1. Optimal transportation set up

Let \( D, D^* \) be two domains on \( S^{n-1} \) or bounded domains in a manifold (\( D \) might be contained in one manifold and \( D^* \) in another) with \( |\partial D| = 0 \).

Let \( N \) be a multi-valued mapping from \( \overline{D} \) onto \( \overline{D^*} \) such that \( N(x) \) is single-valued a.e. on \( \overline{D} \). For \( F \subset \overline{D^*} \), we set

\[
T(F) = N^{-1}(F) = \{ x \in \overline{D} : N(x) \cap F \neq \emptyset \}.
\]

We say \( N \) is measurable if \( T(F) \) is Lebesgue measurable for any Borel set \( F \subset \overline{D^*} \).

For example, if \( N = \partial u \) with \( u \) convex, then \( N \) is measurable (see exercises).

Given nonnegative \( g \in L^1(D) \) and a finite Radon measure \( \Gamma \) on \( \overline{D^*} \) satisfying

\[
\int_{\overline{D}} g(x) \, dx = \Gamma(\overline{D^*}) > 0,
\]

we say \( N \) is measure preserving from \( g(x) \, dx \) to \( \Gamma \) if for any Borel \( F \subset \overline{D^*} \)

\[
\int_{T(F)} g(x) \, dx = \Gamma(F).
\]

**Lemma 1.1.** \( N \) is a measure preserving mapping from \( g(x) \, dx \) to \( \Gamma \) if and only if for any \( v \in C(\overline{D^*}) \)

\[
\int_{\overline{D}} v(N(x)) g(x) \, dx = \int_{\overline{D^*}} v(m) \, d\Gamma(m).
\]

We remark that \( v(N(x)) \) is well defined for \( x \in \overline{D} \setminus S \) where \( N(x) \) is single-valued on \( \overline{D} \setminus S \) and \( |S| = 0 \), and \( \int_{\overline{D}} v(N(x)) g(x) \, dx \) is understood as \( \int_{\overline{D} \setminus S} v(N(x)) g(x) \, dx \).
Proof. Let \( N \) be a measure preserving mapping. To show (1.2), it suffices to prove it for \( v = \chi_F \), the characteristic function of a Borel set \( F \). It is easy to verify that 
\[
\chi_{TF}(x) = \chi_F(N(x)) \text{ for } x \in \overline{D} \setminus S.
\]
Therefore by (1.1)
\[
\int_{D^*} \chi_F(m) \, d\Gamma = \int_{T(F) \cap (D \setminus S)} g \, dx = \int_{D \setminus S} \chi_F(N(x))g(x) \, dx.
\]
To prove the converse, assume that (1.2) holds. We now show for any relatively open set \( G \) in \( D^* \)
(1.3)
\[
\int_{T(G)} g \, dx \leq \Gamma(G).
\]
Indeed, given a compact set \( K \subset G \), choose \( v \in C(D^*) \) such that \( 0 \leq v \leq 1 \), \( v = 1 \) on \( K \), and \( v = 0 \) outside \( G \). By (1.2), one gets
\[
\int_{T(K)} g(x) \, dx \leq \int_{D} v(N(x))g(x) \, dx \leq \Gamma(G),
\]
and (1.3) follows from arbitrariness of \( K \). Because a Borel set can be approximated by open sets, (1.3) is still valid for Borel sets \( F \) in \( D^* \). Noticing \((T(F))^c \subset T(F^c)\), to get the reverse inequality we apply (1.3) to \( D^* \setminus F \) and (1.1) follows. \( \square \)

Consider the general cost function \( c(x, m) \in \text{Lip}(\overline{D} \times D^*) \), the space of Lipschitz functions on \( \overline{D} \times D^* \), and the set of admissible functions

\[
\mathcal{K} = \{(u, v) : u \in C(\overline{D}), v \in C(D^*), u(x) + v(m) \leq c(x, m), \forall x \in D, \forall m \in D^* \}.
\]

