(v), \]
and integrating the derivative in \( v \), we obtain
\[ \psi(R(x, v)) = -\kappa R(m \cdot x(u, v)) + C_1, \]
with \( C_1 \) an arbitrary constant. Writing this in rectangular coordinates
\[ \psi(R(x_1, x_2, x_3)) = -\kappa R(m \cdot (x_1, x_2, x_3)) + C_1 \]
for \(|(x_1, x_2, x_3)| = 1\). We now define \( \psi \) on a neighborhood of \(|z| = R\) so that Eq. (17) holds. Define
\[ \psi(z) = -\kappa R(m \cdot z)|z|^{-1} + C_1, \]
for \(-R < |z| < R + \epsilon. \) (18)

We have
\[ \nabla \psi(Rx) = -\kappa m + \kappa (m \cdot x)x, \]
as desired. Therefore, the phase discontinuity \( \psi \) from Eq. (18) has a gradient tangential to the sphere and can be placed on the spherical interface \(|z| = R\) so that all rays from the origin are refracted into the fixed direction \( m \).

5. METALENSES REFRACTING INTO A SET OF VARIABLE DIRECTIONS

Suppose \( m(u, v) = (m_1(u, v), m_2(u, v), m_3(u, v)) \) is a given \( C^2 \) unit field of directions, and let \( \Gamma \) be a \( C^2 \) surface given parametrically by \( r(u, v) = \rho(u, v)x(u, v) \), where \( x(u, v) \) are spherical coordinates and \( \rho(u, v) > 0 \) is the polar radius. We want to see when it is possible to have a phase discontinuity \( \psi \) on the surface \( \Gamma \), so that each ray from the origin with direction \( x(u, v) \) is refracted into the direction \( m(u, v) \). From Eq. (12),
\[ x(u, v) - \kappa m(u, v) - V(r(u, v)) = \lambda \nu(r(u, v)), \]
so
\[ (x - \kappa m - V) \times \nu = 0. \]

Taking cross-product with \( \nu \) yields
\[ 0 = \nu \times [(x - \kappa m - V) \times \nu] = (x - \kappa m - V)(\nu \times \nu) - \nu [(x - \kappa m - V) \cdot \nu]. \]

If \( V \) is tangential to \( \Gamma \), then \( V \cdot \nu = 0 \), and so
\[ 0 = x - \kappa m - V - [(x - \kappa m) \cdot \nu] \nu, \]
that is,
\[ V = x - \kappa m - [(x - \kappa m) \cdot \nu] \nu. \]

If \( V(r(u, v)) = (\nabla \psi)(r(u, v)) \), then
\[ \psi_x(r(u, v)) = x_x(u, v) - \kappa m_x(u, v), \]
\[ -[(x(u, v) - \kappa m(u, v)) \cdot \nu(r(u, v))] \nu(r(u, v)). \]

Since \( \nu \cdot r_u = \nu \cdot r_v = 0 \) and \( x \cdot x_u = x \cdot x_v = 0, \)
\[ \frac{\partial}{\partial u}(\psi(r(u, v))) = (\nabla \psi)(r(u, v)) \cdot r_u = (x - \kappa m) \cdot r_u - [(x - \kappa m) \cdot \nu] \nu(r(u, v)) \]
\[ = (x - \kappa m) \cdot r_u - [(x - \kappa m) \cdot \nu] \nu(r(u, v)) \]
\[ = \rho_x(u, v)x_x(u, v) + \rho_m(u, v)x_u(u, v) \]
\[ = \rho_x(u, v)x_x(u, v) - \kappa \rho m(u, v) \cdot x_u(u, v) \]
\[ = \rho_x(u, v)x_x(u, v) - \kappa \rho m(u, v) \cdot x_u(u, v) \]
\[ = \rho(1 - \kappa m \cdot x) \nu, \]
as desired, \( \lambda \nu \).

Let us now consider the first order system in \( \Phi \),
\[ \begin{cases} \Phi_u = \kappa m \cdot x, \\ \Phi_v = \kappa \rho m \cdot x, \end{cases} \]
where \( \Phi(u, v) = \psi(r(u, v)) - \rho(1 - \kappa m \cdot x) \).

If the given set of directions \( m(u, v) \) and the surface \( \Gamma \) satisfy
\[ m_u \cdot r_u = m_v \cdot r_v, \]
then by [25] [Chapter 6, pp. 117–118; see also Eq. (35) below], there exists \( \Phi \) solving Eq. (19). By integration, we then obtain that the phase discontinuity \( \psi \) satisfies along \( \Gamma \) that
\[ \psi(r(u, v)) = \rho(1 - \kappa m \cdot x) + \Phi(u, v). \]

To find the gradient of \( \psi \), we need to have \( \psi \) defined in a neighborhood of the surface \( r(u, v) \) such that Eq. (21) holds and that its gradient satisfies on \( r(u, v) \)
\[ (\nabla \psi)(r(u, v)) = x - \kappa m - [(x - \kappa m) \cdot \nu] \nu. \]

Notice that this implies \( (\nabla \psi)(r(u, v)) \perp \nu \). To construct the function \( \psi \) in a neighborhood of the surface \( \Gamma \) (we will construct it in a neighborhood of each point in \( \Gamma \), given parametrically by \( r(u, v) \), we use the notion of envelope from classical differential geometry; see, for example, Chapter 5, Section 4 in [26] or Chapter 3 in [27]. We will actually construct a surface that is developable, in particular, one having zero Gaussian curvature. For a recent reference on developable surfaces, applications, and design, see [28]. Since the required \( \psi \) must satisfy Eq. (21), consider the surface \( \Gamma' \) given parametrically by
\[ P(u, v) = (r(u, v), |r(u, v)| - \kappa (m(u, v) \cdot r(u, v)) + \Phi(u, v)) \]
in four dimensions. At each point \( P(u, v) \), consider the four-dimensional vector
\[ N(u, v) = (x - \kappa m - [(x - \kappa m) \cdot \nu], -1), \]
where \( x = x(u, v) \) and \( \nu \) is the unit normal to the surface \( \Gamma \) at \( r(u, v) \). Next, consider the plane \( \Pi_{uv} \) passing through the point \( P(u, v) \) and with normal \( N(u, v) \), that is, in coordinates \( x_1, x_2, x_3, x_4, \Pi_{uv} \) has equation
\[ F(x_1, x_2, x_3, x_4, u, v) := N(u, v) \cdot ((x_1, x_2, x_3, x_4) - P(u, v)) = 0. \]  
\[ (24) \]

