SOLUTION OF POISSON'S EQUATION

CRISTIAN E. GUTIÉRREZ
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Contents

1. Differentiation under the integral sign

A pretty general result that is often useful is the following.

Theorem 1. *Let $k(x, y)$ be a function defined for all $x \in \mathcal{O} \subset \mathbb{R}^m$ open and for a.e. $y \in \mathbb{R}^n$ such that $k(x, \cdot) \in L^1(\mathbb{R}^n)$ for all $x \in \mathcal{O}$. Then*

(a) *The function*

$$u(x) = \int_{\mathbb{R}^n} k(x, y)\, dy$$

is well defined for all $x \in \mathcal{O}$.

(b) *Fix $1 \le i \le m$. Assume that $\partial_{x_i} k(x, y)$ exists for all $x \in \mathcal{O}$ and for almost all $y \in \mathbb{R}^n$, and there exists $g \in L^1(\mathbb{R}^n)$ such that*

$$|\partial_{x_i} k(x, y)| \le g(y), \qquad \forall x \in \mathcal{O}, \qquad \text{and for almost all } y \in \mathbb{R}^n.$$

Assume further that

$$\partial_{x_i} k(\cdot, y) \in C(\mathcal{O}) \quad \text{for almost all } y \in \mathbb{R}^n.$$

Then the derivative $\partial_{x_i} u$ exists for all $x \in \mathcal{O}$ and

$$\partial_{x_i} u(x) = \int_{\mathbb{R}^n} \partial_{x_i} k(x,y)\,dy,$$

for all $x \in \mathcal{O}$.

(c) *If the gradient $D_x k(\cdot, y) \in C(\mathcal{O})$ and there exists $g \in L^1(\mathbb{R}^n)$ such that*

$$|D_x k(x,y)| \le g(y), \qquad \forall x \in \mathcal{O}, \qquad \text{and for almost all } y \in \mathbb{R}^n,$$

then

$$D_x u(x) = \int_{\mathbb{R}^n} D_x k(x,y)\,dy,$$

for all $x \in \mathcal{O}$.

(d) *If $\partial_{x_j}\partial_{x_i} k(\cdot, y) \in C(\mathcal{O})$ for a.e. $y \in \mathbb{R}^n$, and there exists $g \in L^1(\mathbb{R}^n)$ such that*

$$|\partial_{x_j}\partial_{x_i} k(x,y)| \le g(y), \qquad \forall x \in \mathcal{O}, \qquad \text{and for almost all } y \in \mathbb{R}^n,$$

then

$$\partial_{x_j}\partial_{x_i} u(x) = \int_{\mathbb{R}^n} \partial_{x_j}\partial_{x_i} k(x,y)\,dy.$$

Proof. It follows immediately writing down the differential increment, using the mean value theorem, the continuity assumption and the Lebesgue dominated convergence theorem. □

2. The Newtonian potential is C^1

Theorem 2. *If $\rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then*

$$w(x) = \int_{\mathbb{R}^n} \rho(y)\Gamma(x-y)\,dy$$

is well defined at each $x \in \mathbb{R}^n$ and $w \in C^1(\mathbb{R}^n)$.

Proof. Let $\eta(t)$ be a smooth function such that $\eta(t) = 0$ for $t \le 1$, $0 \le \eta \le 1$ everywhere, and $\eta(t) = 1$ for $t \ge 2$. Let $n \ge 3$, $\Gamma(x) = \dfrac{1}{|x|^{n-2}}$ (for $n = 2$, $\Gamma(x) = \log|x|$) and consider the function

$$\Gamma_\epsilon(x) = \Gamma(x)\,\eta(|x|/\epsilon),$$

where $\epsilon > 0$.

Step 1 If $\rho \in L^1(\mathbb{R}^n)$, then the function

$$w_\epsilon(x) = \int_{\mathbb{R}^n} \rho(y)\Gamma_\epsilon(x-y)\,dy$$

is well defined for all $x \in \mathbb{R}^n$. This follows inmediately since Γ_ϵ is bounded.