Define the dual functional \( I \) for \((u, v) \in C(\overline{D}) \times C(D^*)\)
\[
I(u, v) = \int_{D} u(x)g(x) \, dx + \int_{D^*} v(m) \, d\Gamma,
\]
and define the \( c^- \) and \( c^+ \)-transforms
\[
\quad\quad\quad\quad\quad\quad\quad\quad u^c(m) = \inf_{x \in \overline{D}} \left[ c(x, m) - u(x) \right], \quad m \in D^*; \quad \quad\quad\quad\quad\quad\quad\quad v^c(x) = \inf_{m \in D^*} \left[ c(x, m) - v(m) \right], \quad x \in \overline{D}.
\]

**Definition 1.2.** A function \( \phi \in C(\overline{D}) \) is \( c \)-concave if for \( x_0 \in \overline{D} \), there exist \( m_0 \in \overline{D} \) and \( b \in \mathbb{R} \) such that \( \phi(x) \leq c(x, m_0) - b \) on \( \overline{D} \) with equality held at \( x = x_0 \).

Obviously \( v^c \) is \( c \)-concave for any \( v \in C(D^*) \). We collect the following properties:
(1) For any $u \in C(D)$ and $v \in C(D^*)$, $v_c \in Lip(D)$ and $u_c \in Lip(D^*)$ with Lipschitz constants bounded uniformly by the Lipschitz constant of $c$. Indeed, $(x_0$ the point where the minimum is attained)

$$u_c(m_1) - u_c(m_2) \leq u_c(m_1) - (c(x_0, m_2) - u(x_0)) \leq c(x_0, m_1) - u(x_0) - c(x_0, m_2) + u(x_0) \leq K|m_1 - m_2|.$$

(2) If $(u, v) \in \mathcal{K}$, then $v(m) \leq u_c(m)$ and $u(x) \leq v_c(x)$. Also $(v_c, v), (u, u_c) \in \mathcal{K}$.

(3) $\phi$ is $c$-concave iff $\phi = (\phi^c)_c$.

Indeed, if $\phi(x) \leq c(x, m_0) - b$ on $D$ and the equality holds at $x = x_0$, then $b = \phi^c(m_0)$. So $\phi(x_0) = c(x_0, m_0) - \phi^c(m_0)$ which yields $\phi(x_0) \geq (\phi^c)_c(x_0)$. On the other hand, from the definitions of $c$ and $c^*$ transforms we always have that $(\phi^c)_c \geq \phi$ for any $\phi$.

**Definition 1.3.** Given a function $\phi(x)$, the $c$-normal mapping of $\phi$ is defined by

$$N_{c, \phi}(x) = \{m \in D^* : \phi(x) + \phi^c(m) = c(x, m)\}, \quad \text{for } x \in D,$$

and $T_{c, \phi}(m) = N_{c, \phi}^{-1}(m) = \{x \in D : m \in N_{c, \phi}(x)\}$.

We assume that the cost function $c(x, m)$ satisfies the following:

(1.4) For any $c$-concave function $\phi$, $N_{c, \phi}(x)$ is single-valued a.e. on $D$

and $N_{c, \phi}$ is Lebesgue measurable.

Notice that if $c(x, m) = x \cdot m$, then $N_{c, \phi}(x) = \partial^\ast \phi(x)$, where $\partial^\ast \phi$ is the super-differential of $\phi$

$$\partial^\ast \phi(x) = \{m \in \mathbb{R}^n : \phi(y) \leq \phi(x) + m \cdot (y - x) \forall y \in \Omega\},$$

and we have $\partial^\ast \phi(x) = -\partial(-\phi)(x)$.

**Lemma 1.4.** Suppose that $c(x, m)$ satisfies the assumption (1.4). Then

(i) If $\phi$ is $c$-concave and $N_{c, \phi}$ is measure preserving from $g(x)dx$ to $\Gamma$, then $(\phi, \phi^c)$ is a maximizer of $I(u, v)$ in $\mathcal{K}$. 


Lemma 1.5. There exists a $c$-concave $\varphi$ such that $I(\phi, \phi^c) = \sup\{I(u, v) : (u, v) \in \mathcal{K}\}$.