Therefore, we have a family of planes \( \Pi_{uv} \) depending on the parameters \( u, v \), and we will let \( x_4 = \psi(x_1, x_2, x_3) \) be by definition the envelope to this family of planes. Of course, we need to know under what conditions on \( r(u, v) \) and \( m(u, v) \) this envelope \( \psi \) exists. It will be defined by solving the system of equations

\[ \begin{align*}
F(x_1, x_2, x_3, x_4, u, v) &= 0 \\
\frac{\partial F}{\partial u}(x_1, x_2, x_3, x_4, u, v) &= 0 \\
\frac{\partial F}{\partial v}(x_1, x_2, x_3, x_4, u, v) &= 0
\end{align*} \]
\[ (25) \]

In fact, let us fix values \( u = u_0, v = v_0 \), and let \( P_0 = P(u_0, v_0) = (p_1, p_2, p_3, p_4) \) be the corresponding value on the surface \( \Gamma \); consider the map

\[ G(x_1, x_2, x_3, u, v) = \left( F(x_1, x_2, x_3, x_4, u, v), \frac{\partial F}{\partial u}(x_1, x_2, x_3, x_4, u, v), \times \frac{\partial F}{\partial v}(x_1, x_2, x_3, x_4, u, v) \right). \]

The function \( G \) has continuous partial derivatives in a neighborhood of the point \((p_1, p_2, p_3, p_4, u_0, v_0)\), and

\[ G(p_1, p_2, p_3, p_4, u_0, v_0) = 0. \]

By the implicit function theorem, if the Jacobian determinant

\[ \frac{\partial G}{\partial (x_4, u, v)}(p_1, p_2, p_3, p_4, u_0, v_0) = \begin{vmatrix}
\frac{\partial F}{\partial x_4} & \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\
\frac{\partial^2 F}{\partial x_4 \partial u} & \frac{\partial^2 F}{\partial u^2} & \frac{\partial^2 F}{\partial u \partial v} \\
\frac{\partial^2 F}{\partial x_4 \partial v} & \frac{\partial^2 F}{\partial v^2} & \frac{\partial^2 F}{\partial v^2}
\end{vmatrix}_{(p_1, p_2, p_3, p_4, u_0, v_0)} \neq 0, \tag{26} \]

then there are unique differentiable functions \( g_1, g_2, g_3 \) in the variables \( x_1, x_2, x_3 \) defined in a neighborhood \( U \) of \((p_1, p_2, p_3)\) such that \( p_4 = g_1(p_1, p_2, p_3), u_0 = g_2(p_1, p_2, p_3) \) and \( v_0 = g_3(p_1, p_2, p_3) \) with

\[ G(x_1, x_2, x_3, g_1(x_1, x_2, x_3), g_2(x_1, x_2, x_3), g_3(x_1, x_2, x_3)) = 0 \]

for all \((x_1, x_2, x_3) \in U\). Therefore, if we let \( \psi(x_1, x_2, x_3) = g_1(x_1, x_2, x_3) \) for \((x_1, x_2, x_3) \in U\), then \( \psi \) is the function we need, i.e., \( \psi \) is by construction defined in a neighborhood of the point \((p_1, p_2, p_3) \in \Gamma \) and satisfies Eqs. (21) and (22).

We now analyze under what conditions on the surface \( \Gamma \) and \( m \), Eq. (26), holds. Notice first that since \( \partial_x F = -1 \), the matrix inside the determinant in Eq. (26) equals

\[ \begin{pmatrix}
1 & \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\
0 & \frac{\partial^2 F}{\partial u \partial u} & \frac{\partial^2 F}{\partial u \partial v} \\
0 & \frac{\partial^2 F}{\partial v \partial u} & \frac{\partial^2 F}{\partial v \partial v}
\end{pmatrix}, \]

and therefore, Eq. (26) means

\[ \det \begin{pmatrix}
\frac{\partial^2 F}{\partial u \partial u} & \frac{\partial^2 F}{\partial u \partial v} \\
\frac{\partial^2 F}{\partial v \partial u} & \frac{\partial^2 F}{\partial v \partial v}
\end{pmatrix} \neq 0. \]

Let us find what this means in terms of the initial surface \( \Gamma \) and the field \( m \). To simplify the notation, let \( X = (x_1, x_2, x_3, x_4) \), so we can write Eq. (24) as

\[ F(X, u, v) = N(u, v) \cdot (X - P(u, v)). \]

By calculation,

\[ \begin{align*}
F_u &= N_u \cdot (X - P) - N \cdot P_u \\
F_{uu} &= N_{uu} \cdot (X - P) - 2 N_u \cdot P_u + N_n \cdot P_{uu} \\
F_{uv} &= N_{uv} \cdot (X - P) - N_u \cdot P_v + N_n \cdot P_{uv} \\
F_{vv} &= N_{vv} \cdot (X - P) - 2 N_v \cdot P_v + N_{vv} \cdot P_{vv} \tag{27}
\end{align*} \]

We first show that

\[ N \cdot P_u = N \cdot P_v = 0. \tag{28} \]