Step 2 If $\rho \in L^1(\mathbb{R}^n)$, then $w_\epsilon \in C^1(\mathbb{R}^n)$. First prove using the previous theorem that w_ϵ has derivative at each point taking $k(x,y) = \rho(y)\Gamma_\epsilon(x-y)$. Notice that the function $F(x,y) = |D_x\left(\Gamma(x-y)\eta\left(|x-y|/\epsilon\right)\right)| \le C(\epsilon)$ for $\epsilon \le |x-y| \le 2\epsilon$ and $F(x,y) = 0$ otherwise. Therefore $|D_x k(x,y)| \le C(\epsilon)\rho(y)$ for all $x, y \in \mathbb{R}^n$. Second we show that $\partial_{x_i} w(x)$ is continuous in x for each ϵ fixed, $i = 1, \cdots, n$. Write

$$\begin{aligned}\partial_j w_\epsilon(x) - \partial_j w_\epsilon(z) &= \int_{\mathbb{R}^n} \rho(y)\left(\partial_{x_j}\Gamma_\epsilon(x-y) - \partial_{x_j}\Gamma_\epsilon(z-y)\right) dy \\ &= \int_{\mathbb{R}^n} \rho(y)\left(D\left(\partial_{x_j}\Gamma_\epsilon\right)(tx + (1-t)z - y)\cdot(x-z)\right) dy.\end{aligned}$$

Differentiating we obtain as before that $|D\left(\partial_{x_j}\Gamma_\epsilon\right)(x)| \le C(\epsilon)$ for all $x \in \mathbb{R}^n$, and therefore

$$|\partial_j w_\epsilon(x) - \partial_j w_\epsilon(z)| \le C(\epsilon)\int_{\mathbb{R}^n} |\rho(y)|\, dy\, |x-z|$$

and the continuity of $\partial_j w_\epsilon$ follows.

Step 3 If $\rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then

$$w_\epsilon(x) \to w(x) = \int_{\mathbb{R}^n} \rho(y)\Gamma(x-y)\, dy$$

uniformly for $x \in \mathbb{R}^n$, where $w(x)$ is well defined for all x. Clearly, $w(x)$ is well defined since Γ is integrable near the origin, Γ is bounded away from the origin, ρ is bounded and integrable. Write

$$w(x) - w_\epsilon(x) = \int_{|x-y|\le 2\epsilon} \rho(y)\,\Gamma(x-y)\,\left(1 - \eta(|x-y|/\epsilon)\right)\, dy.$$

So

$$|w(x) - w_\epsilon(x)| \le \int_{|x-y|\le 2\epsilon} |\rho(y)|\,\Gamma(x-y)\, dy \le C\,\|\rho\|_\infty\,\epsilon^2.$$

Step 4 If $\rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then

$$Dw_\epsilon \to v$$

uniformly in $\mathbb{R}^n$, where

$$v(x) = \int_{\mathbb{R}^n} \rho(y) D_x(\Gamma(x-y))\, dy = (v_1(x), \cdots, v_n(x)),$$

where the last integral is well defined. It is clear that v is well defined. Write

$$
\begin{aligned}
&\partial_j w_\epsilon(x) - v_j(x)\\
&= \int_{\mathbb{R}^n} \rho(y)\left(\partial_{x_j}\Gamma_\epsilon(x-y) - \partial_{x_j}\Gamma(x-y)\right)dy\\
&= \int_{\mathbb{R}^n} \rho(y)\left(\partial_{x_j}\Gamma(x-y)\eta(|x-y|/\epsilon) + \Gamma(x-y)\partial_{x_j}\left(\eta(|x-y|/\epsilon)\right) - \partial_{x_j}\Gamma(x-y)\right)dy\\
&= \int_{\mathbb{R}^n} \rho(y)\left(\partial_{x_j}\Gamma(x-y)\left(\eta(|x-y|/\epsilon)-1\right) + \Gamma(x-y)\partial_{x_j}\left(\eta(|x-y|/\epsilon)\right)\right)dy.
\end{aligned}
$$

So

$$
\begin{aligned}
&|\partial_j w_\epsilon(x) - v_j(x)|\\
&\leq \int_{|x-y|\leq 2\epsilon} |\rho(y)|\left(|\partial_{x_j}\Gamma(x-y)| + |\Gamma(x-y)|\frac{1}{\epsilon}\eta'(|x-y|/\epsilon)\right)dy\\
&\leq C_n\|\rho\|_\infty \int_{|x-y|\leq 2\epsilon}\left(\frac{1}{|x-y|^{n-1}} + \frac{1}{|x-y|^{n-2}}\frac{1}{\epsilon}\right)dy = C\,\|\rho\|_\infty\,\epsilon.
\end{aligned}
$$

Step 5 As a consequence of steps 2-4, the functions w and v are continous when $\rho \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$.

