1. Let \( a < b, c < d \). Define
\[
f(x) = \begin{cases} 
ax \sin \frac{1}{x} + bx \cos \frac{1}{x}, & \text{for } x > 0 \\
0, & \text{for } x = 0 \\
cx \sin \frac{1}{x} + dx \cos \frac{1}{x}, & \text{for } x > 0.
\end{cases}
\]
Calculate \( D^- f, D^+ f, D_- f, D_+ f \) at \( x = 0 \).

2. Let \( f \) be a continuous function in \([-1,2]\). Given \( 0 \leq x \leq 1 \), and \( n \geq 1 \) define the sequence of functions
\[
f_n(x) = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t) \, dt.
\]
Show that \( f_n \) is continuous in \([0,1]\) and \( f_n \) converges uniformly to \( f \) in \([0,1]\).

3. Let \( f_n : \mathbb{R} \to \mathbb{R} \) be a uniformly bounded sequence of functions. Show that for each countable subset \( S \subset \mathbb{R} \) there exists a subsequence of \( f_n \) which converges in \( S \).
Hint: select the subsequence by using a diagonal process

4. Let \( f_n \) be absolutely continuous functions in \([a,b]\), \( f_n(a) = 0 \). Suppose \( f'_n \) is a Cauchy sequence in \( L^1([a,b]) \). Show that there exists \( f \) absolutely continuous in \([a,b]\) such that \( f_n \to f \) uniformly in \([a,b]\).

5. Let \( f_n(x) = \cos(nx) \) on \( \mathbb{R} \). Prove that there is no subsequence \( f_{n_k} \) converging uniformly in \( \mathbb{R} \).

6. Let \( f_1, \ldots, f_k \) be continuous real valued functions on the interval \([a,b]\). Show that the set \( \{f_1, \ldots, f_k\} \) is linearly dependent on \([a,b]\) if and only if the \( k \times k \) matrix with entries
\[
\langle f_i, f_j \rangle = \int_a^b f_i(x) f_j(x) \, dx
\]
has determinant zero.

7. Let \( f : [0, +\infty) \to \mathbb{R} \) be continuously differentiable with compact support in \([0, +\infty)\); and \( 0 < a < b < \infty \). Prove that
\[
\int_0^\infty \frac{f(bx) - f(ax)}{x} \, dx = -f(0) \ln(b/a).
\]

8. Find all the values of \( p \) such that the integral
\[
\int_0^\infty \int_0^{\pi/2} e^{-xy} \sin \theta \, y \, dx \, dy
\]
converges.

9. If \( E \subset [0,1] \) is a measurable set such that \( |E \cap I| \geq \alpha|I| \) for some \( \alpha > 0 \) and for all intervals \( I \subset [0,1] \), then \( |E| = 1 \).
10. If $F \subset \mathbb{R}$ is closed and $\delta(x) = \text{dist}(x, F)$, then $\delta(x + y) \leq |y|$ for all $x \in F$. Moreover, $\frac{\delta(x + y)}{|y|} \to 0$ as $|y| \to 0$ for a.e. $x \in F$.

11. If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then $|f(E)| = 0$ for all $E \subset [a, b]$ with $|E| = 0$.

12. Let $\alpha > 0$ and $H_\alpha$ be the Hausdorff outer measure in $\mathbb{R}^n$. Given $A \subset \mathbb{R}^n$ and $t > 0$, let $\delta_t A = \{tx : x \in A\}$. Prove that $H_\alpha(\delta_t A) = t^\alpha H_\alpha(A)$.

13. Prove that
   
   1. $H_\alpha(\mathbb{R}^n) = +\infty$ for all $0 < \alpha \leq n$.
   
   2. if $\alpha \leq \beta$, then $H_\alpha(A) \geq H_\beta(A)$ for all $A \subset \mathbb{R}^n$.

14. Prove that the Hausdorff measure is not $\sigma$-finite for $0 < \alpha < n$.

15. Prove that the Hausdorff dimension of the set $A$ equals $\dim_H(A) = \inf\{\alpha : H_\alpha(A) < +\infty\} = \sup\{\alpha : H_\alpha(A) = +\infty\}$, and $\dim_H(\bigcup_{j=1}^\infty A_j) = \sup_j \dim_H(A_j)$.

16. Determine all values of $p$ such that $\lim_{x \to 0} \frac{\sin(|\sin x|^p)}{x}$ exists and calculate its value.

17. Show that the series $\sum_{n=1}^{\infty} (\cos(1/n))^{n^2}$ diverges.
   
   HINT: $\cos 1/n = \sqrt{1 - (\sin 1/n)^2}$, $0 \leq \sin 1/n \leq 1/n$.

18. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of equicontinuous functions. Prove that the set
   
   $\{x \in \mathbb{R} : \{f_n(x)\} \text{ is a Cauchy sequence}\}$
   
   is closed.

19. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be continuous with compact support. Define the following multiplication:
   
   $(x, y, t) \circ (x_0, y_0, t_0) = (x + x_0, y + y_0, t + t_0 + xy_0 - yx_0)$. Prove that
   
   $$\int_{\mathbb{R}^3} f((x, y, t) \circ (x_0, y_0, t_0)) \, dxdydt = \int_{\mathbb{R}^3} f(x, y, t) \, dxdydt$$
   
   for all $(x_0, y_0, t_0) \in \mathbb{R}^3$.