Indeed, we have

\[ P(u, v) = \rho(x, 1 - km \cdot x) + (0, \Phi), \]

so

\[ P_u = \rho_u(x, 1 - km \cdot x) + \rho(x_{uw} - km \cdot x_{uw} - km \cdot x + 0, \Phi_u), \]
\[ P_v = \rho_v(x, 1 - km \cdot x) + \rho(x_{vw} - km \cdot x_{vw} - km \cdot x + 0, \Phi_v). \tag{29} \]

Hence,

\[ N \cdot P_u = \{ \rho_u(x, 1 - km \cdot x) + \rho(x_{uw} - km \cdot x_{uw} - km \cdot x + 0, \Phi_u) \} \cdot (x - km - [(x - km) \cdot \nu] \cdot 1) \]
\[ = (\rho_u x + \rho x_u) \cdot (x - km - [(x - km) \cdot \nu]) - \rho_u (1 - km \cdot x) \]
\[ + \rho \cdot (k m \cdot x_{uw} + km \cdot x_{uw} - km \cdot x_{uw} - km \cdot x_{uw} - km \cdot \nu) \]
\[ = 0. \]

since \((\rho_u x + \rho x_u) \cdot \nu = r_u \cdot \nu = 0 \) and \( x_{uw} \cdot \nu = 0 \). The same calculation with \( P_v \) instead of \( P_u \) yields the second identity in Eq. (28).

Next, differentiating Eq. (28) with respect to \( u \) and \( v \) yields

\[ N \cdot P_{uu} = -N_u \cdot P_u, \quad N \cdot P_{uv} = -N_u \cdot P_v = -N_v \cdot P_u, \quad N \cdot P_{vv} = -N_v \cdot P_v, \]

since \( P_{uw} = P_{uw} \). Hence, letting \( X = X \) in Eq. (29) yields

\[ F_{uu} = -N_u \cdot P_u, \quad F_{uv} = -N_u \cdot P_v = -N_v \cdot P_u, \quad F_{vv} = -N_v \cdot P_v. \]
Now let us calculate these dot products. First set
\[ B = (x - km) \cdot \nu \]
and write
\[
N_u \cdot P_u = \{\rho_u(x, 1 - km \cdot x) + \rho(x_{uv} - km \cdot x_u) + (0, \Phi_u)\}
\]
\[
\cdot [x_u - km_u - [(x - km) \cdot \nu u] - (x - km) \cdot \nu_{uv} 0] = (\rho_x + \rho x_u) \cdot (x_u - km_u - B_x \cdot \nu - B_{uv}) = (\rho_x + \rho x_u) \cdot (x_u - km_u - B_x \cdot \nu - B_{uv}) = (\sin^2 v) - \kappa (\rho x + \rho x_u) \cdot m_u - B(\rho x + \rho x_u) \cdot \nu_u = (\sin^2 v) - \kappa (\rho x + \rho x_u) \cdot m_u - B(\rho x + \rho x_u) \cdot \nu_u = (\sin^2 v) - \kappa (\rho x + \rho x_u) \cdot m_u - B(\rho x + \rho x_u) \cdot \nu_u
\]
\[
\text{since } x \cdot x_u = 0, x_u \cdot x_u = \sin^2 v, \text{ and } (\rho x + \rho x_u) \cdot \nu = r_u \cdot \nu = 0. \text{ Also } x_v \cdot x_v = 1 \text{ and } x_u \cdot x_v = 0, \text{ so we obtain similarly}
\]
\[
N_v \cdot P_v = \rho - kr_v \cdot m_u - B r_v \cdot \nu_{uv}
\]
\[
N_u \cdot P_v = -kr_u \cdot m_u - B r_u \cdot \nu_{uv}
\]
Next, differentiating \( r_u \cdot \nu = r_v \cdot \nu = 0 \) yields
\[
r_u \cdot \nu_{uv} = -r_{uv} \cdot \nu, \quad r_u \cdot \nu_v = -r_{uv} \cdot \nu, \quad r_v \cdot \nu_v = -r_{uv} \cdot \nu. \tag{30}
\]
Therefore,
\[
(F_{uu} \quad F_{uv}) \tag{31}
\]
\[
= \begin{pmatrix} -\sin^2 v \rho + kr_v \cdot m_u - B r_v \cdot \nu & kr_u \cdot m_v - B r_u \cdot \nu \\ kr_v \cdot m_u - B r_v \cdot \nu & -\rho + kr_u \cdot m_u - B r_u \cdot \nu \end{pmatrix}
\]
\[
= \begin{pmatrix} x_u \cdot x_u & x_u \cdot x_v \\ x_v \cdot x_u & x_v \cdot x_v \end{pmatrix} + \kappa \begin{pmatrix} r_u \cdot m_u & r_u \cdot m_v \\ r_v \cdot m_u & r_v \cdot m_v \end{pmatrix}
\]
\[
- B \begin{pmatrix} r_{uv} \cdot \nu \\ r_{uv} \cdot \nu_v \\ r_{uv} \cdot \nu_v \\ r_{uv} \cdot \nu_v \\ r_{uv} \cdot \nu_v \end{pmatrix},
\]
with \( B = (x - km) \cdot \nu \). Notice that the first and third matrices in the last line are, respectively, the first fundamental form of the 2-sphere, and the second fundamental form of the surface \( \Gamma \). Therefore, we have proved the following: if a variable field \( m \) and a surface \( \Gamma \) satisfy the compatibility condition Eq. (20), and the determinant of the matrix Eq. (31) is zero at a point \( (u_0, v_0) \), then there is a neighborhood \( U \) of the point \( r(u_0, v_0) \) and a phase discontinuity function \( \psi \) defined in \( U \) for the surface \( \Gamma \), with gradient \( \nabla \psi \) tangential to \( \Gamma \), so that it yields the desired refraction job. That is, each ray emanating in the direction \( x(u, v) \), for \( (u, v) \) in a neighborhood of \( (u_0, v_0) \), is refracted by the metasurface \( (\Gamma, \psi) \) into the direction \( m(u, v) \).