Proof. First prove (i). Given $(u, v) \in \mathcal{K}$, obviously

$$u(x) + v(N_{c, \phi}(x)) \leq c(x, N_{c, \phi}(x)) = \phi(x) + \phi^c(N_{c, \phi}(x)), \quad \text{a.e. } x \text{ on } \overline{D}.$$ Integrating the above inequality with respect to $g(x) dx$ yields

$$\int_{\overline{D}} u g dx + \int_{\overline{D}} v(N_{c, \phi}(x)) g(x) dx \leq \int_{\overline{D}} \phi g dx + \int_{\overline{D}} \phi^c(N_{c, \phi}(x)) g(x) dx.$$ By Lemma 1.1, it yields $I(u, v) \leq I(\phi, \phi^c)$ and from (2) above the conclusion follows.

To prove (ii), let $\psi = \phi^c$, and for $v \in C(\overline{D}^c)$, let $\psi_\theta(m) = \psi(m) + \theta v(m)$ where $0 < |\theta| \leq \epsilon_0$ with $\epsilon_0$ small, and let $\phi_\theta = (\psi_\theta)_c$. It suffices to show

$$(1.5) \quad 0 = \lim_{\theta \to 0} \frac{I(\phi_\theta, \psi_\theta) - I(\phi, \psi)}{\theta} = \int_{\overline{D}} -v(N_{c, \phi}(x)) g dx + \int_{\overline{D}} v(m) d\Gamma.$$ Since $(\phi_\theta, \psi_\theta) \in \mathcal{K}, I(\phi_\theta, \psi_\theta) \leq I(\phi, \psi)$. So the limit must be zero if it exists. We have

$$\frac{I(\phi_\theta, \psi_\theta) - I(\phi, \psi)}{\theta} = \int_{\overline{D}} \frac{\phi_\theta - \phi}{\theta} g dx + \int_{\overline{D}} v(m) d\Gamma.$$ To prove (1.5), one only needs to show that $\frac{\phi_\theta(x) - \phi(x)}{\theta}$ is uniformly bounded and $\frac{\phi_\theta(x) - \phi(x)}{\theta} \to -v(N_{c, \phi}(x))$ for all $x \in D \setminus S$ where $N_{c, \phi}(x)$ is single-valued on $D \setminus S$ and $|S| = 0$. Indeed, for $x \in D \setminus S$, we have $\phi_\theta(x) = c(x, m_\theta) - \psi_\theta(m_\theta)$ and $\phi(x) = c(x, m_1) - \psi(m_1)$ for some $m_\theta, m_1 \in \overline{D}^c$. Then we get

$$-\theta v(m_\theta) \leq \phi_\theta(x) - \phi(x) \leq -\theta v(m_1).$$

Moreover, $m_1 = N_{c, \phi}(x)$ due to $\psi = \phi^c$. To finish the proof, we show that $m_\theta$ converges to $m_1$ as $\theta \to 0$. Otherwise, there exists a sequence $m_{\theta_k}$ such that $m_{\theta_k} \to m_\infty \neq m_1$. So $\phi(x) = c(x, m_\infty) - \psi(m_\infty)$, which yields $m_\infty \in N_{c, \phi}(x)$. We then get $m_1 = m_\infty$, a contradiction. The proof is complete. \qed

Lemma 1.5. There exists a $c$-concave $\phi$ such that $I(\phi, \phi^c) = \sup\{I(u, v) : (u, v) \in \mathcal{K}\}$. 
Proof. Let
\[ I_0 = \sup \{ I(u, v) : (u, v) \in \mathcal{K} \}, \]
and let \( (u_k, v_k) \in \mathcal{K} \) be a sequence such that \( I(u_k, v_k) \to I_0 \). Set \( \bar{u}_k = (v_k)_c \) and \( \bar{v}_k = (\bar{u}_k)^c \). From property (2) above, \( (\bar{u}_k, \bar{v}_k) \in \mathcal{K} \) and \( I(\bar{u}_k, \bar{v}_k) \to I_0 \). Let \( c_k = \min_D \bar{u}_k \) and define
\[ u^s_k = \bar{u}_k - c_k, \quad v^s_k = \bar{v}_k + c_k. \]
Obviously \( (u^s_k, v^s_k) \in \mathcal{K} \) and by the mass conservation of \( gdx \) and \( \Gamma \), \( I(\bar{u}_k, \bar{v}_k) = I(u^s_k, v^s_k) \). Since \( \bar{u}_k \) are uniformly Lipschitz, \( u^s_k \) are uniformly bounded. In addition, \( v^s_k = (\bar{u}_k)^c + c_k = (u^s_k)^c \) and consequently \( v^s_k \) are also uniformly bounded. By Arzelà-Ascoli’s theorem, \( (u^s_k, v^s_k) \) contains a subsequence converging uniformly to \( (\phi, \psi) \) on \( \overline{D} \times \overline{D} \). We then obtain that \( (\phi, \psi) \in \mathcal{K} \) and \( I_0 = \sup \{ I(u, v) : (u, v) \in \mathcal{K} \} = I(\phi, \psi) \).