Step 6 If $\rho \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, then $w \in C^1(\mathbb{R}^n)$, and $Dw(x) = v(x)$.

By the fundamental theorem of calculus,

$$
w_\epsilon(x_1, x_2, \cdots, x_n) = \int_{y_1}^{x_1} \partial_{x_1} w_\epsilon(t, x_2, \cdots, x_n)\,dt + w_\epsilon(y_1, x_2, \cdots, x_n),
$$

and from the uniform convergence

$$
w(x_1, x_2, \cdots, x_n) = \int_{y_1}^{x_1} v_1(t, x_2, \cdots, x_n)\,dt + w(y_1, x_2, \cdots, x_n).
$$

□

3. The Newtonian potential from the 3rd Green formula

Let $n \geq 3$ and $\Gamma(x) = \dfrac{1}{\omega_{n-1}(2-n)}|x|^{2-n}$, where ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$. The third Green formula reads

$$
u(y) = \int_{\partial\Omega}\left(u(x)\frac{\partial\Gamma(x-y)}{\partial\eta(x)} - \Gamma(x-y)\frac{\partial u}{\partial\eta}(x)\right)d\sigma(x) + \int_\Omega \Gamma(x-y)\,\Delta u(x)\,dx,
$$

where $u \in C^2(\Omega)\cap C^1(\bar{\Omega})$, and $\Delta u \in L^1(\Omega)$ and the domain Ω is sufficiently regular.

Suppose $u \in C^2(\mathbb{R}^n)$ solves the Poisson equation

$$
\Delta u = \rho
$$

in all space, with $\rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Applying the third Green formula in the ball $B_R(y)$ yields

$$u(y) = \int_{\partial B_R(y)} \left(u(x) \frac{\partial \Gamma(x-y)}{\partial \eta(x)} - \Gamma(x-y) \frac{\partial u}{\partial \eta}(x) \right) d\sigma(x) + \int_{B_R(y)} \Gamma(x-y)\, \rho(x)\, dx.$$

Suppose in addition that

$$|u(x)| = O(1/|x|), \qquad |Du(x)| = O(1/|x|^2),$$

when $|x| \to \infty$. Then, when $R \to \infty$, the surface integral tends to zero and we obtain

$$u(y) = \int_{\mathbb{R}^n} \Gamma(x-y)\, \rho(x)\, dx,$$

for each $y \in \mathbb{R}^n$.

4. Second derivatives

We assume $f : \Omega \to \mathbb{R}$ is locally Hölder continuous, that is, there exists $0 < \alpha \leq 1$ such that for each $K \subset \Omega$ compact there is a constant $C_K > 0$ such that

$$|f(x) - f(y)| \leq C_K\, |x-y|^\alpha, \quad \forall x, y \in K.$$

We also assume f is bounded in Ω.

The goal is to prove the representation formula

$$D_{ij}w(x) = \int_{\Omega_0} (D_{ij}\Gamma)(x-y)(\bar{f}(y) - f(x))\, dy - f(x) \int_{\partial\Omega_0} D_i\Gamma(x-y)\nu_j(y)\, d\sigma(y), \tag{1}$$

for the second derivatives of the Newtonian potential, where Ω_0 is any domain for which the divergence theorem holds, with $\Omega \subset \Omega_0$; and $\bar{f}(x) = f(x)$ for $x \in \Omega$ and $\bar{f}(x) = 0$ in $\Omega_0 \setminus \Omega$.