**Remark 3 (case when \( m \) is a constant vector).** If \( m(u, v) = (m_1, m_2, m_3) \) is constant, then Eq. (20) is clearly satisfied by any \( \Gamma \) and, in condition (31), the second matrix on the right-hand side is zero.

**Remark 4.** To illustrate the determinant condition (31), let us consider the special case when \( \Gamma \) is a sphere centered at the origin, and \( m \) is a constant vector. We have \( r(u, v) = R(u, v) \), and \( \nu = x(u, v) \). So \( r_{uv} = R x_{uv} \) and similarly for \( r_{uv} \) and \( r_{uv} \). Also \( B = 1 - km \cdot x \), \( x_{uv} \cdot x = -\sin^2 v \), \( x_{uv} \cdot x = 0 \), and \( x_{uv} \cdot x = -R \sin^2 v \). Therefore, the determinant in Eq. (31) equals
\[
R^2 \sin^2 v (1 - B)^2 = R^2 kl^2 (\sin^2 v)(m \cdot x)^2.
\]
For example, if \( m = (0, 0, 1) \), i.e., all rays are refracted vertically, then the determinant equals
\[
R^2 k^2 (\sin v \cos v)^2 = R^2 / 4 \sin^2 (2v),
\]
which is not zero as long as \( v \neq \pi / 2 \) or zero. This shows also that for the sphere, the phase discontinuity \( \psi \) exists and can be obtained by solving the system of Eq. (25). Notice that in this case, a phase discontinuity \( \psi \) was calculated explicitly in Section 4.B and given by Eq. (18).

**Remark 5 (case when \( \Gamma \) is off centered).** A case considered in Section 3 [4]: a sphere of radius \( R \) is centered at a point \( (0, 0, a) \) with \( a > R \), and the authors claim there that it is not possible to find a phase discontinuity on such a sphere so that all rays from the origin are refracted into the vertical direction. We believe this claim is in error and in fact, with the method above, will show that for each unit \( m = (m_1, m_2, m_3) \) with \( m_3 > 0 \), there is a phase discontinuity \( \psi \) defined in a neighborhood of such a sphere so that its gradient is tangential to the sphere and so that radiation from the origin is refracted into a fixed direction \( m \) (see Fig. 3). In particular, when \( m \) is vertical, a phase discontinuity exists. By reversibility of optical paths, this shows that the conclusion in Section 3 [4] is incorrect.

First, the lower part of the sphere with center at \( (0, 0, a) \) and radius \( R \) is parametrized by the vector \( r(u, v) = \rho(u, v)x(u, v) \) with
\[
\rho(u, v) = a \cos v - \sqrt{R^2 - a^2\sin^2 v},
\]
where \( 0 \leq v \leq \arcsin(R/a) \), and the unit normal to the sphere pointing upward is
\[
\nu = \frac{(0, 0, a) - \rho(u, v)x(u, v)}{R}.
\]
To show our claim, we need to verify that the determinant in Eq. (31) is not zero. From Eq. (30), we obtain by simple calculations that

\[
\begin{align*}
    r_{uu} \cdot \nu &= -r_u \cdot \nu_u = \frac{1}{R} (\sin^2 \nu) \rho \nu \\
    r_{uv} \cdot \nu &= -r_u \cdot \nu_v = \frac{1}{R} \rho \rho_v = 0 \\
    r_{vv} \cdot \nu &= -r_v \cdot \nu_v = \frac{1}{R} (\rho_v^2 + \nu^2).
\end{align*}
\]

Therefore, the determinant in Eq. (31) equals

\[
\det \begin{pmatrix} F_{uu} & F_{uv} \\ F_{uv} & F_{vv} \end{pmatrix} = \rho (\sin^2 \nu) \left( 1 + \frac{B}{R} \right) \left( \rho + \frac{B}{R} (\rho^2 + (\rho_v)^2) \right)
\]

(32)

with

\[
B = (x - km) \cdot \nu = \frac{1}{R} (x - km) \cdot ((0, 0, a) - \rho x)
\]

\[
= \frac{1}{R} \left( \sqrt{R^2 - a^2 \sin^2 \nu - k a m_3 + k \rho (m \cdot x)} \right).
\]

The last determinant is not zero for \(u, v\), such that \(\sin^2 \nu \neq 0\), \(1 + \frac{B}{R} \rho \neq 0\), and \(\rho + \frac{B}{R} (\rho^2 + (\rho_v)^2) \neq 0\).

Let us take, for example, \(m = (0, 0, 1)\), i.e., rays are refracted vertically, then we get

\[
B = \frac{1}{R} \left( (1 - k \cos \nu) \sqrt{R^2 - a^2 \sin^2 \nu - k a m_3 + k \rho (m \cdot x)} \right),
\]

so \(B\) is independent of \(u\). If \(\nu \approx 0\), then \(B \approx 1 - k\), \(\rho \approx a - R\) and \(\rho_v \approx 0\), so

\[
1 + \frac{B}{R} \rho \approx 1 + (1 - k) \left( \frac{a}{R} - 1 \right)
\]

\[
\rho + \frac{B}{R} (\rho^2 + (\rho_v)^2) \approx (a - R) \left[ 1 + (1 - k) \left( \frac{a}{R} - 1 \right) \right].
\]

Recall that \(k = n_3/n_1\). If \(k < 1\), since \(a > R\), we obtain that \(1 + (1 - k) \left( \frac{a}{R} - 1 \right) \neq 0\). If \(k > 1\), then \(1 + (1 - k) \left( \frac{a}{R} - 1 \right) \neq 0\) if and only if \(k \neq 1 + \frac{a}{R} \). This shows that in these cases, the determinant in Eq. (32) is not zero for \(\nu \neq 0\) with \(\nu\) close to zero. Therefore, there exists a phase discontinuity \(w_r\) on the sphere centered at \((0, 0, a)\) with radius \(R\), defined in a neighborhood of each point of the form \(\rho(u, v) x(u, v)\) with \(\nu\) close to zero.

