You may not consult in any form with any other person, except your instructor, while doing this take-home test.

1. Solve the Cauchy problem

\[ x^2 u_x + y^2 u_y = u^2, \quad u(x, 2x) = 1. \]

2. Let \( \alpha \in \mathbb{R} \).

(a) If \( f \in C^1(\mathbb{R}) \), then show that the function \( u(x, y) = f(x - \alpha y) \) is a solution of the first order pde

\[ \alpha u_x + u_y = 0. \]

(b) If \( f \in C(\mathbb{R}) \), then show that \( u(x, y) = f(x - \alpha y) \) is a weak solution, that is, for every \( \phi \in C^1_0(\mathbb{R}^2) \)

\[ \int \int (\alpha \phi_x(x, y) + \phi_y(x, y)) u(x, y) \, dx \, dy = 0. \]

HINT: make the change of variables \( x' = x - \alpha y, \quad y' = y \) and integrate.

3. Suppose \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with smooth boundary. If \( u \in C^2(\Omega) \) satisfies the pde

\[ (e^x u_x)_x + (e^y u_y)_y = 0 \]

in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \), then \( u = 0 \) in \( \Omega \).

HINT: use the divergence theorem with the field \( F(x, y) = u(x, y)(e^x u_x(x, y), e^y u_y(x, y)) \).

4. Let \( u \) be harmonic in \( \Omega \) bounded regular domain in \( \mathbb{R}^n \), \( u \in C(\bar{\Omega}) \). Prove that

\[ |Du(x)| \leq \frac{C_n}{\text{dist}(x, \partial \Omega)} \max_{\Omega} |u| \]

for all \( x \in \Omega \), where \( C_n \) is a constant depending only on the dimension \( n \).

HINT: use that the derivatives of \( u \) are also harmonic, the mean value property and the divergence theorem.

5. Let \( u \in C^2(\bar{B}_n(0)) \) in \( \mathbb{R}^3 \) be a solution of \( \Delta u + u = w \) where \( w \) is continuous and \( u = 0 \) on \( \partial B_n(0) \). Prove that

\[ \int_{B_n(0)} w(x) \frac{\sin |x|}{|x|} \, dx = 0. \]

Hint: Use the second Green formula and notice that \( \Delta \frac{\sin |x|}{|x|} = -\frac{\sin |x|}{|x|} \).

6. Prove that the solid and surface mean value properties for harmonic functions are equivalent. That is,
(a) If \( u \in C(\Omega) \) and \( u(x_0) = \frac{1}{\text{area}(B_R(x_0))} \int_{\partial B_R(x_0)} u(x) \, d\sigma(x) \) for each ball \( B_R(x_0) \subset \Omega \), then
\[
u(x_0) = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u(x) \, dx.
\]

(b) If \( u \in C(\Omega) \) and \( u(x_0) = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u(x) \, dx \) for each ball \( B_R(x_0) \subset \Omega \), then \( u(x_0) = \frac{1}{\text{area}(B_R(x_0))} \int_{\partial B_R(x_0)} u(x) \, d\sigma(x) \).

HINT: For (a) use the formula of integration in polar coordinates: if \( f \) is a continuous and integrable function in \( \mathbb{R}^n \), then
\[
\int_{\mathbb{R}^n} f(x) \, dx = \int_0^\infty \left( \int_{|x|=r} f(x) \, d\sigma(x) \right) \, dt = \int_0^\infty t^{n-1} \left( \int_{|x|=1} f(tx) \, d\sigma(x) \right) \, dt.
\]
Use the fact that if \( \Omega_n \) and \( \omega_{n-1} \) denote the volume and surface area of the unit sphere in \( \mathbb{R}^n \) respectively, then \( \omega_{n-1} = n \, \Omega_n \).

For (b) differentiate \( \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} u(x) \, dx \) with respect to \( R \), and show once again integrating in polar coordinates that \( \int \frac{d}{dR} \int_{B_R(x_0)} u(x) \, dx = \int_{\partial B_R(x_0)} u(x) \, d\sigma(x) \).