Notice that this shows in particular that the supremum of \( I \) over \( \mathcal{K} \) is finite. From property (2) above, \((\psi^c, (\psi^c)^c)\) is the sought maximizer of \( I(u, v) \), and \( \psi^c \) is \( c\)-concave. \( \Box \)

Lemma 1.6. Suppose that \( c(x, m) \) satisfies the assumption (1.4). Let \( (\phi, \phi^c) \) with \( \phi \) \( c\)-concave be a maximizer of \( I(u, v) \) in \( \mathcal{K} \). Then \( \inf_{s \in S} \int_{\overline{D}} c(x, s(x))g(x)\,dx \) is attained at \( s = N_{c, \phi} \), where \( S \) is the class of measure preserving mappings from \( g(x)dx \) to \( \Gamma \). Moreover
\[
\tag{1.6} \inf_{s \in S} \int_{\overline{D}} c(x, s(x))g(x)\,dx = \sup \{ I(u, v) : (u, v) \in \mathcal{K} \}. 
\]

Proof. Let \( \psi = \phi^c \). For \( s \in S \), we have
\[
\int_{\overline{D}} c(x, s(x))g(x)\,dx \geq \int_{\overline{D}} (\phi(x) + \psi(s(x)))g(x)\,dx \\
= \int_{\overline{D}} \phi(x)g(x)\,dx + \int_{\overline{D}} \psi(s(x))g(x)\,dx \\
= \int_{\overline{D}} \phi(x)g(x)\,dx + \int_{\overline{D}} \psi(m)\,d\Gamma = I(\phi, \psi) \\
= \int_{\overline{D}} (\phi(x) + \psi(N_{c, \phi}(x)))g(x)\,dx, \text{ from Lemma 1.4(ii)} \\
= \int_{\overline{D}} c(x, N_{c, \phi}(x))g(x)\,dx.
\]
Obviously, for any $c$-concave function $\phi$, $N_{c,\phi}$ has the following converging property (C): if $m_k \in N_{c,\phi}(x_k)$, $x_k \rightarrow x_0$ and $m_k \rightarrow m_0$, then $m_0 \in N_{c,\phi}(x_0)$.

Lemma 1.7. Assume that $c(x, m)$ satisfies the assumption (1.4) and that $\int_G g \, dx > 0$ for any open $G \subset D$. Then the minimizing mapping of $\inf_{s \in \mathcal{S}} \int_D c(x, s(x)) g(x) \, dx$ is unique in the class of measure preserving mappings from $g(x) dx$ to $\Gamma$ with the converging property (C).

Proof. From Lemmas 1.5 and 1.6, let $N_{c,\phi}$ be a minimizing mapping associated with a maximizer $(\phi, \phi^c)$ of $I(u, v)$ with $\phi$-c-concave. Suppose that $N_0$ is another minimizing mapping with the converging property (C). Clearly

$$
\int_D \left( c(x, N_0(x)) - \phi(x) - \phi^c(N_0(x)) \right) g(x) \, dx = \inf_{s \in \mathcal{S}} \int_D c(x, s(x)) g(x) \, dx - \left( \int_D \phi(x) g(x) \, dx + \int_{\partial D} \phi^c(m) \, d\Gamma \right) = 0,
$$

and since $\phi(x) + \phi^c(N_0(x)) \leq c(x, N_0(x))$, it follows that $\phi(x) + \phi^c(N_0(x)) = c(x, N_0(x))$ on the set $\{ x \in D : g(x) > 0 \}$ which is dense in $D$. Hence from (1.4) and the converging property (C), we get $N_0(x) = N_{c,\phi}(x)$ a.e. on $D$. \hfill \Box

We remark from the above proof that if $g(x) > 0$ on $D$, then the minimizing mapping of $\inf_{s \in \mathcal{S}} \int_D c(x, s(x)) g(x) \, dx$ is unique in the class of measure preserving mappings from $g(x) dx$ to $\Gamma$.