6. GIVEN A PHASE DISCONTINUITY, FIND AN ADMISSIBLE SURFACE

We now turn to the second question proposed at the beginning of Section 4, that is, of finding the surface \(\Gamma\) when the field \(V = (V_1, V_2, V_3)\) is given. The unknown surface is given parametrically by

\[
r(u, v) = \rho(u, v) x(u, v),
\]

where \(x(u, v)\) are spherical coordinates as before, and we seek the polar radius \(\rho\); the value of \(V\) along the surface is \(V(r(u, v))\). From Eq. (12), \(x(u, v) - km - V(r(u, v))\) is a multiple of the normal \(\nu\) at \(r(u, v)\), so

\[
r_u(u, v) \cdot (x(u, v) - km - V(r(u, v))) = 0 \quad \text{and} \quad r_v(u, v) \cdot (x(u, v) - km - V(r(u, v))) = 0.
\]

We have

\[
r_u(u, v) = [\rho(u, v)]_u x(u, v) + \rho(u, v) \kappa x_u(u, v),
\]

\[
r_v(u, v) = [\rho(u, v)]_v x(u, v) + \rho(u, v) \kappa x_v(u, v),
\]

so

\[
0 = r_u(u, v) \cdot (x(u, v) - km - V(r(u, v)))
\]

\[
= [\rho(u, v)]_u x(u, v) + \rho(u, v) \kappa x_u(u, v)
\]

\[
\cdot (x(u, v) - km - V(r(u, v))
\]

\[
= [\rho(u, v)]_u (1 - x(u, v) \cdot [km + V(r(u, v))])
\]

\[
- \rho(u, v) x_u(u, v) \cdot [km + V(r(u, v))]
\]

and a similar equation for \(r_v\). That is, \(\rho(u, v)\) satisfies the following first order nonlinear system of pdes, depending on \(V\) (we are assuming that \(1 - x(u, v) \cdot [km + V(\rho(u, v) x(u, v))] \neq 0\):

\[
\begin{align*}
\rho_u(u, v) - \frac{x_u(u, v) \cdot [km + V(\rho(u, v) x(u, v))]}{1 - x(u, v) \cdot [km + V(\rho(u, v) x(u, v))]} \rho(u, v) &= 0,
\end{align*}
\]

\[
\begin{align*}
\rho_v(u, v) - \frac{x_v(u, v) \cdot [km + V(\rho(u, v) x(u, v))]}{1 - x(u, v) \cdot [km + V(\rho(u, v) x(u, v))]} \rho(u, v) &= 0.
\end{align*}
\]

(33)

If \(F = (F_1, F_2)\) with

\[
F_1(u, v, \rho) = \frac{x_u(u, v) \cdot [km + V(\rho x(u, v))]}{1 - x(u, v) \cdot [km + V(\rho x(u, v))]} \rho(u, v),
\]

\[
F_2(u, v, \rho) = \frac{x_v(u, v) \cdot [km + V(\rho x(u, v))]}{1 - x(u, v) \cdot [km + V(\rho x(u, v))]} \rho(u, v),
\]

then Eq. (33) can be written as

\[
\nabla \rho = F(u, v, \rho).
\]

(34)

To solve the system Eq. (34), we need an initial condition, say, \(\rho(u_0, v_0) = \rho_0\), and use a result from [25] (Chapter 6, pp. 117–118); that is, if

\[
\frac{\partial F_1}{\partial \nu} (u, v, \rho) + \frac{\partial F_1}{\partial \rho} (u, v, \rho) F_2(u, v, \rho) = \frac{\partial F_2}{\partial \nu} (u, v, \rho) + \frac{\partial F_2}{\partial \rho} (u, v, \rho) F_1(u, v, \rho)
\]

(35)

holds for all \((u, v, \rho)\) in an open set \(O\), then for each \((u_0, v_0, \rho_0) \in O\), there is neighborhood \(U\) of \((u_0, v_0)\) and a unique solution \(\rho(u, v)\) defined for \((u, v) \in U\) solving the system Eq. (34) and satisfying \(\rho(u_0, v_0) = \rho_0\).

We will see under what circumstances on the field \(V\) condition Eq. (35) is satisfied, and therefore, the existence of the desired surface \(r(u, v)\) will be guaranteed. Set

\[
W(u, v, \rho) = km + V(\rho x(u, v)),
\]

(36)

then

\[
F_1(u, v, \rho) = \frac{x_u(u, v) \cdot W(u, v, \rho)}{1 - x(u, v) \cdot [W(u, v, \rho)]} \rho,
\]

\[
F_2(u, v, \rho) = \frac{x_v(u, v) \cdot W(u, v, \rho)}{1 - x(u, v) \cdot [W(u, v, \rho)]} \rho.
\]
We have
\[ \partial F_1 = (x_{uv} \cdot W + x_u \cdot W_u)(1-x \cdot W)^{-1} \rho \]
\[ + (x_v \cdot W + x_u \cdot W_u)(1-x \cdot W)^{-2} \rho, \]
\[ \partial F_2 = (x_{uv} \cdot W + x_v \cdot W_v)(1-x \cdot W)^{-1} \rho \]
\[ + (x_u \cdot W + x_v \cdot W_v)(1-x \cdot W)^{-2} \rho, \]
\[ \frac{\partial F_1}{\partial \rho} = (x_u \cdot W)(1-x \cdot W)^{-1} \]
\[ + [(x_u \cdot W)(1-x \cdot W)^{-1} + (x_v \cdot W)(1-x \cdot W)^{-2}] \rho, \]
\[ \frac{\partial F_2}{\partial \rho} = (x_v \cdot W)(1-x \cdot W)^{-1} \]
\[ + [(x_v \cdot W)(1-x \cdot W)^{-1} + (x_u \cdot W)(1-x \cdot W)^{-2}] \rho. \]