Assume $n \geq 3$, let $\Gamma(x) = C_n|x|^{2-n}$ be the fundamental solution, $\eta : \mathbb{R} \to \mathbb{R}$ is smooth, $\eta(t) = 0$ for $t \leq 1$, $0 \leq \eta \leq 1$, $\eta(t) = 1$ for $t \geq 2$ and $0 \leq \eta' \leq 2$.

Define for $x \in \Omega$

$$u(x) = \int_{\Omega_0} (D_{ij}\Gamma)(x-y)(\bar{f}(y) - f(x))\, dy - f(x) \int_{\partial\Omega_0} D_i\Gamma(x-y)\nu_j(y)\, d\sigma(y).$$

Since $|D_{ij}\Gamma(x)| \leq C_n|x|^{-n}$, and f is bounded and locally Hölder continuous, then the function $u(x)$ is well defined for all $x \in \Omega$. Let $\epsilon > 0$ and define

$$v_\epsilon(x) = \int_\Omega D_i\Gamma(x-y)\eta((x-y)/\epsilon) f(y)\, dy,$$

and let $v(x) = D_i w(x)$, where

$$w(x) = \int_\Omega \Gamma(x-y) f(y)\, dy.$$

If f is integrable in Ω, then, as in Theorem 2, $v_\epsilon \in C^1(\mathbb{R}^n)$, and if in addition f is bounded, $w \in C^1(\mathbb{R}^n)$. We have

$$\begin{aligned}
D_j v_\epsilon(x) &= \int_\Omega D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right) f(y)\, dy \\
&= \int_{\Omega_0} D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right) \bar{f}(y)\, dy \\
&= \int_{\Omega_0} D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right) \left(\bar{f}(y) - f(x)\right) dy \\
&\qquad + f(x)\int_{\Omega_0} D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right)\, dy \\
&= \int_{\Omega_0} D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right) \left(\bar{f}(y) - f(x)\right) dy \\
&\qquad - f(x)\int_{\Omega_0} D_{y_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right)\, dy \\
&= \int_{\Omega_0} D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right) \left(\bar{f}(y) - f(x)\right) dy \\
&\qquad - f(x)\int_{\partial\Omega_0} D_i\Gamma(x-y)\eta((x-y)/\epsilon)\nu_j(y)\, d\sigma(y)
\end{aligned}$$

from the divergence theorem. Since $x \in \Omega$, if we take $\epsilon \leq \text{dist}(x, \partial\Omega_0)/2$, then $|x-y| \geq 2\epsilon$ for $y \in \partial\Omega_0$ and so $\eta((x-y)/\epsilon) = 1$. So

$$\begin{aligned}
D_j v_\epsilon(x) &= \int_{\Omega_0} D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right) \left(\bar{f}(y) - f(x)\right) dy \\
&\qquad - f(x)\int_{\partial\Omega_0} D_i\Gamma(x-y)\nu_j(y)\, d\sigma(y).
\end{aligned}$$

Then subtracting we get

$$
\begin{aligned}
&u(x) - D_j v_\epsilon(x)\\
&= \int_{\Omega_0} \Big(D_{ij}\Gamma(x-y) - D_{x_j}\left(D_i\Gamma(x-y)\eta((x-y)/\epsilon)\right)\Big)(\bar{f}(y) - f(x))\,dy\\
&= \int_{\Omega_0} D_{x_j}\left[(1-\eta((x-y)/\epsilon))\,D_i\Gamma(x-y)\right](\bar{f}(y)-f(x))\,dy\\
&= \int_{|x-y|\le 2\epsilon} D_{x_j}\left[(1-\eta((x-y)/\epsilon))\,D_i\Gamma(x-y)\right](\bar{f}(y)-f(x))\,dy\\
&= \int_{|x-y|\le 2\epsilon}\left[D_{ij}\Gamma(x-y)\,(1-\eta((x-y)/\epsilon)) - D_i\Gamma(x-y)\frac{1}{\epsilon}\eta'(|x-y|/\epsilon)\frac{x_j-y_j}{x-y}\right](\bar{f}(y)-f(x))\,dy.
\end{aligned}
$$