2. The refractor problem $\kappa < 1$

Let $n_1$ and $n_2$ be the indexes of refraction of two homogeneous and isotropic media I and II, respectively. Suppose that from a point $O$ inside medium I light emanates with intensity $f(x)$ for $x \in \Omega$. We want to construct a refracting surface $\mathcal{R}$ parameterized as $\mathcal{R} = \{ \rho(x)x : x \in \overline{\Omega} \}$, separating media I and II, and such that all rays refracted by $\mathcal{R}$ into medium II have directions in $\Omega^*$ and the prescribed illumination intensity received in the direction $m \in \Omega^*$ is $f^*(m)$. 

\hfill \Box
We first introduce the notions of refractor mapping and measure, and weak solution. In the next section we then convert the refractor problem into an optimal mass transport problem from $\Omega$ to $\Omega^*$ with the cost function $\log \frac{1}{1 - \kappa x \cdot m}$ and establish existence and uniqueness of weak solutions.

Let $\Omega$, $\Omega^*$ be two domains on $S^{n-1}$, the illumination intensity of the emitting beam is given by nonnegative $f(x) \in L^1(\Omega)$, and the prescribed illumination intensity of the refracted beam is given by a nonnegative Radon measure $\mu$ on $\Omega^*$. Throughout this section, we assume that $|\partial \Omega| = 0$ and the physical constraint

\[ \inf_{x \in \Omega, m \in \Omega^*} x \cdot m \geq \kappa. \]

We further suppose that the total energy conservation

\[ \int_{\Omega} f(x) \, dx = \mu(\Omega^*) > 0, \]

and for any open set $G \subset \Omega$

\[ \int_{G} f(x) \, dx > 0, \]

where $dx$ denotes the surface measure on $S^{n-1}$.

2.1. **Refractor measure and weak solutions.** We begin with the notions of refractor and supporting semi-ellipsoid.

**Definition 2.1.** A surface $R$ parameterized by $\rho(x) \in C(\overline{\Omega})$ is a refractor from $\overline{\Omega}$ to $\overline{\Omega}^*$ for the case $\kappa < 1$ (often simply called as refractor in this section) if for any $x_0 \in \overline{\Omega}$ there exists a semi-ellipsoid $E(m, b)$ with $m \in \overline{\Omega}^*$ such that $\rho(x_0) = \frac{b}{1 - \kappa m \cdot x_0}$ and $\rho(x) \leq \frac{b}{1 - \kappa m \cdot x}$ for all $x \in \overline{\Omega}$. Such $E(m, b)$ is called a supporting semi-ellipsoid of $R$ at the point $\rho(x_0)x_0$.

From the definition, any refractor is globally Lipschitz on $\overline{\Omega}$.

**Definition 2.2.** Given a refractor $R = \{\rho(x) : x \in \overline{\Omega}\}$, the refractor mapping of $R$ is the multi-valued map defined by for $x_0 \in \overline{\Omega}$

\[ \mathcal{N}_R(x_0) = \{m \in \overline{\Omega}^* : E(m, b) \text{ supports } R \text{ at } \rho(x_0)x_0 \text{ for some } b > 0\}. \]
Given \( m_0 \in \overline{\Omega} \), the tracing mapping of \( \mathcal{R} \) is defined by
\[
\mathcal{T}_\mathcal{R}(m_0) = N_\mathcal{R}^{-1}(m_0) = \{ x \in \overline{\Omega} : m_0 \in N_\mathcal{R}(x) \}.
\]

**Definition 2.3.** Given a refractor \( \mathcal{R} = \{ \rho(x) : x \in \overline{\Omega} \} \), the Legendre transform of \( \mathcal{R} \) is defined by
\[
\mathcal{R}^* = \{ \rho^*(m) : \rho^*(m) = \inf_{x \in \Omega} \frac{1}{\rho(x)(1 - \kappa x \cdot m)} , m \in \overline{\Omega} \}.
\]

We now give some basic properties of Legendre transforms.

