1. Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ and $\rho$ be bounded and integrable over $\Omega$. Let

$$F(x) = \int_{\Omega} \rho(y) \frac{y - x}{|y - x|^3} \, dy.$$  

$F(x)$ is the gravitational force (up to a multiplicative constant) felt at the point $x$ by the presence of the solid $\Omega$ having density $\rho$.

Prove that there exists a constant $C$ such that

$$|F(x) - F(z)| \leq C |x - z| \ln \left( \frac{1}{|x - z|} + 1 \right)$$

for all $x, y \in \mathbb{R}^3$.

It was proved in class that $F(x)$ is well defined for all $x \in \mathbb{R}^3$.

Proceed in steps:

Step 1. Prove that $|D_j \left( x_j/|x|^3 \right)| \leq C |x|^{-3}$ for $j = 1, 2, 3$.

Step 2. Let

$$I_j = \int_{\Omega} \rho(y) \left( \frac{y_j - x_j}{|y_j - x_j|^3} - \frac{y_j - z_j}{|y_j - z_j|^3} \right) \, dy,$$

that is, $I_j$ is the $j$-component of $F(x) - F(z)$, $j = 1, 2, 3$. Let $\bar{x}$ be the midpoint between $x$ and $z$ and consider the ball $B_{2|x|}(\bar{x})$. Extend $\rho$ to be zero outside $\Omega$, and write

$$I_j = \int_{B_{2|x|}(\bar{x})} \cdots \, dy + \int_{|y - \bar{x}| > 2|x|} \cdots \, dy = A_j + B_j.$$

Step 3. Since $\rho$ is bounded, comparing the size of the balls show that

$$|A_j| \leq C \left( \int_{B_{5|x|}(\bar{x})} \frac{1}{|y - x|^2} \, dy + \int_{B_{5|x|}(\bar{x})} \frac{1}{|y - z|^2} \, dy \right) = C |x - z|.$$

Step 4. Use the mean value theorem and Step 1 to show that

$$|B_j| \leq C \int_{|y - \bar{x}| > 2|x|} \rho(y) \frac{|z - x|}{|y - \bar{x}|^3} \, dy$$

and integrate to obtain the desired estimate.