Let $K \subset \Omega$ be compact, and $K' = \{y : \operatorname{dist}(y,K) \le \operatorname{dist}(K,\partial\Omega)/2\}$. We have K' is compact, $K' \subset \Omega$, and if $\epsilon < \operatorname{dist}(K,\partial\Omega)/2$, then $B_\epsilon(x) \subset K'$ for all $x \in K$. Therefore estimating the last integral we obtain for $x \in K$

$$
\begin{aligned}
|u(x) - D_j v_\epsilon(x)| &\le \int_{|x-y|\le 2\epsilon}\left[\left|D_{ij}\Gamma(x-y)\right| + \left|D_i\Gamma(x-y)\right|\frac{C}{\epsilon}\right]|f(y)-f(x)|\,dy\\
&\le C_{K'}\int_{|x-y|\le 2\epsilon}\left[\frac{C}{|x-y|^n} + \frac{C}{|x-y|^{n-1}}\frac{C}{\epsilon}\right]|x-y|^\alpha\,dy \le C'_{K'}\epsilon^\alpha.
\end{aligned}
$$

Therefore $D_j v_\epsilon \to u$ uniformly on compact subsets of Ω as $\epsilon \to 0$, so u is continuous in Ω. In addition, $v_\epsilon \to v$ uniformly in $\mathbb{R}^n$ (this follows directly by subtracting). This implies that $w \in C^2(\Omega)$ and $u(x) = D_{ij}w(x)$ for $x \in \Omega$ by the fundamental theorem of calculus, because we write

$$
v_\epsilon(x_1,\cdots,x_{j-1},y_j,x_{j+1},\cdots,x_n) = \int_{x_j}^{y_j} D_j v_\epsilon(x_1,\cdots,x_{j-1},t,x_{j+1},\cdots,x_n)\,dt + v_\epsilon(x),
$$

and pass to the limit as $\epsilon \to 0$. So we have proved the following representation formula for the second derivatives of the Newtonian potential w for $x \in \Omega$:

$$
D_{ij}w(x) = \int_{\Omega_0}(D_{ij}\Gamma)(x-y)(\bar{f}(y)-f(x))\,dy - f(x)\int_{\partial\Omega_0} D_i\Gamma(x-y)\nu_j(y)\,d\sigma(y), \tag{2}
$$

under the assumption that f is bounded in Ω and locally Hölder continuous for some $0 < \alpha \le 1$.

We next prove that $\Delta w = f$ in Ω. Let $x \in \Omega$, take R sufficiently large such that $\Omega \subset B_R(x)$, and apply (2) with $\Omega_0 = B_R(x)$, then

$$
D_{ii}w(x) = \int_{|x-y|<R}(D_{ij}\Gamma)(x-y)(\bar{f}(y)-f(x))\,dy - f(x)\int_{|x-y|=R} D_i\Gamma(x-y)\nu_j(y)\,d\sigma(y),
$$

adding over $1 \le i \le n$ and using that $\Delta\Gamma(x) = 0$ for $x \neq 0$, we obtain

$$\Delta w(x) = -f(x) \int_{|x-y|=R} D\Gamma(x-y) \cdot \nu(y)\, d\sigma(y).$$

We have $D\Gamma(x) = C_n(2-n)\dfrac{x}{|x|^n}$ and $\nu(y) = \dfrac{y-x}{|x-y|}$. Then $\Delta w(x) = f(x)C_n(2-n)\omega_n$, where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. Since $C_n = \dfrac{1}{(2-n)\omega_n}$, we are done.

We also have that $\Delta w(x) = 0$ for $x \notin \bar{\Omega}$, which follows differentiating twice under the integral sign (the justification follows from Theorem 1). So we have proved the following theorem.

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, ρ be a bounded function in Ω. Then*

(1) $w \in C^1(\mathbb{R}^n)$;
(2) $w \in C^2(\mathbb{R}^n \setminus \bar{\Omega})$;
(3) $\Delta w = 0$ *in* $\mathbb{R}^n \setminus \bar{\Omega}$;
(4) *if in addition ρ is Hölder continuous, then $w \in C^2(\Omega)$ and $\Delta w = \rho$ in Ω.*
(5) *the second derivatives of w might not exist in $\partial\Omega$.*