**Lemma 2.4.** Let \( \mathcal{R} \) be a refractor from \( \overline{\Omega} \) to \( \overline{\Omega}^* \). Then

(i) \( \mathcal{R}^* \) is a refractor from \( \overline{\Omega}^* \) to \( \overline{\Omega} \).

(ii) \( \mathcal{R}^{**} = (\mathcal{R}^*)^* = \mathcal{R} \).

(iii) If \( x_0 \in \overline{\Omega} \) and \( m_0 \in \overline{\Omega}^* \), then \( x_0 \in N_\mathcal{R}(m_0) \) iff \( m_0 \in N_\mathcal{R}(x_0) \).

**Proof.** Given \( m_0 \in \overline{\Omega}^* \), \( \rho(x)(1 - \kappa x \cdot m) \) must attain the maximum over \( \overline{\Omega} \) at some \( x_0 \in \overline{\Omega} \). Then \( \rho^*(m_0) = 1/[\rho(x_0)(1 - \kappa x_0 \cdot m_0)] \). We always have
\[
(2.4) \quad \rho^*(m) = \inf_{x \in \overline{\Omega}} \frac{1}{\rho(x)(1 - \kappa m \cdot x)} \leq \frac{1}{\rho(x_0)(1 - \kappa x_0 \cdot m)}, \quad \forall m \in \overline{\Omega}.
\]
Hence \( E(x_0, 1/\rho(x_0)) \) is a supporting semi-ellipsoid to \( \mathcal{R}^* \) at \( \rho^*(m_0)m_0 \). Thus, (i) is proved.

To prove (ii), from the definitions of Legendre transform and refractor mapping we have
\[
(2.5) \quad \rho(x_0) \rho^*(m_0) = \frac{1}{1 - \kappa m_0 \cdot x_0} \quad \text{for} \quad m_0 \in N_\mathcal{R}(x_0).
\]
For \( x_0 \in \overline{\Omega} \), there exists \( m_0 \in N_\mathcal{R}(x_0) \) and so from (2.5) \( \rho^*(m_0) = \frac{1/\rho(x_0)}{1 - \kappa x_0 \cdot m_0} \). By (2.4), \( \rho^*(m)(1 - kx_0 \cdot m) \) attains the maximum \( 1/\rho(x_0) \) at \( m_0 \). Thus,
\[
\rho^{**}(x_0) = \inf_{m \in \overline{\Omega}} \frac{1}{\rho^*(m)(1 - kx_0 \cdot m)} = \frac{1}{\rho(x_0)^{-1}}.
\]

To prove (iii), we get from the proof of (ii) that if \( m_0 \in N_\mathcal{R}(x_0) \), then the semi-ellipsoid \( E(x_0, 1/\rho(x_0)) \) supports \( \mathcal{R}^* \) at \( \rho^*(m_0)m_0 \) and so \( x_0 \in N_\mathcal{R}(m_0) \). On the other hand, if \( x_0 \in N_\mathcal{R}(m_0) \), we get that \( m_0 \in N_\mathcal{R}^{**}(x_0) \), and since \( \mathcal{R}^{**} = \mathcal{R} \), \( m_0 \in N_\mathcal{R}(x_0) \). □
The next two lemmas discuss the refractor measure.

**Lemma 2.5.** \( C = \{ F \subset \Omega^c : \mathcal{T}_R(F) \text{ is Lebesgue measurable} \} \) is a \( \sigma \)-algebra containing all Borel sets in \( \Omega^c \).