Hence,
\[ \frac{\partial F_1}{\partial v} - \frac{\partial F_2}{\partial u} = (x_{uv} \cdot W - x_u \cdot W_u)(1-x \cdot W)^{-1} \]
\[ + [(x \cdot W_u)(x_u \cdot W) - (x \cdot W_u)(x_v \cdot W)](1-x \cdot W)^{-2} \rho, \]
and
\[ \frac{\partial F_1}{\partial \rho} \frac{\partial F_2}{\partial \rho} - \frac{\partial F_2}{\partial v} \frac{\partial F_1}{\partial u} = [(x_u \cdot W - x_v \cdot W_v)(1-x \cdot W)^{-1} \]
\[ + [(x \cdot W_u)(x_u \cdot W) - (x \cdot W_u)(x_v \cdot W)](1-x \cdot W)^{-2} \rho = 0. \]

Since we assume \(1-x \cdot W \neq 0\) and \(\rho > 0\), this is equivalent to
\[ (x_u \cdot W - x_v \cdot W_v)(1-x \cdot W) \]
\[ + [(x \cdot W_u)(x_u \cdot W) - (x \cdot W_u)(x_v \cdot W)] + [(x_u \cdot W)(x_u \cdot W) - (x_v \cdot W)(x_v \cdot W)] = 0, \]
which means
\[ (x_u \cdot W) - [(x \cdot W_u)(x_u \cdot W) - (x_v \cdot W)(x_v \cdot W)] = 0, \]  
(37)

We have
\[ W_u = \rho(\nabla V_1 \cdot x_u, \nabla V_2 \cdot x_u, \nabla V_3 \cdot x_u), \]
\[ W_v = \rho(\nabla V_1 \cdot x_v, \nabla V_2 \cdot x_v, \nabla V_3 \cdot x_v), \]
\[ W_p = (\nabla V_1 \cdot x, \nabla V_2 \cdot x, \nabla V_3 \cdot x). \]

Now
\[ x \cdot W_v = \rho \sum_{k=1}^{3} x_k(\nabla V_k \cdot x_v) = \rho \sum_{k=1}^{3} x_k \sum_{j=1}^{3} \frac{\partial V_k}{\partial y_j}(x_j)y_k, \]
\[ = \rho \sum_{k=1}^{3} \frac{\partial V_k}{\partial y_j}(x_j)y_k. \]

If we let
\[ A = \begin{pmatrix} \frac{\partial V_1}{\partial y_1} & \frac{\partial V_1}{\partial y_2} & \frac{\partial V_1}{\partial y_3} \\ \frac{\partial V_2}{\partial y_1} & \frac{\partial V_2}{\partial y_2} & \frac{\partial V_2}{\partial y_3} \\ \frac{\partial V_3}{\partial y_1} & \frac{\partial V_3}{\partial y_2} & \frac{\partial V_3}{\partial y_3} \end{pmatrix}, \]
then
\[ x \cdot W_v = \rho x A(x_v)^{T}, \]

where \(x, x_v\) are row vectors and \(t\) denotes the transpose. Similarly,
\[ x \cdot W_u = \rho x A(x_u)^{T}, \quad W_u = \rho x A(x_u)^{T}, \quad W_v = \rho x A(x_v)^{T}, \quad x_v \cdot W_v = x_v x A(x_v)^{T}. \]

Suppose \(V = \nabla \psi\), then \(A = \nabla^2 \psi\) is a symmetric matrix, so
\[ x_u \cdot W_v = x_v \cdot W_u \]
\[ (x \cdot W_v) - (x \cdot W_u) = (\rho - 1) x A(x_u)^{T} = \frac{\rho - 1}{\rho} (x \cdot W_u), \]
\[ (x \cdot W_u) - (x \cdot W_u) = (\rho - 1) x A(x_u)^{T} = \frac{\rho - 1}{\rho} (x \cdot W_u), \]
and Eq. (37) reads
\[ (\rho - 1)(x_u \cdot W_v) + (x_v \cdot W_u) = 0, \]
which can be written as
\[ \det \begin{pmatrix} x_u A(x_u)^{T} & x_u A(x_u)^{T} \\ x_u \cdot W_v & x_u \cdot W_v \end{pmatrix} = 0. \]

From the Cauchy–Binet formula for cross-products \([a \times b] \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)\), this means that
\[ (x_u \cdot x_v) \cdot (A x \times W) = 0, \]
and since \(x_u \times x_v \mid x, Eqs. (38) is equivalent to the following geometric condition:
\[ x \cdot (A x \times W) = 0. \]  
(39)

Equivalently,
\[ W \cdot (A x \times x) = A x \cdot (x \times W) = 0. \]  
(40)

Therefore, if the field \(V = \nabla \psi\), \(W\) is given in Eq. (36), and Eq. (38)—or equivalently, Eqs. (39) and (40)—holds in an open set \(O\) in the variables \((p, u, v)\), then for each \((p_0, u_0, v_0) \in O\), the system Eq. (33) has a unique solution \(p(u, v)\) defined in a
neighborhood of \((u_0, v_0)\) and satisfying the initial condition 
\[ \rho(u_0, v_0) = \rho_0. \]
Notice that if \(V = V_0\) is a constant field, then \(A = 0\), and so Eq. (38) obviously holds. In this case, Eq. (33) can be easily integrated, and the solution is 
\[ \rho(u, v) = \frac{C_1}{1 - x(u, v) \cdot (\kappa m + V_0)} + C_2 \]
with \(C_i\) constants.