Actually, the last item was not proved and we leave it as an exercise. For that consider in $\mathbb{R}^3$ the ball $B(0,R)$. Assume that the ball is homogeneous with density λ. The Newtonian potential due to this ball is given by

$$U(x) = G \int_{B(0,R)} \lambda \frac{1}{|y-x|}\, dy.$$

(a) Show that

$$U(x) = \begin{cases} G\lambda \frac{4}{3}\pi R^3 \frac{1}{|x|}, & \text{for } |x| > R \\ G\lambda \frac{2}{3}\pi (3R^2 - |x|^2), & \text{for } 0 \le |x| \le R. \end{cases}$$

HINT: to simplify the calculation notice that $U(Ox) = U(x)$ for any rotation O around the origin, that is, U is a radial function.

(b) Prove that the first order derivatives of U exist and are continuous everywhere.
(c) Prove that all the second derivatives of U exist and are continuous in $\mathbb{R}^3 \setminus \{x : |x| = R\}$. Prove that they do not exist on $|x| = R$.

(d) Prove that

$$\Delta U(x) = \operatorname{div} DU(x) = \begin{cases} -G\lambda 4\pi, & \text{for } 0 \le |x| < R \\ 0, & \text{for } |x| > R. \end{cases}$$

5. Solution in all space

Let $w(x) = \int_{\mathbb{R}^n} \Gamma(x-y)\rho(y)\,dy$. We shall prove that under some conditions on ρ, we have that w is C^2 everywhere and $\Delta w(x) = \rho(x)$ for all $x \in \mathbb{R}^n$. Fix $R > 0$ and let $\phi \in C^\infty(\mathbb{R}^n)$ be such that $\phi(y) = 1$ for $|y| \le 2R$, $\phi(y) = 0$ for $|y| \ge 3R$ and $0 \le \phi \le 1$.

Write

$$\begin{aligned} w(x) &= \int_{\mathbb{R}^n} \Gamma(x-y)\rho(y)\phi(y)\,dy + \int_{\mathbb{R}^n} \Gamma(x-y)\rho(y)(1-\phi(y))\,dy \\ &:= w_1(x) + w_2(x). \end{aligned}$$

We have

$$w_1(x) = \int_{B_{3R}(0)} \Gamma(x-y)\rho(y)\phi(y)\,dy$$

and from what we proved before we have that if ρ is locally Hölder continuous in $\mathbb{R}^n$, then $w_1 \in C^2(B_{3R}(0))$ and $\Delta w_1(x) = \rho(x)\phi(x)$ for all $x \in B_{3R}(0)$. Therefore $\Delta w_1(x) = \rho(x)$ for $x \in B_{2R}(0)$.

We shall prove that $\Delta w_2(x) = 0$ for all $x \in B_R(0)$. If $\rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then from Theorem 2 we have that $w_2 \in C^1(\mathbb{R}^n)$ and

$$Dw_2(x) = \int_{\mathbb{R}^n} D_x\Gamma(x-y)\rho(y)(1-\phi(y))\,dy. \tag{3}$$

Let $x_0 \in B_R(0)$, then $B_{R/2}(x_0) \subset B_{3R/2}(0)$. Notice that the integrand in w_2 vanishes for $|y| \le 2R$, and therefore the domain of integration reduces to the set $|y| \ge 2R$. We want to differentiate under the integral sign in (3), and for that we need to verify the hypotheses in Theorem 1. Indeed, we have that $|\partial^2_{x_jx_j}\Gamma(x-y)| \le \dfrac{C}{|x-y|^n}$, for $x \ne y$, and since $|x-y| \ge R/2$ for $x \in B_R(x_0)$ and $|y| \ge 2R$

$$|\partial_{x_jx_j}\Gamma(x-y)| \le \frac{2^nC}{R^n}$$

there. So if we let $g(y) = \dfrac{2^nC}{R^n}\rho(y)(1-\phi(y))$, then we obtain that

$$|h(x,y) := \partial_{x_jx_j}\Gamma(x-y)\rho(y)(1-\phi(y))| \le g(y)$$

for all $x \in B_{R/2}(x_0)$ and for all $y \in \mathbb{R}^n$. In addition, $h(\cdot, y) \in C(B_{R/2}(x_0))$ for all $y \in \mathbb{R}^n$. Therefore, we can differentiate under the integral sign and we obtain

$$\partial_{x_j x_j} w_2(x) = \int_{\mathbb{R}^n} \partial_{x_j x_j} \Gamma(x-y)\rho(y)(1-\phi(y))\, dy,$$

for all $x \in B_{R/2}(x_0)$ and so

$$\Delta w_2(x) = \int_{\mathbb{R}^n} \Delta\Gamma(x-y)\rho(y)(1-\phi(y))\, dy = 0,$$

for all $x \in B_{R/2}(x_0)$, for each $x_0 \in B_R(0)$.