**Proof.** Obviously, \( \mathcal{T}_R(\emptyset) = \emptyset \) and \( \mathcal{T}_R(\Omega^c) = \Omega^c \). Since \( \mathcal{T}_R(\cup_{i=1}^{\infty} F_i) = \cup_{i=1}^{\infty} \mathcal{T}_R(F_i) \), \( C \) is closed under countable unions. Clearly for \( F \subset \Omega^c \)

\[
\mathcal{T}_R(F^c) = \{ x \in \Omega^c : N_R(x) \cap F^c \neq \emptyset \}
= \{ x \in \Omega^c : N_R(x) \cap F = \emptyset \} \cup \{ x \in \Omega^c : N_R(x) \cap F^c \neq \emptyset, N_R(x) \cap F \neq \emptyset \}
(2.6) \quad = [\mathcal{T}_R(F)]^c \cup [\mathcal{T}_R(F^c) \cap \mathcal{T}_R(F)].
\]

If \( x \in \mathcal{T}_R(F^c) \cap \mathcal{T}_R(F) \cap \Omega \), then \( \mathcal{R} \) parameterized by \( \rho \) has two distinct supporting semi-ellipsoids \( E(m_1, b_1) \) and \( E(m_2, b_2) \) at \( \rho(x)x \). We show that \( \rho(x)x \) is a singular point of \( \mathcal{R} \). Otherwise, if \( \mathcal{R} \) has the tangent hyperplane \( \Pi \) at \( \rho(x)x \), then \( \Pi \) must coincide both with the tangent hyperplane of \( E(m_1, b_1) \) and that of \( E(m_2, b_2) \) at \( \rho(x)x \). It follows from the Snell law that \( m_1 = m_2 \). Therefore, the area measure of \( \mathcal{T}_R(F^c) \cap \mathcal{T}_R(F) \) is 0. So \( C \) is closed under complements, and we have proved that \( C \) is a \( \sigma \)-algebra.

To prove that \( C \) contains all Borel subsets, it suffices to show that \( \mathcal{T}_R(K) \) is compact if \( K \subset \Omega^c \) is compact. Let \( x_i \in \mathcal{T}_R(K) \) for \( i \geq 1 \). There exists \( m_i \in N_R(x_i) \cap K \). Let \( E(m_i, b_i) \) be the supporting semi-ellipsoid to \( \mathcal{R} \) at \( \rho(x_i)x_i \). We have

\[
\rho(x)(1 - \kappa m_i \cdot x) \leq b_i \quad \text{for } x \in \Omega,
(2.7)
\]

where the equality in \( (2.7) \) occurs at \( x = x_i \). Assume that \( a_1 \leq \rho(x) \leq a_2 \) on \( \Omega \) for some constants \( a_2 \geq a_1 > 0 \). By \( (2.7) \) and \( (2.1) \), \( a_1(1 - \kappa) \leq b_i \leq a_2(1 - \kappa^2) \). Assume through subsequence that \( x_i \to x_0, m_i \to m_0 \in K, b_i \to b_0 \), as \( i \to \infty \). By taking limit in \( (2.7) \), one obtains that the semi-ellipsoid \( E(m_0, b_0) \) supports \( \mathcal{R} \) at \( \rho(x_0)x_0 \) and \( x_0 \in \mathcal{T}_R(m_0) \). This proves \( \mathcal{T}_R(K) \) is compact. □

**Lemma 2.6.** Given a nonnegative \( f \in L^1(\Omega) \), the set function

\[
\mathcal{M}_{R,f}(F) = \int_{\mathcal{T}_R(F)} f \, dx
\]
is a finite Borel measure defined on $C$ and is called the refractor measure associated with $R$ and $f$.

**Proof.** Let $\{F_i\}_{i=1}^\infty$ be a sequence of pairwise disjoint sets in $C$. Let $H_1 = T_R(F_1)$, and $H_k = T_R(F_k) \setminus \bigcup_{i=1}^{k-1} T_R(F_i)$, for $k \geq 2$. Since $H_i \cap H_j = \emptyset$ for $i \neq j$ and $\bigcup_{k=1}^\infty H_k = \bigcup_{k=1}^\infty T_R(F_k)$, it is easy to get

$$M_{R,f}(\bigcup_{k=1}^\infty F_k) = \int_{\bigcup_{k=1}^\infty H_k} f \, dx = \sum_{k=1}^\infty \int_{H_k} f \, dx.$$ 

Observe that $T_R(F_k) \setminus H_k = T_R(F_k) \cap (\bigcup_{i=1}^{k-1} T_R(F_i))$ is a subset of the singular set of $R$ and has area measure 0 for $k \geq 2$. Therefore, $\int_{H_k} f \, dx = M_{R,f}(F_k)$ and the $\sigma$-additivity of $M_{R,f}$ follows. \hfill \qed

The notion of weak solutions is introduced through the conservation of energy.