Notice also that with the choice \(V\), as in Eq. (15), with \(h \neq 0\) so \(1 - x \cdot \nabla \neq 0\), the system of Eq. (33) becomes
\[
\begin{align*}
\rho_a(u, v) - \frac{\sin u}{\cos u} \rho(u, v) &= 0, \\
\rho_s(u, v) + \frac{\sin v}{\cos v} \rho(u, v) &= 0,
\end{align*}
\]
whose solution is 
\[ \rho(u, v) = \frac{C}{\cos u \sin v}, \]
where the constant \(C\) is determined by the point where the solution passes through. This is in agreement with Eq. (13).

7. Near Field Refracting Metasurfaces

The near field case can be regarded as a special case from Section 5, when the vector field \(m(u, v)\) points toward a fixed point \(Q\), and therefore, the method from that section can be used to derive conditions for the existence of the desired metasurface. In fact, if the surface \(\Gamma\) is parameterized by \(r(u, v)\) and \(m(u, v) = \frac{Q - r(u, v)}{|Q - r(u, v)|}\), then it is easy to see that the compatibility condition Eq. (20) holds. The existence of the phase discontinuity then follows when the determinant in Eq. (31) is not zero.

However, the phase discontinuities in the planar and spherical cases can be obtained explicitly as follows; see Fig. 4.

A. Case of a Plane Interface

Let \(O\) be the origin in medium I with index \(n_1\), and let \(Q = (q_1, q_2, q_3)\) be a point in medium II with index \(n_2\). Denote by \(\Gamma\) the plane with equation \(x_1 = a\) so that it separates the points \(O\) and \(Q\). We find the field \(V\) so that rays from \(O\) are refracted into \(Q\). We know from Section 4.A that \(\Gamma\) is given parametrically by Eq. (13); the normal \(\nu = (1, 0, 0)\). So we seek \(V\) such that Eq. (12) holds. Since the refracted vector from each point \(r(u, v)\) on the plane interface to the point \(Q\) has unit direction \(\frac{Q - r(u, v)}{|Q - r(u, v)|}\), \(V\) must satisfy

\[ \cos u \sin v - \kappa \frac{q_1 - a}{|Q - r(u, v)|} = \lambda + V_1 \]
\[ \sin u \sin v - \kappa \frac{q_2 - a \tan u}{|Q - r(u, v)|} = V_2 \]
\[ \cos v - \kappa \frac{q_3 - a / \cos u \tan v}{|Q - r(u, v)|} = V_3. \]

Rewriting these equations in rectangular coordinates yields
\[
\begin{align*}
\frac{a}{\sqrt{a^2 + x_2^2 + x_3^2}} = \kappa \frac{q_1 - a}{|Q - (x_1, x_2, x_3)|} & = \lambda + V_1, \\
\frac{x_2}{\sqrt{a^2 + x_2^2 + x_3^2}} = \kappa \frac{q_2 - x_2}{|Q - (x_1, x_2, x_3)|} & = V_2, \\
\frac{x_3}{\sqrt{a^2 + x_2^2 + x_3^2}} = \kappa \frac{q_3 - x_3}{|Q - (x_1, x_2, x_3)|} & = V_3.
\end{align*}
\]

Therefore, \(V_i, i = 1, 2, 3\), are determined:
\[
\begin{align*}
V_1(a, x_2, x_3) &= \partial_{x_1} \left( \sqrt{x_1^2 + x_2^2 + x_3^2} \right)_{x_1 = a} \\
&+ \kappa \frac{\partial}{\partial x_1} |Q - (x_1, x_2, x_3)|_{x_1 = a} - \lambda, \\
V_2(a, x_2, x_3) &= \partial_{x_2} \left( \sqrt{x_1^2 + x_2^2 + x_3^2} \right)_{x_1 = a} \\
&+ \kappa \frac{\partial}{\partial x_2} |Q - (x_1, x_2, x_3)|_{x_1 = a}, \\
V_3(a, x_2, x_3) &= \partial_{x_3} \left( \sqrt{x_1^2 + x_2^2 + x_3^2} \right)_{x_1 = a} \\
&+ \kappa \frac{\partial}{\partial x_3} |Q - (x_1, x_2, x_3)|_{x_1 = a},
\end{align*}
\]
where \(\lambda\) is chosen arbitrarily. Notice that if we let
\[ \psi(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2} + \kappa |Q - (x_1, x_2, x_3)| + C \]
with \(C\) a constant of integration, and choose \(\lambda = 0\), then \(V = \nabla \psi\), and so the plane with the phase discontinuity function \(\psi\) does the desired refraction job.

B. Case of a Spherical Interface

If \(\Gamma\) is the sphere of radius \(R\) centered at the origin, that is, \(r(u, v) = R(x(u, v))\), then the normal \(\nu = x\), and from Eq. (12), we get
\[ \left( x - \kappa \frac{Q - r(u, v)}{|Q - r(u, v)|} \right) \times x = 0. \]

As before, taking cross-product with \(x\) yields
\[ V + \kappa \frac{Q - r(u, v)}{|Q - r(u, v)|} \left[ \kappa \frac{Q - r(u, v)}{|Q - r(u, v)|} \cdot x \right] + V \cdot x \right] x = 0. \]