Since R is arbitrary, we therefore have proved the following theorem:

Theorem 4. *If ρ defined in $\mathbb{R}^n$ is locally Hölder continuous of order α, for some $0 < \alpha \leq 1$, and $\rho \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then the Newtonian potential*

$$w(x) = \int_{\mathbb{R}^n} \Gamma(x-y)\rho(y)\, dy$$

is C^2 in all space, and satisfies the Poisson equation $\Delta w(x) = \rho(x)$ in $\mathbb{R}^n$.

6. The Newtonian potential is not necessarily 2nd order differentiable

We show that there exists f continuous in $B_2(0)$ such that

$$w(x) = \int_{B_2(0)} \Gamma(x-y)\, f(y)\, dy$$

is not second order differentiable at the origin. The example is from [GT83, Exercise 4.9]. Assume $n \geq 3$. Let c_k be a sequence such that $c_k \to 0$ and $k \to \infty$ and $\sum_{k=0}^{\infty} c_k = +\infty$. Let $\eta \in C_0^\infty(B_2(0))$ with $\eta(x) = 1$ for $|x| < 1$, and let P be a harmonic polynomial of degree two in $\mathbb{R}^n$ with $D^\alpha P \neq 0$ for some $|\alpha| = 2$, e.g., $P = x_1 x_2$. Let

$$f(x) = \sum_{k=0}^{\infty} c_k\, \Delta(\eta P)(2^k x).$$

Given $x \neq 0$ in $B_2(0)$ there exists a unique integer $N \geq 0$ such that $2^{-N} \leq |x| < 2^{1-N}$. Notice that $\Delta(\eta P)(x) = 0$ for $|x| > 2$ and for $|x| < 1$. We have

$$f(x) = \sum_{\{k: 2^{-k} \leq |x| < 2^{1-k}\}} \cdots = c_N\, \Delta(\eta P)(2^N x).$$

This function is clearly continuous in $B_2(0)$. We write

$$\begin{aligned} w(x) &= \sum_{k=0}^{\infty} \int_{2^{-k}\le|y|<2^{1-k}} \Gamma(x-y)\, f(y)\, dy \\ &= \sum_{k=0}^{\infty} c_k \int_{2^{-k}\le|y|<2^{1-k}} \Gamma(x-y)\, \Delta(\eta P)(2^k y)\, dy = \sum_{k=0}^{\infty} c_k\, I_k(x). \end{aligned}$$

Changing variables we have

$$I_k(x) = 2^{-2k} \int_{1\le|z|<2} \Gamma(2^k x - z)\, \Delta(\eta P)(z)\, dz = 2^{-2k} \int_{\mathbb{R}^n} \Gamma(2^k x - z)\, \Delta(\eta P)(z)\, dz$$

and from the 3rd Green formula since ηP has compact support we get as in Section 3 that

$$\int_{\mathbb{R}^n} \Gamma(2^k x - z)\, \Delta(\eta P)(z)\, dz = (\eta P)(2^k x).$$

Therefore

$$w(x) = \sum_{k=0}^{\infty} c_k\, 2^{-2k}\, (\eta P)(2^k x).$$

Given $|x| < 2$ there exists a unique $N \ge 0$ integer such that $2^{-N} \le |x| < 2^{1-N}$. So $|2^k x| \ge 2^{k-N} \ge 2$ for $k \ge N+1$, and so $(\eta P)(2^k x) = 0$ for $k \ge N+1$. On the other hand, $|2^k x| < 2^k 2^{1-N} \le 1$ if $k \le N-1$ and hence $\eta(2^k x) = 1$ for $k \le N-1$. Therefore

$$w(x) = \sum_{k=0}^{N-1} c_k\, 2^{-2k}\, P(2^k x) + c_N\, 2^{-2N}\, (\eta P)(2^N x), \quad \text{for } 2^{-N} \le |x| < 2^{1-N}.$$

