Homework 2, First order pdes II (Due on 9/19/2013)

1. Solve the following Cauchy problems

   1. \( u_x = (u_y)^2, u(0, y) = y^2/2; \) ANS: \( u = y^2/(2(1 - 2x)) \);
   2. \( xu_x + yu_y + u_xu_y = u, u(s, 0) = s^2; \) ANS: \( u = (4x - y)^2/16 \);
   3. \( x(u_x)^2 + yu_y = 0, u(s, 1) = -s; \) ANS: \( u = x/(\ln y - 1) \);
   4. \( x(u_x)^2 + (u_y)^3 = 1, u(s, 0) = \sqrt{s}, s > 0; \) ANS: \( u = \sqrt{x} + (3/4)^{1/3}y \).

2. The continuity of the coefficients of a linear pde does not guarantee existence of solutions. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function that is differentiable nowhere and let \( g : \mathbb{R} \to \mathbb{R} \) be any \( C^1 \) function. Show that the pde
   
   \[ u_t(x, y) - u_y(x, y) = f(x + y), \quad \text{with } u(0, y) = g(y) \]

   has no \( C^1 \) solutions.
   HINT: suppose there is a \( C^1 \) solution \( u \) and consider \( v(x, y) = u(x, y - x) \).

3. Let \( u = u(x) \) with \( x \in \mathbb{R}^n \) be a solution of the pde

   \[ F(x, Du(x)) = \sum_{k=1}^{n} a_k(x) \frac{\partial u}{\partial x_k}(x) = 0. \]

   Prove that if \( x(t) = (x_1(t), \cdots, x_n(t)) \) is a characteristic curve, i.e., \( x'_k(t) = a_k(x(t)), k = 1, \cdots, n \), then \( u(x(t))=\text{constant} \), that is, the characteristic curves are contained in the level surfaces of the solution \( u \).

4. The Hamilton-Jacobi equation has the form

   \[ u_{x_{n+1}} + H(x, x_{n+1}, D_xu) = 0 \]

   where \( u = u(x, x_{n+1}), x_{n+1} \in \mathbb{R}, x = (x_1, \cdots, x_n), D_xu = (u_{x_1}, \cdots, u_{x_n}) \); the function \( H \) is called the Hamiltonian, \( H = H(x, x_{n+1}, p), p = (p_1, \cdots, p_n) \).

   (a) Find the characteristic equations for the Hamilton-Jacobi equation.
   (b) Show that to solve these equations if is enough to solve the system:

   \[ x'_i(t) = H_{p_i}, \quad p'_i(t) = -H_{x_i}, \quad i = 1, \cdots, n; \]

   this system is called the Hamilton canonical system and appears in mechanics and the calculus of variations, see [CH62, pp. 106-131, Vol. II] for applications.

References