Assuming \(V\) is tangential to the sphere,
\[
\begin{align*}
V &= -\kappa \frac{Q - r(u, v)}{|Q - r(u, v)|} + \kappa \left( \frac{Q - r(u, v)}{|Q - r(u, v)|} \cdot x \right) x.
\end{align*}
\]
If $V(R(x, u, v)) = (\nabla \psi)(R(x, u, v))$, then

$$\psi_j(R(x, u, v)) = -\kappa \frac{q_j - R_j(x, u, v)}{|Q - R(x, u, v)|} + \kappa \left( \left| \frac{Q - R(x, u, v)}{|Q - R(x, u, v)|} \right| \right) x_j$$

$$j = 1, 2, 3.$$  \hfill (41)

Hence,

$$\frac{\partial}{\partial u} (\psi(R(x, u, v))) = (\nabla \psi)(R(x, u, v)) \cdot R_x$$

$$= -\kappa R \frac{Q - R(x, u, v)}{|Q - R(x, u, v)|} \cdot x_u,$$

and similarly,

$$\frac{\partial}{\partial v} (\psi(R(x, u, v))) = -\kappa R \frac{Q - R(x, u, v)}{|Q - R(x, u, v)|} \cdot x_v.$$  \hfill (42)

since $x \cdot x_u = x \cdot x_v = 0$. Since $\psi$ is assumed $C^2$, we get

$$\left( \frac{Q - R(x, u, v)}{|Q - R(x, u, v)|} \right)_u \cdot x_v = \left( \frac{Q - R(x, u, v)}{|Q - R(x, u, v)|} \right)_v \cdot x_u.$$  \hfill (43)

Integrating Eq. (42) in $u$ yields

$$\psi(R(x, u, v)) = -\kappa R \int \frac{Q - R(x', u', v)}{|Q - R(x', u', v)|} \cdot x_v(u', v) d u' + b(v)$$

for some function $b$. To calculate $b$, we differentiate the integral with respect to $v$ and use Eq. (44):

$$\frac{\partial}{\partial v} (\psi(R(x, u, v)))$$

$$= -\kappa R \int \frac{\partial}{\partial v} \left( \frac{Q - R(x', u', v)}{|Q - R(x', u', v)|} \cdot x_v(u', v) \right) d u' + b'(v)$$

$$= -\kappa R \int \left\{ \frac{\partial}{\partial v} \left( \frac{Q - R(x', u', v)}{|Q - R(x', u', v)|} \right) \cdot x_v(u', v)$$

$$+ \frac{Q - R(x', u', v)}{|Q - R(x', u', v)|} \cdot x_v(u', v) \right\} d u' + b'(v)$$

$$= -\kappa R \int \left\{ \frac{\partial}{\partial v} \left( \frac{Q - R(x', u', v)}{|Q - R(x', u', v)|} \right) \cdot x_v(u', v)$$

$$+ \frac{Q - R(x', u', v)}{|Q - R(x', u', v)|} \cdot x_v(u', v) \right\} d u' + b'(v)$$

$$= -\kappa R \int \left\{ \frac{\partial}{\partial u} \left( \frac{Q - R(x', u', v)}{|Q - R(x', u', v)|} \right) \cdot x_v(u', v)$$

$$+ \frac{Q - R(x', u', v)}{|Q - R(x', u', v)|} \cdot x_v(u', v) \right\} d u' + b'(v)$$

which implies $b'(v) = 0$ from Eq. (43). Therefore, the phase discontinuity $\psi$ on the sphere satisfies

$$\psi(R(x, u, v)) = -\kappa R \int \frac{Q - R(x', u', v)}{|Q - R(x', u', v)|} \cdot x_v(u', v) d u' + C$$

$$= \kappa \int \partial_j((Q - R(x', u', v)) d u' + C$$

$$= \kappa |Q - R(x, u, v)| + C,$$

with $C$ a constant. Writing this in rectangular coordinates yields

$$\psi(R(x_1, x_2, x_3)) = \kappa |Q - R(x_1, x_2, x_3)| + C,$$

for $|(x_1, x_2, x_3)| = 1$.

We now define $\psi$ on a neighborhood of $|z| = R$ so that Eq. (41) holds. Let

$$\psi(z) = \kappa \left| \frac{Q - R \frac{z}{|z|}}{|Q - R \frac{z}{|z|}|} \right| + C, \quad \text{for } R - \epsilon < |z| < R + \epsilon. \hfill (45)$$

We have

$$\nabla \psi(z) = -\kappa R \frac{Q - R \frac{z}{|z|}}{|Q - R \frac{z}{|z|}|} + \kappa \left( \frac{Q - R \frac{z}{|z|}}{|Q - R \frac{z}{|z|}|} \cdot \frac{z}{|z|^2} \right) \cdot x,$$

so for $z = R$, with $|x| = 1$, we obtain

$$\nabla \psi(R) = -\kappa \frac{Q - R}{|Q - R|} + \kappa \left( \frac{Q - R}{|Q - R|} \cdot x \right),$$

as desired. Therefore, the phase discontinuity $\psi$ in Eq. (45) has a gradient tangential to the sphere and can be placed on the spherical interface $|z| = R$ so that all rays from the origin are refracted into the point $Q$.

8. CONCLUSION

A rigorous mathematical foundation of general metasurfaces is provided. The starting point is the derivation of a generalized Snell’s law in the presence of a phase discontinuity using wave fronts. This is used also to derive all possible critical angles. We solve, under appropriate curvature type conditions on the surface $\Gamma$, the problem of finding a phase discontinuity, so that the pair (surface and phase discontinuity) refracts light in a desired manner. When a phase discontinuity is given, we derive conditions so that a surface is admissible for that phase discontinuity in the far field setting. Extensions to the case when the far field is a set of variable directions are given, and examples and explicit calculations of phase discontinuities are also provided. The near field case is also studied.

Funding. National Science Foundation (NSF) (DMS-1600578).

Acknowledgment. The first author was partially supported by the National Science Foundation. The authors thank the referees for useful comments and references. We also thank Professor Andrea Alù for useful comments and Refs. [16,17].

REFERENCES AND NOTES