Thus

$$D^\alpha w(x) = \sum_{k=0}^{N-1} c_k\, D^\alpha P(2^k x) + c_N\, D^\alpha(\eta P)(2^N x), \quad |\alpha| = 2, \text{ for } 2^{-N} \le |x| < 2^{1-N}.$$

If $P = x_1 x_2$, then

$$D_{12} w(x) = \sum_{k=0}^{N-1} c_k + c_N\, D_{12}(\eta P)(2^N x), \quad \text{for } 2^{-N} \le |x| < 2^{1-N}.$$

If $x \to 0$, then $N \to \infty$ and therefore $D_{12} w(x) \to +\infty$.

7. Helmholtz decomposition

Let F be a vector field in $\mathbb{R}^3$ we shall prove that under appropriate hypotheses

$$F = R + S$$

with div $S = 0$ and curl $R = 0$. This means that

$$F = Du + \text{curl } H$$

for some function u and some field H.

Recall that *if G is a vector field with* div $G = 0$ *in* $\mathbb{R}^3$*, then there exists a field H such that* $G =$ curl H. Also, if G is a field with curl $G = 0$, then there exists a function u such that $G = Du$. In addition, if $G =$ curl $H_1 =$ curl H_2, then $H_1 - H_2 = D\phi$ for some function ϕ.

Assuming the desired decomposition, we have div $F =$ div $S +$ div $R =$ div R. Since curl $R = 0$, there exists u with $R = Du$, and so div $F =$ div $Du = \Delta u$.

Therefore, if div F satisfies the hypotheses of Theorem 4 and let u be the Newtonian potential of div F, then $\Delta u =$ div F. So if we let

$$R = Du, \text{ and } S = F - R,$$

then curl $R = 0$, and div $S =$ div $F -$ div $R =$ div $F -$ div $Du = 0$.

So we have:

Theorem 5 (Helmholtz decomposition). *Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a field such that* div F *is locally Hölder continuous in $\mathbb{R}^3$, uniformly bounded and integrable over all space. If*

$$w(x) = \int_{\mathbb{R}^3} \Gamma(x - y) \,\text{div}\, F(y)\, dy,$$

then

$$F(x) = Dw(x) + \text{curl } H(x)$$

for some field H. The field H is determined up to the gradient of an arbitrary function ϕ.

If H is a field we define the Laplacian of F by

$$\Delta H = D(\text{div } H) - \text{curl } (\text{curl } H).$$

If $H = a_1(x)\mathbf{i} + a_2(x)\mathbf{j} + a_3(x)\mathbf{k}$, then is easy to see calculating the components of ΔH that

$$\Delta H = \Delta a_1(x)\mathbf{i} + \Delta a_2(x)\mathbf{j} + \Delta a_3(x)\mathbf{k}.$$

If we assume div $H = 0$, then $\Delta H = -$curl (curl H) $= -$curl F by the Helmholtz decomposition. But the equation $\Delta H = -$curl F is a Poisson equation now between vectors, that is, three scalar Poisson equations $\Delta a_i = -(\text{curl } F)_i$, $i = 1, 2, 3$. Solving each one as before with the Newtonian potential we get that

$$H(x) = -\int_{\mathbb{R}^3} \Gamma(x-y) \,\text{curl}\, F(y)\, dy,$$

that is, H is the Newtonian potential (now a vector) of $-\text{curl } F$. This means that if for a given field F we know div F and curl F, then F is automatically determined by the formula

$$F(x) = D\left(\int_{\mathbb{R}^3} \Gamma(x-y) \,\text{div}\, F(y)\, dy\right) + \text{curl}\left(-\int_{\mathbb{R}^3} \Gamma(x-y) \,\text{curl}\, F(y)\, dy\right).$$

References

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Department of Mathematics, Temple University, Philadelphia, PA 19122
E-mail address: gutierre@temple.edu