**Definition 2.7.** A refractor $R$ is a weak solution of the refractor problem for the case $\kappa < 1$ with emitting illumination intensity $f(x)$ on $\Omega$ and prescribed refracted illumination intensity $\mu$ on $\Omega^*$ if for any Borel set $F \subset \Omega^*$

$$M_{R,f}(F) = \int_{T_R(F)} f \, dx = \mu(F).$$

2.2. Solution of the refractor problem. We introduce the cost

$$c(x,m) = \frac{1}{1 - \kappa x \cdot m}$$

for $x \in \Omega$ and $m \in \Omega^*$ where we assume $\Omega \cdot \Omega^* \geq \kappa$. From Definitions 1.2 and 2.1 $R = \{\rho(x) x : x \in \Omega\}$ is a refractor iff $\log \rho$ is $c$-concave. Using Definitions 1.3 and 2.2 we get that

$$N_{c,\phi}(x) = N_R(x), \quad R = \{\rho(x)x : x \in \Omega\}, \quad \rho(z) = e^{\phi(z)}.$$ 

Furthermore, $\log \rho^* = (\log \rho)^\circ$, $\log \rho = (\log \rho)^*$, by Remark (3) after Definition 1.2 and $N_R(x_0) = N_{c,\log \rho}(x_0)$ by (2.5). By the Snell law and Lemma 2.5 $c(x,m)$ satisfies (1.4). From the definitions, $R$ is a weak solution of the refractor problem iff $\log \rho$ is $c$-concave and $N_{c,\log \rho}$ is a measure preserving mapping from $f(x)dx$ to $\mu$. }
By Lemma 1.5 there exists a $c$-concave $\phi(x)$ such that $(\phi, \phi^c)$ maximizes

$$I(u, v) = \int_{\Omega} u f \, dx + \int_{\Omega^*} v \, d\mu(m)$$

in $\mathcal{K} = \{(u, v) \in C(\overline{\Omega}) \times C(\overline{\Omega}^*) : u(x) + v(m) \leq c(x, m), \text{ for } x \in \overline{\Omega}, m \in \overline{\Omega^*}\}$. Then by Lemma 1.4, $\mathcal{N}_{c, \phi}(x)$ is a measure preserving mapping from $f dx$ to $\mu$. Therefore, $\mathcal{R} = \{e^{\phi(x)} x : x \in \overline{\Omega}\}$ is a weak solution of the refractor problem.

It remains to prove the uniqueness of solutions up to dilations. Let $\mathcal{R}_i = \{\rho_i(x) x : x \in \overline{\Omega}\}, i = 1, 2$, be two weak solutions of the refractor problem. Obviously, $\mathcal{N}_{c, \log \rho_i}$ have the converging property (C) stated before Lemma 1.7. It follows from Lemmas 1.4, 1.6 and 1.7 that $\mathcal{N}_{c, \log \rho_1}(x) = \mathcal{N}_{c, \log \rho_2}(x)$ a.e. on $\Omega$. That is, $\mathcal{N}_{\mathcal{R}_1}(x) = \mathcal{N}_{\mathcal{R}_2}(x)$ a.e. on $\Omega$. From the Snell law $\nu_i(x) = \frac{x - \kappa \mathcal{N}_{\mathcal{R}_i}(x)}{|x - \kappa \mathcal{N}_{\mathcal{R}_i}(x)|}$ is the unit normal to $\mathcal{R}_i$ towards medium II at $\rho_i(x) x$ where $\mathcal{R}_i$ is differentiable. So $\nu_1(x) = \nu_2(x)$ a.e. and consequently $\rho_1(x) = C \rho_2(x)$ for some $C > 0$.

References

Department of Mathematics, Temple University, Philadelphia, PA 19122

E-mail address: gutierre@temple.edu