20. Let $f : [0, 1] \to \mathbb{R}$ be continuous and let $g_n(x) = x^n f(x)$ for $0 \leq x \leq 1$. Prove that $g_n$ converges uniformly in $[0, 1]$ iff $f(1) = 0$.

21. Let $f$ be non-negative and measurable on $[0, 1]$. Prove that
   
   $$\int_0^1 \int_x^y f(x) f(y) f(z) \, dz \, dy \, dx = \frac{1}{3!} \left( \int_0^1 f(x) \, dx \right)^3.$$
22. An ellipsoid in \( \mathbb{R}^n \) centered at \( x_0 \) is a set of the form
\[
E = \{ x \in \mathbb{R}^n : \langle A(x - x_0), x - x_0 \rangle \leq 1 \}
\]
where \( A \) in an \( n \times n \) positive definite symmetric matrix and \( \langle , \rangle \) denotes the standard Euclidean product. Prove that
\[
|E| = \frac{\omega_n}{(\det A)^{1/2}}
\]
where \( \omega_n \) is the volume of the unit ball.

23. Let \( f_n(x) = \frac{1}{\ln(n+1)} \frac{nx}{1 + n^2 x^4} \) for \( 0 \leq x \leq 1 \). Prove that \( f_n \) converges pointwise in \([0, 1]\) and not uniformly. Prove that \( f_n \to 0 \) in measure and \( \int_0^1 f_n(x) \, dx \to 0 \) as \( n \to \infty \).

24. Let \( 0 < x_n \leq a \) and \( x_n \to a \) as \( n \to \infty \). Prove that the series \( \sum_{n=1}^{\infty} (a - x_n) \) converges if and only if \( \frac{x_1 + \cdots + x_N}{a^N} \to \ell \) for some \( \ell > 0 \) as \( N \to \infty \). HINT: \( \ln x \sim (x - 1) \) as \( x \to 1^- \).

25. Let \( f : [a, b] \to \mathbb{R} \) integrable. Prove that the functions \( f_n(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) \, dt \) are well defined for \( a \leq x \leq b, n = 1, 2, \cdots \) and satisfy \( \int_a^b f_n(t) \, dt = f_{n+1}(x) \).

26. Investigate the convergence of the integral
\[
\int_0^{\infty} \frac{|\sin x|}{e^{x^2}} \, dx.
\]
HINT: write the integral as \( \sum_{n=0}^{\infty} \int_\pi^{(n+1)\pi} \cdots \); next make the change of variables \( x = y + n\pi \) and break the domain of integration into \( |\sin y| \geq \pi/\sqrt{n} \) and \( |\sin y| \leq \pi/\sqrt{n} \). Show that the first piece adds ok, and for the second piece use that \( \sin y \sim y \), dominate the integrand and change variables to show that it is \( O(1/n^2) \).

27. Let \( \Omega \) be an uncountable set and let \( \mathcal{F} = \{ A \subset \Omega : A \text{ is countable or } A^c \text{ is countable} \} \). Prove that \( \mathcal{F} \) is a \( \sigma \)-algebra. Define the measures \( \mu(A) = +\infty \) if \( A \) is an infinite set, \( \mu(A) = \#(A) \) if \( A \) is finite; \( \nu(A) = 0 \) if \( A \) is countable and \( \nu(A) = 1 \) if \( A \) is uncountable. Prove that \( \nu \) is absolutely continuous with respect to \( \mu \) but an integral representation of the form \( \nu(A) = \int_A f \, d\mu \) is not valid. Why this does not contradicts Radon-Nikodym’s theorem?

28. Let \( (X, \Sigma, \mu) \) be a finite measure space, \( \{E_j\}_{j=1}^N \subset \Sigma \) and \( \{c_j\}_{j=1}^N \subset \mathbb{R} \). For \( E \in \Sigma \) define
\[
\nu(E) = \sum_{j=1}^N c_j \mu(E \cap E_j).
\]
Prove that \( \nu \) is absolutely continuous with respect to \( \mu \) and find the Radon-Nikodym derivative \( \frac{d\nu}{d\mu} \).
29. Let $\mu_k$ be a sequence of measures on a $\sigma$-algebra $\Sigma$ of a set $X$ such that $\nu_k(X) \leq C$ for all $k$. Define $\mu = \sum_{k=1}^{\infty} \frac{\mu_k}{2^k}$. Prove that $\mu$ is a measure in $\Sigma$ and $\mu_k$ is absolutely continuous with respect to $\mu$ for each $k$.

30. Let $\mu$ be a finite Borel measure on $[a, b]$ that is absolutely continuous with respect to Lebesgue measure. Prove that $g(x) = \mu([a, x])$ is an absolutely continuous function in $[a, b]$.

31. Let $\mu$ be a finite Borel measure on the interval $[a, b]$. Prove that the function $g(x) = \mu([a, x])$ is non decreasing and continuous function from the right on $[a, b]$.

32. If $A, B$ are $\mu$-measurable and $\mu(A \cup B) = \mu(A) + \mu(B)$, then $\mu(A \cap B) = 0$. HINT: use the Carathéodory condition.